MATH 450, HOMEWORK 3

DUE MARCH 2, 2018

Part I. Theory

Problem 1 (Iserles, Exercise 3.4)**.** Restricting your attention to scalar autonomous equations $y' = f(y)$, prove that the ERK method with the tableau

$$\begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ 1 & 0 & 0 & 1 & \\ \hline & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array}$$

is of order four. (Note: This is a long calculation, so start early!)

Problem 2 (Iserles, Exercise 3.7)**.** Write the theta method, (1.13), as a Runge–Kutta method.

Problem 3 (Iserles, Exercise 4.6)**.** Evaluate explicitly the function r for the following Runge–Kutta methods:

$$\textbf{a}.\ \begin{array}{c|cc} 0 & 0 & 0 \\ \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \hline & \frac{1}{4} & \frac{3}{4} \end{array}, \qquad \textbf{b}.\ \begin{array}{c|cc} \frac{1}{6} & \frac{1}{6} & 0 \\ \frac{5}{6} & \frac{2}{3} & \frac{1}{6} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}, \qquad \textbf{c}.\ \begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 \\ 1 & 0 & 1 & 0 \\ \hline & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{array}.$$

Part II. Programming

Download the sample code, `hw3.py`, which contains implementations of Euler's method (`euler`) and the explicit trapezoid rule (`etrap`). For `etrap`, observe that after computing `xi1` $= \xi_1$ and `xi2` $= \xi_2$, we immediately evaluate and store the function values `f1` $= f(t_n + c_1 h, \xi_1)$ and `f2` $= f(t_n + c_2 h, \xi_2)$. This lets us reuse these function values at later stages without having to evaluate f again. (Remember: function evaluation is expensive!)

Create functions implementing the following ERK methods:

- explicit midpoint: `emid(f,t0,y0,h,N)`
- classical 3-stage Runge–Kutta: `rk3(f,t0,y0,h,N)`
- RK4 (tableau given in Problem 1): `rk4(f,t0,y0,h,N)`

Problem 4. For this problem, you will be solving the scalar IVP

$$y' = y, \qquad y(0) = 1,$$

numerically on the interval $[0, 1]$. The exact solution $y(t) = e^t$ has $y(1) = e = 2.7182818284590\ldots$. Approximate e by solving this IVP with $h = 0.01$ for each of the following explicit Runge–Kutta methods:

a. `euler`
b. `emid`
c. `rk3`
d. `rk4`

Problem 5. The function `errorPlot(method)` applies `method` (which can be any function for solving ODEs) to solve the IVP from Problem 4 for various choices of h, then creates a log-log plot of the absolute error vs. h.

Create error plots for the methods `euler`, `emid`, `rk3` and `rk4`. What is the relationship between the plot and the order of each method?

Problem 6. The *Lorenz system* is a famous system of nonlinear ODEs, whose study (numerically, at first) helped launch Chaos Theory. Consider the system of ODEs

$$\begin{aligned} x' &= 10(y - x), \\ y' &= x(28 - z) - y, \\ z' &= xy - \tfrac{8}{3}z, \end{aligned}$$

which is a special case of the Lorenz system. If $\boldsymbol{y} = (x, y, z)$, then this is in the usual form $\boldsymbol{y}' = \boldsymbol{f}(t, \boldsymbol{y})$. Create a function `fLorenz(t,y)` corresponding to this $\boldsymbol{f}$.

The function `lorenzPlot` uses your `rk4` and `fLorenz` to solve the Lorenz system for $t \in [0, 100]$, with $\boldsymbol{y}_0 = (0, 2, 20)$, and creates a 3D plot of the numerical solution. Run `lorenzPlot()`, and print out your plot.