HOMEWORK 1

MATH 5051, FALL 2012
DUE WEDNESDAY, SEPTEMBER 5

Exercise 1. Let X be an infinite set, and define

$$\mathcal{A} = \{E \subset X \mid E \text{ is finite or } E^c \text{ is finite}\}.$$

Then $\mathcal{A}$ is an algebra (called the *finite-cofinite algebra*).

Exercise 2. Let $\{\mathcal{M}_\alpha\}_{\alpha \in A}$ be a nonempty family of σ-algebras on a set X. Then $\mathcal{M} = \bigcap_{\alpha \in A} \mathcal{M}_\alpha$ is also a σ-algebra. (Do not assume that the index set A is countable.)

Exercise 3 (Folland, Exercise 1.3)**.** Let $\mathcal{M}$ be an infinite σ-algebra.

a. $\mathcal{M}$ contains an infinite sequence of nonempty, disjoint sets. (*Hint*: If $\mathcal{M}$ contains an infinite sequence of strictly nested sets, then we're done, so assume that no such sequence exists. Next, use this assumption to find a nonempty set in $\mathcal{M}$ with no nonempty proper subsets in $\mathcal{M}$. Finally, show that this can be done infinitely many times.)

b. $\operatorname{card}(\mathcal{M}) \geq \mathfrak{c}$. (*Note:* In Folland, $\mathfrak{c}$ denotes $\operatorname{card}(\mathbb{R})$.)

Exercise 4 (Folland, Exercise 1.4)**.** An algebra $\mathcal{A}$ is a σ-algebra if and only if $\mathcal{A}$ is closed under countable increasing unions (i.e., if $\{E_j\}_{j=1}^{\infty} \subset \mathcal{A}$ and $E_1 \subset E_2 \subset \cdots$, then $\bigcup_{j=1}^{\infty} E_j \in \mathcal{A}$).

Exercise 5 (Folland, Exercise 1.5)**.** If $\mathcal{M}$ is the σ-algebra generated by $\mathcal{E}$, then $\mathcal{M}$ is the union of the σ-algebras generated by $\mathcal{F}$ as $\mathcal{F}$ ranges over all countable subsets of $\mathcal{E}$. (*Hint*: Show that the latter object is a σ-algebra.)