MATH 449, HOMEWORK 7

DUE NOVEMBER 20, 2017

Part I. Theory

Problem 1. Prove that Simpson's rule is exact for degree ≤ 3 polynomials.

Problem 2.

a. Find a function f on the interval $[-1, 1]$ where

$$\left|E_1(f)\right| = \frac{(b-a)^3}{12} M_2 > 0,$$

i.e., where the trapezoid rule attains the maximum error allowed by Theorem 7.1, but does not integrate f exactly.

b. Find a function f on the interval $[-1, 1]$ where

$$\left|E_2(f)\right| = \frac{(b-a)^5}{2880} M_4 > 0,$$

i.e., where Simpson's rule attains the maximum error allowed by Theorem 7.2, but does not integrate f exactly.

Problem 3 (Süli–Mayers, Exercise 7.3)**.** A quadrature formula on the interval $[-1, 1]$ uses the quadrature points $x_0 = -\alpha$ and $x_1 = \alpha$, where $0 < \alpha \leq 1$:

$$\int_{-1}^{1} f(x)\,\mathrm{d}x \approx w_0 f(-\alpha) + w_1 f(\alpha).$$

The formula is required to be exact whenever f is a polynomial of degree 1.

a. Show that $w_0 = w_1 = 1$, independent of the value of α.

b. Show also that there is one particular value of α for which the formula is exact for all polynomials of degree 2. Find this α, and

c. show that, for this value, the formula is also exact for all polynomials of degree 3.

Part II. Programming

Like last week, there is no sample code for this assignment, so you should create a new file `hw7.py` beginning with the usual lines:

```
from __future__ import division
from pylab import *
```

Hand in a printed copy of your code, as well as a printout of the IPython terminal session(s) containing the commands and output you used to get your answers.

Problem 4. Write functions `mid(f,a,b)`, `trap(f,a,b)`, and `simp(f,a,b)` implementing the midpoint rule, trapezoid rule, and Simpson's rule for a function f on the interval $[a, b]$. Use each of these quadrature rules to approximate $\frac{\pi}{4} = \int_0^1 \sqrt{1 - x^2}\,\mathrm{d}x$.

Problem 5. Next, write functions `midc(f,a,b,m)`, `trapc(f,a,b,m)`, and `simpc(f,a,b,m)` implementing the *composite* midpoint rule, trapezoid rule, and Simpson's rule for a function f on the interval $[a, b]$, where this interval is subdivided into m equally-sized subintervals. As in Problem 4, use each of these to approximate $\frac{\pi}{4} = \int_0^1 \sqrt{1 - x^2}\,\mathrm{d}x$ for the following parameters m:

a. $m = 16$,
b. $m = 256$,
c. $m = 4096$.