10. $f(x,y,z) = y^2 e^{xyz}$

(a) $\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle = \left\langle y^2 e^{xyz}(yz), y^2 \cdot e^{xyz}(xz) + e^{xyz} \cdot 2y, y^2 e^{xyz}(xy)\right\rangle$

$= \left\langle y^3 z e^{xyz}, (xy^2 z + 2y)e^{xyz}, xy^3 e^{xyz}\right\rangle$

(b) $\nabla f(0,1,-1) = \langle -1, 2, 0\rangle$

(c) $D_{\mathbf{u}} f(0,1,-1) = \nabla f(0,1,-1) \cdot \mathbf{u} = \langle -1,2,0\rangle \cdot \left\langle \frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right\rangle = -\frac{3}{13} + \frac{8}{13} + 0 = \frac{5}{13}$

15. $f(x,y,z) = x^2 y + y^2 z \quad \Rightarrow \quad \nabla f(x,y,z) = \left\langle 2xy, x^2 + 2yz, y^2\right\rangle, \quad \nabla f(1,2,3) = \langle 4, 13, 4\rangle$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{4+1+4}}\langle 2,-1,2\rangle = \frac{1}{3}\langle 2,-1,2\rangle$, so

$D_{\mathbf{u}}\, f(1,2,3) = \nabla f(1,2,3) \cdot \mathbf{u} = \langle 4,13,4\rangle \cdot \frac{1}{3}\langle 2,-1,2\rangle = \frac{1}{3}(8 - 13 + 8) = \frac{3}{3} = 1.$

21. $f(x,y) = 4y\sqrt{x} \quad \Rightarrow \quad \nabla f(x,y) = \left\langle 4y \cdot \frac{1}{2}x^{-1/2}, 4\sqrt{x}\right\rangle = \langle 2y/\sqrt{x}, 4\sqrt{x}\,\rangle.$

$\nabla f(4,1) = \langle 1, 8\rangle$ is the direction of maximum rate of change, and the maximum rate is $|\nabla f(4,1)| = \sqrt{1+64} = \sqrt{65}.$

22. $f(s,t) = te^{st} \quad \Rightarrow \quad \nabla f(s,t) = \left\langle te^{st}(t), te^{st}(s) + e^{st}(1)\right\rangle = \left\langle t^2 e^{st}, (st+1)e^{st}\right\rangle.$

$\nabla f(0,2) = \langle 4, 1\rangle$ is the direction of maximum rate of change, and the maximum rate is $|\nabla f(0,2)| = \sqrt{16+1} = \sqrt{17}.$

29. The direction of fastest change is $\nabla f(x,y) = (2x-2)\,\mathbf{i} + (2y-4)\,\mathbf{j}$, so we need to find all points (x,y) where $\nabla f(x,y)$ is parallel to $\mathbf{i}+\mathbf{j} \quad \Leftrightarrow \quad (2x-2)\,\mathbf{i} + (2y-4)\,\mathbf{j} = k\,(\mathbf{i}+\mathbf{j}) \quad \Leftrightarrow \quad k = 2x-2$ and $k = 2y-4$. Then $2x - 2 = 2y - 4 \quad \Rightarrow$ $y = x+1$, so the direction of fastest change is $\mathbf{i}+\mathbf{j}$ at all points on the line $y = x+1$.

42. Let $F(x,y,z) = y^2 + z^2 - x$. Then $x = y^2 + z^2 + 1 \quad \Leftrightarrow \quad y^2 + z^2 - x = -1$ is a level surface of F.

$F_x(x,y,z) = -1 \quad \Rightarrow \quad F_x(3,1,-1) = -1, \quad F_y(x,y,z) = 2y \quad \Rightarrow \quad F_y(3,1,-1) = 2,$ and $F_z(x,y,z) = 2z \quad \Rightarrow$ $F_z(3,1,-1) = -2.$

(a) By Equation 19, an equation of the tangent plane at $(3,1,-1)$ is $(-1)(x-3) + 2(y-1) + (-2)[z-(-1)] = 0$ or $-x + 2y - 2z = 1$ or $x - 2y + 2z = -1$.

(b) By Equation 20, the normal line has symmetric equations $\dfrac{x-3}{-1} = \dfrac{y-1}{2} = \dfrac{z-(-1)}{-2}$ or equivalently $x - 3 = \dfrac{y-1}{-2} = \dfrac{z+1}{2}$ and parametric equations $x = 3 - t,\ y = 1 + 2t,\ z = -1 - 2t.$

43. Let $F(x,y,z) = xy^2z^3$. Then $xy^2z^3 = 8$ is a level surface of F and $\nabla F(x,y,z) = \left\langle y^2z^3, 2xyz^3, 3xy^2z^2\right\rangle.$

(a) $\nabla F(2,2,1) = \langle 4, 8, 24\rangle$ is a normal vector for the tangent plane at $(2,2,1)$, so an equation of the tangent plane is $4(x-2) + 8(y-2) + 24(z-1) = 0$ or $4x + 8y + 24z = 48$ or equivalently $x + 2y + 6z = 12$.

