MATH 449, HOMEWORK 1

DUE SEPTEMBER 4, 2015

Part I. Theory

Problem 1. Brouwer's fixed point theorem (Süli and Mayers, Theorem 1.2) only holds for *closed* intervals $[a, b]$. For each of the following intervals I, find a continuous function $g\colon I \to I$ that does *not* have a fixed point in I.

a. $I = (0, 1]$
b. $I = [0, 1)$
c. $I = [0, \infty)$

Problem 2. The simple iteration $x_{k+1} = \cos x_k$ always converges to a unique fixed point $\xi = \cos \xi$, no matter which x_0 you start with. (If you have a scientific calculator or calculator app, you can see this by putting in any number and hitting `cos` repeatedly—in radians mode, of course.) Prove this using the contraction mapping theorem.

Hint: Use the mean value theorem, as in Example 1.2 in Süli and Mayers, to show that $x \mapsto \cos x$ is a contraction on some appropriately chosen interval.

Part II. Programming

To get started, download and install the Anaconda software package from `http://continuum.io/downloads`. Next, start the Anaconda Launcher, and install and launch the Spyder application. (You can also launch *Spyder* by typing `spyder` in a command line terminal.) Finally, open the file `hw1.py` (available on the class web page) in Spyder, and click the green "play" button to run the code in the IPython console.

Problem 3. Starting with $x_0 = 1$, how many iterations n of Heron's method are needed so that $x_n \approx \sqrt{2}$ is correct to 6 decimal places? What about $\sqrt{0}$?

Problem 4. The function `plotLogError` plots $\log|x_k - \sqrt{y}|$ for Heron's method. Look at the cases $y = 2, 3$ when $x_0 = 1$. For each of these plots, you should notice that something strange happens after a few steps. Can you explain the observed behavior?

Problem 5. What happens when you try to apply Heron's method to $y < 0$? In particular, can you explain what happens for $y = -1, -2, -3$ when $x_0 = 1$?

Problem 6. Create a new function `cosArray(x0,n)`, which implements the recursion $x_{k+1} = \cos x_k$ from Problem 2 but is otherwise similar to `heronArray`. Run the command `plot(cosArray(1,40))`, and print the resulting plot. Give the fixed point $\xi = \cos \xi$ to 6 decimal places.