MATH 449, HOMEWORK 6

DUE NOVEMBER 5, 2014

Part I. Theory

Problem 1. Given $x_0, x_1, \dots, x_n \in \mathbb{R}$, define the *Vandermonde matrix*

$$V = \begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix}.$$

Prove that V is invertible if and only if $x_0, \dots, x_n$ are distinct. Hint: Let $p_n(x) = a_0 + a_1 x + \cdots + a_n x^n$, and consider the meaning of the linear system

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}.$$

Problem 2. Recall that the interpolating polynomial, in Lagrange form, is

$$p_n(x) = \sum_{k=0}^{n} L_k(x) y_k, \quad \text{where} \quad L_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{x - x_i}{x_k - x_i}. \tag{1}$$

This formula, however, does not give the most efficient or numerically stable way to evaluate p_n at a point x. In this problem, we will explore an alternate approach, called *barycentric Lagrange interpolation*[1], which is generally preferred for numerical implementation.

a. Explain why evaluating (1), as written, requires $\mathcal{O}(n^2)$ operations.

b. For $k = 0, \dots, n$, define the *barycentric weights*

$$w_k = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{1}{x_k - x_i}.$$

Show that the interpolating polynomial can be rewritten as

$$p_n(x) = \pi_{n+1}(x) \sum_{k=0}^{n} \frac{w_k}{x - x_k} y_k, \qquad x \neq x_0, \dots, x_n, \tag{2}$$

where (following Süli–Mayers notation) $\pi_{n+1}(x) = \prod_{i=0}^{n} (x - x_i)$.

[1] Berrut and Trefethen, SIAM Rev., 46 (3), 501–517, 2004.

c. Show that for every $x \neq x_0, \ldots, x_n$,

$$1 = \pi_{n+1}(x) \sum_{k=0}^{n} \frac{w_k}{x - x_k}$$

(hint: interpolate the constant function 1), and combine this with (2) to deduce the *barycentric interpolation formula*

$$p_n(x) = \frac{\displaystyle\sum_{k=0}^{n} \frac{w_k}{x - x_k} y_k}{\displaystyle\sum_{k=0}^{n} \frac{w_k}{x - x_k}}, \qquad x \neq x_0, \ldots, x_n. \tag{3}$$

d. Show that computing the barycentric weights requires $\mathcal{O}(n^2)$ operations, but that subsequently evaluating (2) or (3) requires only $\mathcal{O}(n)$ operations. (Note that the barycentric weights only need to be computed once, since they are independent of x.)

Part II. Programming

For this part of the assignment, you will program the barycentric interpolation method described in Problem 2. There is no sample code for this assignment, so you should create a new file `hw6.py` beginning with the usual lines:

```
from __future__ import division
from pylab import *
```

Hand in a printed copy of your code, as well as a printout of the IPython terminal session(s) containing the commands and output you used to get your answers.

Problem 3. Create a function `weights(xk)` that takes the array of interpolation points `xk` $= (x_0, \ldots, x_n)$ and returns the array of barycentric weights `wk` $= (w_0, \ldots, w_n)$. Print the output of the following command:

```
weights(linspace(-5,5,11))
```

Problem 4. Create a function `interpolate(x, xk, yk, wk)` that uses the barycentric interpolation formula (3) to return the value of $p_n(x)$. Here, `x` is the evaluation point, `xk` is the array of interpolation points $(x_0, \ldots, x_n)$, `yk` is the array of interpolation values $(y_0, \ldots, y_n)$, and `wk` is the array of barycentric weights $(w_0, \ldots, w_n)$.

a. Let $f(x) = \dfrac{1}{1 + x^2}$. Evaluate $p_{10}(x)$ at $x = 4.75$, where the interpolation points are $-5, -4, \ldots, 4, 5$, i.e., `xk = linspace(-5,5,11)`, and compare this with the actual value of $f(x)$.

b. Repeat this for $p_{20}(x)$, with `xk = linspace(-5,5,21)`, and $p_{50}(x)$, with `xk = linspace(-5,5,51)`. Is the interpolated value $p_n(x)$ getting more accurate as n increases?