(b) The normal line has direction $\nabla F(2,2,1) = \langle 4,8,24\rangle$ or equivalently $\langle 1,2,6\rangle$, so parametric equations are $x = 2+t,$ $y = 2+2t,\ z = 1+6t$, and symmetric equations are $x - 2 = \dfrac{y-2}{2} = \dfrac{z-1}{6}$.

44. Let $F(x,y,z) = xy + yz + zx$. Then $xy + yz + zx = 5$ is a level surface of F and $\nabla F(x,y,z) = \langle y+z, x+z, x+y\rangle.$

(a) $\nabla F(1,2,1) = \langle 3,2,3\rangle$ is a normal vector for the tangent plane at $(1,2,1)$, so an equation of the tangent plane is $3(x-1) + 2(y-2) + 3(z-1) = 0$ or $3x + 2y + 3z = 10$.

(b) The normal line has direction $\langle 3,2,3\rangle$, so parametric equations are $x = 1+3t,\ y = 2+2t,\ z = 1+3t$, and symmetric equations are $\dfrac{x-1}{2} = \dfrac{y-2}{1} = \dfrac{z-1}{3}$.

54. Let $F(x, y, z) = x^2 + y^2 + 2z^2$; then the ellipsoid $x^2 + y^2 + 2z^2 = 1$ is a level surface of F. $\nabla F(x, y, z) = \langle 2x, 2y, 4z \rangle$ is a normal vector to the surface at (x, y, z) and so it is a normal vector for the tangent plane there. The tangent plane is parallel to the plane $x + 2y + z = 1$ when the normal vectors of the planes are parallel, so we need a point (x_0, y_0, z_0) on the ellipsoid where $\langle 2x_0, 2y_0, 4z_0 \rangle = k \langle 1, 2, 1 \rangle$ for some $k \neq 0$. Comparing components we have $2x_0 = k \quad \Rightarrow \quad x_0 = k/2$,

$2y_0 = 2k \quad \Rightarrow \quad y_0 = k, \quad 4z_0 = k \quad \Rightarrow \quad z_0 = k/4$. $(x_0, y_0, z_0) = (k/2, k, k/4)$ lies on the ellipsoid, so

$(k/2)^2 + k^2 + 2(k/4)^2 = 1 \quad \Rightarrow \quad \frac{11}{8}k^2 = 1 \quad \Rightarrow \quad k^2 = \frac{8}{11} \quad \Rightarrow \quad k = \pm 2\sqrt{\frac{2}{11}}$. Thus the tangent planes at the points

$\left(\sqrt{\frac{2}{11}}, 2\sqrt{\frac{2}{11}}, \frac{1}{2}\sqrt{\frac{2}{11}}\right)$ and $\left(-\sqrt{\frac{2}{11}}, -2\sqrt{\frac{2}{11}}, -\frac{1}{2}\sqrt{\frac{2}{11}}\right)$ are parallel to the given plane.

57. Let (x_0, y_0, z_0) be a point on the cone [other than $(0, 0, 0)$]. The cone is a level surface of $F(x, y, z) = x^2 + y^2 - z^2$ and $\nabla F(x, y, z) = \langle 2x, 2y, -2z \rangle$, so $\nabla F(x_0, y_0, z_0) = \langle 2x_0, 2y_0, -2z_0 \rangle$ is a normal vector to the cone at this point and an equation of the tangent plane there is $2x_0 (x - x_0) + 2y_0 (y - y_0) - 2z_0 (z - z_0) = 0$ or

$x_0 x + y_0 y - z_0 z = x_0^2 + y_0^2 - z_0^2$. But $x_0^2 + y_0^2 = z_0^2$ so the tangent plane is given by $x_0 x + y_0 y - z_0 z = 0$, a plane which always contains the origin.

58. Let (x_0, y_0, z_0) be a point on the sphere. Then the normal line is given by $\dfrac{x - x_0}{2x_0} = \dfrac{y - y_0}{2y_0} = \dfrac{z - z_0}{2z_0}$. For the center $(0, 0, 0)$ to be on the line, we need $-\dfrac{x_0}{2x_0} = -\dfrac{y_0}{2y_0} = -\dfrac{z_0}{2z_0}$ or equivalently $1 = 1 = 1$, which is true.

60. The ellipsoid is a level surface of $F(x, y, z) = 4x^2 + y^2 + 4z^2$ and $\nabla F(x, y, z) = \langle 8x, 2y, 8z \rangle$, so $\nabla F(1, 2, 1) = \langle 8, 4, 8 \rangle$ or equivalently $\langle 2, 1, 2 \rangle$ is a normal vector to the surface. Thus the normal line to the ellipsoid at $(1, 2, 1)$ is given by $x = 1 + 2t$, $y = 2 + t$, $z = 1 + 2t$. Substitution into the equation of the sphere gives

$(1+2t)^2 + (2+t)^2 + (1+2t)^2 = 102 \quad \Leftrightarrow \quad 6 + 12t + 9t^2 = 102 \quad \Leftrightarrow \quad 9t^2 + 12t - 96 = 0 \quad \Leftrightarrow \quad 3(t+4)(3t-8) = 0.$

Thus the line intersects the sphere when $t = -4$, corresponding to the point $(-7, -2, -7)$, and when $t = \frac{8}{3}$, corresponding to the point $\left(\frac{19}{3}, \frac{14}{3}, \frac{19}{3}\right)$.