MATH 450, HOMEWORK 2

DUE FEBRUARY 9, 2018

Part I. Theory

Problem 1. In the book, the proof that the trapezoid rule is convergent (Iserles, Theorem 1.2) begins by showing that the error satisfies the recursion

$$\|e_{n+1,h}\| \leq \left(\frac{1 + \frac{1}{2}h\lambda}{1 - \frac{1}{2}h\lambda} \right) \|e_{n,h}\| + \left(\frac{c}{1 - \frac{1}{2}h\lambda} \right) h^3, \tag{1.10}$$

for $n = 0, \ldots, N-1$. It is "left as an exercise for the reader" (that's you!) to show that

$$\|e_{n,h}\| \leq \frac{c}{\lambda} \left[\left(\frac{1 + \frac{1}{2}h\lambda}{1 - \frac{1}{2}h\lambda} \right)^n - 1 \right] h^2, \tag{1.11}$$

for $n = 0, \ldots, N$. Prove, using induction, that (1.10) implies (1.11).

Problem 2. Consider the theta method,

$$y_{n+1} = y_n + h\big[\theta f(t_n, y_n) + (1-\theta) f(t_{n+1}, y_{n+1})\big].$$

- **a.** Find the domain of linear stability $\mathcal{D}_\theta \subset \mathbb{C}$. Sketch this in the complex plane for $\theta = 1/3$ and $\theta = 2/3$.
- **b.** For which values $\theta \in [0,1]$ is the method A-stable? Give a proof.

Problem 3 (Iserles, Exercise 2.4)**.** Determine the order of the three-step method

$$y_{n+3} - y_n = h\big[\tfrac{3}{8} f(t_{n+3}, y_{n+3}) + \tfrac{9}{8} f(t_{n+2}, y_{n+2}) + \tfrac{9}{8} f(t_{n+1}, y_{n+1}) + \tfrac{3}{8} f(t_n, y_n)\big],$$

the *three-eighths* scheme. Is it convergent?

Part II. Programming

Download the sample code `hw2.py`, which contains implementations of the Euler and backward Euler methods for *systems* of ODEs. (The code from HW1 was only designed for scalar ODEs.) The backward Euler code uses the nonlinear root-finder `fsolve` from the `scipy.optimize` library to solve for y_{n+1} at each step. Specifically,

$$y_{n+1} = y_n + hf(t_{n+1}, y_{n+1}) \quad \Leftrightarrow \quad 0 = -y_{n+1} + y_n + hf(t_{n+1}, y_{n+1}),$$

so at each step, we can use `fsolve` to find a root of the function

$$F(y_{n+1}) = -y_{n+1} + y_n + hf(t_{n+1}, y_{n+1}),$$

with initial guess y_n.

Problem 4. Consider the simple harmonic oscillator

$$x'' = -x,$$

which can be written as the first-order linear system

$$\begin{pmatrix} x' \\ v' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix}.$$

- **a.** Apply the Euler method to this problem for $t \in [0, 25]$, with initial condition $\begin{pmatrix} x_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Plot $x(t)$ vs. t for $h = 0.25$ and $h = 0.01$.
- **b.** Repeat part (a) for the backward Euler method.

Problem 5. Create a function `trapezoid(f, t0, y0, h, N)` that implements the trapezoid method. (Hint: This function should be very similar to `backwardEuler`, but with a different choice of `F`.) Use this to repeat Problem 4(a) for the trapezoid method.

Problem 6. Create a function `ab2(f, t0, y0, y1, h, N)` that implements the two-step Adams–Bashforth (AB2) method. Use this to repeat Problem 4(a) for the AB2 method. To get the method started, take y_1 to be the value of the exact solution, $y_1 = y(t_1) = (\cos h, -\sin h)$.