

6. $w = \ln \sqrt{x^2 + y^2 + z^2} = \frac{1}{2} \ln(x^2 + y^2 + z^2)$, $x = \sin t$, $y = \cos t$, $z = \tan t \Rightarrow$

$$\begin{aligned} \frac{dw}{dt} &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt} = \frac{1}{2} \cdot \frac{2x}{x^2 + y^2 + z^2} \cdot \cos t + \frac{1}{2} \cdot \frac{2y}{x^2 + y^2 + z^2} \cdot (-\sin t) + \frac{1}{2} \cdot \frac{2z}{x^2 + y^2 + z^2} \cdot \sec^2 t \\ &= \frac{x \cos t - y \sin t + z \sec^2 t}{x^2 + y^2 + z^2} \end{aligned}$$

9. $z = \ln(3x + 2y)$, $x = s \sin t$, $y = t \cos s \Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} = \frac{3}{3x + 2y} (\sin t) + \frac{2}{3x + 2y} (-t \sin s) = \frac{3 \sin t - 2t \sin s}{3x + 2y}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} = \frac{3}{3x + 2y} (s \cos t) + \frac{2}{3x + 2y} (\cos s) = \frac{3s \cos t + 2 \cos s}{3x + 2y}$$

10. $z = \sqrt{x} e^{xy}$, $x = 1 + st$, $y = s^2 - t^2 \Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} = \left(\sqrt{x} \cdot e^{xy}(y) + e^{xy} \cdot \frac{1}{2} x^{-1/2} \right) (t) + \sqrt{x} e^{xy}(x) (2s) = \left(yt\sqrt{x} + \frac{t}{2\sqrt{x}} + 2x^{3/2}s \right) e^{xy}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} = \left(\sqrt{x} \cdot e^{xy}(y) + e^{xy} \cdot \frac{1}{2} x^{-1/2} \right) (s) + \sqrt{x} e^{xy}(x) (-2t) = \left(ys\sqrt{x} + \frac{s}{2\sqrt{x}} - 2x^{3/2}t \right) e^{xy}$$

14. $R(s, t) = G(u(s, t), v(s, t)) \Rightarrow \frac{\partial R}{\partial s} = \frac{\partial R}{\partial u} \frac{\partial u}{\partial s} + \frac{\partial R}{\partial v} \frac{\partial v}{\partial s}$ and $\frac{\partial R}{\partial t} = \frac{\partial R}{\partial u} \frac{\partial u}{\partial t} + \frac{\partial R}{\partial v} \frac{\partial v}{\partial t}$ by the

Chain Rule (3). When $s = 1$ and $t = 2$, $u(1, 2) = 5$ and $v(1, 2) = 7$.

Thus $R_s(1, 2) = G_u(5, 7) u_s(1, 2) + G_v(5, 7) v_s(1, 2) = (9)(4) + (-2)(2) = 32$ and

$R_t(1, 2) = G_u(5, 7) u_t(1, 2) + G_v(5, 7) v_t(1, 2) = (9)(-3) + (-2)(6) = -39$.

15. $g(u, v) = f(x(u, v), y(u, v))$ where $x = e^u + \sin v$, $y = e^u + \cos v \Rightarrow$

$$\frac{\partial x}{\partial u} = e^u, \quad \frac{\partial x}{\partial v} = \cos v, \quad \frac{\partial y}{\partial u} = e^u, \quad \frac{\partial y}{\partial v} = -\sin v. \text{ By the Chain Rule (3), } \frac{\partial g}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u}. \text{ Then}$$

$$g_u(0, 0) = f_x(x(0, 0), y(0, 0)) x_u(0, 0) + f_y(x(0, 0), y(0, 0)) y_u(0, 0) = f_x(1, 2)(e^0) + f_y(1, 2)(e^0) = 2(1) + 5(1) = 7.$$

Similarly, $\frac{\partial g}{\partial v} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v}$. Then

$$\begin{aligned} g_v(0, 0) &= f_x(x(0, 0), y(0, 0)) x_v(0, 0) + f_y(x(0, 0), y(0, 0)) y_v(0, 0) = f_x(1, 2)(\cos 0) + f_y(1, 2)(-\sin 0) \\ &= 2(1) + 5(0) = 2 \end{aligned}$$

