

## MATH 450, HOMEWORK 4

DUE FRIDAY, APRIL 1, 2016

### Part I. Theory

**Problem 1.** Given  $z \in \mathbb{R}^d$ , we say that  $z^T y$  is a *linear invariant* of the ODE  $y' = f(t, y)$  if  $z^T f(t, y) = 0$  for all  $t, y$ . A one-step method *preserves linear invariants* if  $z^T y_{n+1} = z^T y_n$  whenever  $z^T y$  is a linear invariant of  $y' = f(t, y)$ . Prove that every RK method preserves linear invariants.

**Problem 2.** Recall that the condition

$$b_i a_{ij} + b_j a_{ji} = b_i b_j, \quad i, j = 1, \dots, \nu,$$

implies that an RK method preserves quadratic invariants. Prove that no ERK method can satisfy this condition, except for the order 0 method  $y_{n+1} = y_n$ .

### Part II. Programming

Download `hw4.py`. The function `euler(f, g, q0, p0, h, N)` applies  $N$  steps of Euler's method with time step size  $h$  to the partitioned ODE

$$\begin{aligned} q' &= f(p), \\ p' &= g(q), \end{aligned}$$

with initial conditions  $q(0) = q_0$  and  $p(0) = p_0$ .

**Problem 3.** Recall that the *simple harmonic oscillator* (SHO) has the Hamiltonian  $H(q, p) = \frac{1}{2}p^2 + \frac{1}{2}q^2$ , corresponding to the Hamiltonian system

$$\begin{aligned} q' &= p \\ p' &= -q. \end{aligned}$$

A “phase plot” visualizes solutions as parametric curves in the  $(q, p)$ -plane. The function `shoPhasePlot(method)` creates a phase plot for the numerical solutions using `method`, along with the phase plot for the exact solution. (Since the exact solutions satisfy  $H(q, p) = \text{const}$ , they correspond to level sets of  $H$  in the  $(q, p)$ -plane, i.e., the exact phase plot is just a contour plot of  $H$ .)

- Create a phase plot for the method `euler`.
- Modify `euler` to create two new functions, `symplecticEuler1` and `symplecticEuler2`, corresponding to the two symplectic Euler methods. (It doesn't matter which is which.) Create phase plots for each of these.
- Write a function `stoermerVerlet` which implements the Störmer/Verlet method, and create the corresponding phase plot.

**Problem 4.** Consider the nonlinear oscillator defined by the Hamiltonian  $H(q, p) = \frac{1}{2}p^2 - \cos q$ , whose corresponding Hamiltonian system is

$$\begin{aligned}q' &= p, \\p' &= -\sin q.\end{aligned}$$

This actually describes the motion of a simple pendulum, where  $q$  is angle in radians and  $p$  corresponds to angular momentum. For very small angles  $q$ , we have  $\sin q \approx q$ , so the motion of the pendulum resembles that of the (linear) simple harmonic oscillator. For large angles, though, the behavior is quite different.

Like in the previous problem, use `pendulumPhasePlot(method)` to create phase plots for the following methods:

- a. Euler's method,
- b. the two symplectic Euler methods, and
- c. the Störmer/Verlet method.