MATH 450, HOMEWORK 1

DUE JANUARY 26, 2018

Part I. Theory

Problem 1. Given $a \in \mathbb{R}$, consider the scalar initial value problem

$$y' = ay, \qquad y(0) = 1. \tag{1}$$

a. Prove that $f(t, y) = ay$ is Lipschitz. What is the constant λ?

b. Recall that the Picard iteration for (1) is defined recursively by

$$y_0(t) = 1, \qquad y_{k+1}(t) = 1 + \int_0^t a y_k(s)\,\mathrm{d}s.$$

Find the first three iterates $y_1(t)$, $y_2(t)$, and $y_3(t)$.

c. Find a general expression for $y_k(t)$, and use this to prove directly that $\lim_{k\to\infty} y_k(t) = e^{at}$.

Problem 2. Suppose Euler's method is applied to (1) for $t \in [0, 1]$ by taking N time steps of size $h = 1/N$. Find a general expression for y_n, and use this to prove directly that y_N converges to $y(1)$ as $N \to \infty$.
Hint: Use the identity $e^a = \lim_{N\to\infty}(1 + a/N)^N$.

Part II. Programming

To get started, download and install the Anaconda software package from `https://www.anaconda.com/download/`. Next, start the Anaconda Launcher, and install and launch the Spyder application. (You can also launch *Spyder* by typing `spyder` in a command line terminal.) Finally, open the file `hw1.py` (available on the class web page) in Spyder, and click the green "play" button to run the code in the IPython console.

The file `hw1.py` contains a single function, `euler(f, t0, y0, h, N)`, which computes an approximate solution to the initial value problem

$$y' = f(t, y), \qquad y(t_0) = y_0,$$

by performing N steps of Euler's method with time step size h. The function returns the array $(y_0, y_1, \ldots, y_N)$.

Problem 3. Approximate $e = 2.71828\ldots$ by applying `euler` to (1) with $a = 1$ on the interval $t \in [0, 1]$. Use $h = 1, 0.1, 0.01, 0.001$, and 0.0001.

Problem 4. Reproduce the first plot at the top of p. 11 by (a) applying `euler` to the initial value problem

$$y' = -y + 2e^{-t}\cos 2t, \qquad y(0) = 0,$$

on the interval $t \in [0, 10]$ with $h = \frac{1}{2}, \frac{1}{10}$, and $\frac{1}{50}$; and (b) plotting the log absolute error, $\ln\lvert y_n - y(t_n)\rvert$, where $y(t) = e^{-t}\sin 2t$ is the exact solution.

(Don't worry about the formatting of the plot: axis limits, aspect ratio, line styles, etc. Just focus on getting the plot itself correct.)