39. (a) $V = \ell wh$, so by the Chain Rule,

$$\frac{dV}{dt} = \frac{\partial V}{\partial \ell} \frac{d\ell}{dt} + \frac{\partial V}{\partial w} \frac{dw}{dt} + \frac{\partial V}{\partial h} \frac{dh}{dt} = wh \frac{d\ell}{dt} + \ell h \frac{dw}{dt} + \ell w \frac{dh}{dt} = 2 \cdot 2 \cdot 2 + 1 \cdot 2 \cdot 2 + 1 \cdot 2 \cdot (-3) = 6 \text{ m}^3/\text{s}.$$

(b) $S = 2(\ell w + \ell h + wh)$, so by the Chain Rule,

$$\begin{aligned} \frac{dS}{dt} &= \frac{\partial S}{\partial \ell} \frac{d\ell}{dt} + \frac{\partial S}{\partial w} \frac{dw}{dt} + \frac{\partial S}{\partial h} \frac{dh}{dt} = 2(w + h) \frac{d\ell}{dt} + 2(\ell + h) \frac{dw}{dt} + 2(\ell + w) \frac{dh}{dt} \\ &= 2(2 + 2)2 + 2(1 + 2)2 + 2(1 + 2)(-3) = 10 \text{ m}^2/\text{s} \end{aligned}$$

(c) $L^2 = \ell^2 + w^2 + h^2 \Rightarrow 2L \frac{dL}{dt} = 2\ell \frac{d\ell}{dt} + 2w \frac{dw}{dt} + 2h \frac{dh}{dt} = 2(1)(2) + 2(2)(2) + 2(2)(-3) = 0 \Rightarrow$

$$dL/dt = 0 \text{ m/s}.$$

43. Let x be the length of the first side of the triangle and y the length of the second side. The area A of the triangle is given by

$A = \frac{1}{2}xy \sin \theta$ where θ is the angle between the two sides. Thus A is a function of x , y , and θ , and x , y , and θ are each in turn

functions of time t . We are given that $\frac{dx}{dt} = 3$, $\frac{dy}{dt} = -2$, and because A is constant, $\frac{dA}{dt} = 0$. By the Chain Rule,

$$\frac{dA}{dt} = \frac{\partial A}{\partial x} \frac{dx}{dt} + \frac{\partial A}{\partial y} \frac{dy}{dt} + \frac{\partial A}{\partial \theta} \frac{d\theta}{dt} \Rightarrow \frac{dA}{dt} = \frac{1}{2}y \sin \theta \cdot \frac{dx}{dt} + \frac{1}{2}x \sin \theta \cdot \frac{dy}{dt} + \frac{1}{2}xy \cos \theta \cdot \frac{d\theta}{dt}.$$

When $x = 20$, $y = 30$, and $\theta = \pi/6$ we have

$$\begin{aligned} 0 &= \frac{1}{2}(30)(\sin \frac{\pi}{6})(3) + \frac{1}{2}(20)(\sin \frac{\pi}{6})(-2) + \frac{1}{2}(20)(30)(\cos \frac{\pi}{6}) \frac{d\theta}{dt} \\ &= 45 \cdot \frac{1}{2} - 20 \cdot \frac{1}{2} + 300 \cdot \frac{\sqrt{3}}{2} \cdot \frac{d\theta}{dt} = \frac{25}{2} + 150\sqrt{3} \frac{d\theta}{dt} \end{aligned}$$

Solving for $\frac{d\theta}{dt}$ gives $\frac{d\theta}{dt} = \frac{-25/2}{150\sqrt{3}} = -\frac{1}{12\sqrt{3}}$, so the angle between the sides is decreasing at a rate of

$$1/(12\sqrt{3}) \approx 0.048 \text{ rad/s.}$$

45. (a) By the Chain Rule, $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cos \theta + \frac{\partial z}{\partial y} \sin \theta$, $\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x} (-r \sin \theta) + \frac{\partial z}{\partial y} r \cos \theta$.

$$(b) \left(\frac{\partial z}{\partial r}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 \cos^2 \theta + 2 \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} \cos \theta \sin \theta + \left(\frac{\partial z}{\partial y}\right)^2 \sin^2 \theta,$$

$$\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 r^2 \sin^2 \theta - 2 \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} r^2 \cos \theta \sin \theta + \left(\frac{\partial z}{\partial y}\right)^2 r^2 \cos^2 \theta. \text{ Thus}$$

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial z}{\partial \theta}\right)^2 = \left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 \right] (\cos^2 \theta + \sin^2 \theta) = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2.$$