MATH 450, HOMEWORK 4

DUE **WEDNESDAY**, MARCH 18, 2015

Part I. Theory

Problem 1. Given $z \in \mathbb{R}^d$, we say that $z^T y$ is a *linear invariant* of the ODE $y' = f(t, y)$ if $z^T f(t, y) = 0$ for all t, y. A one-step method *preserves linear invariants* if $z^T y_{n+1} = z^T y_n$ whenever $z^T y$ is a linear invariant of $y' = f(t, y)$. Prove that every RK method preserves linear invariants.

Problem 2. Recall that the condition

$$b_i a_{ij} + b_j a_{ji} = b_i b_j, \quad i, j = 1, \ldots, \nu,$$

implies that an RK method preserves quadratic invariants and is symplectic. Prove that no ERK method can satisfy this condition, except for the order 0 method $y_{n+1} = y_n$.

Part II. Programming

Download `hw4.py`. The function `euler(f, g, q0, p0, h, N)` applies N steps of Euler's method with time step size h to the partitioned ODE

$$\begin{aligned} q' &= f(p), \\ p' &= g(q), \end{aligned}$$

with initial conditions $q(0) = q_0$ and $p(0) = p_0$.

Problem 3. Recall that the *simple harmonic oscillator* (SHO) has the Hamiltonian $H(q, p) = \frac{1}{2}p^2 + \frac{1}{2}q^2$, corresponding to the Hamiltonian system

$$\begin{aligned} q' &= p \\ p' &= -q. \end{aligned}$$

A "phase plot" visualizes solutions as parametric curves in the (q, p)-plane. The function `shoPhasePlot(method)` creates a phase plot for the numerical solutions using `method`, along with the phase plot for the exact solution. (Since the exact solutions satisfy $H(q, p) = \text{const}$, they correspond to level sets of H in the (q, p)-plane, i.e., the exact phase plot is just a contour plot of H.)

- **a.** Create a phase plot for the method `euler`.
- **b.** Modify `euler` to create two new functions, `symplecticEuler1` and `symplecticEuler2`, corresponding to the two symplectic Euler methods. (It doesn't matter which is which.) Create phase plots for each of these.
- **c.** Write a function `stoermerVerlet` which implements the Störmer/Verlet method, and create the corresponding phase plot.

Problem 4. Consider the nonlinear oscillator defined by the Hamiltonian $H(q,p) = \frac{1}{2}p^2 - \cos q$, whose corresponding Hamiltonian system is

$$q' = p,$$
$$p' = -\sin q.$$

This actually describes the motion of a simple pendulum, where q is angle in radians and p corresponds to angular momentum. For very small angles q, we have $\sin q \approx q$, so the motion of the pendulum resembles that of the (linear) simple harmonic oscillator. For large angles, though, the behavior is quite different.

Like in the previous problem, use `pendulumPhasePlot(method)` to create phase plots for the following methods:

- **a.** Euler's method,
- **b.** the two symplectic Euler methods, and
- **c.** the Störmer/Verlet method.