SPACES OF RATIONAL CURVES ON COMPLETE INTERSECTIONS

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Abstract. We prove that the space of smooth rational curves of degree e on a general complete intersection of multidegree $(d_1, \ldots, d_m)$ in $\mathbb{P}^n$ is irreducible of the expected dimension if $\sum_{i=1}^m d_i < \frac{2n+m+1}{3}$ and n is sufficiently large. This generalizes a result of Harris, Roth and Starr [9], and is achieved by proving that the space of conics passing through any point of a general complete intersection has constant dimension if $\sum_{i=1}^m d_i$ is small compared to n.

1. Introduction

Throughout this paper, we work over the field of complex numbers. For a smooth projective variety $X \subset \mathbb{P}^n$ and an integer $e \geq 1$, we denote by $\text{Hilb}_{et+1}(X)$ the Hilbert scheme parametrizing subschemes of X with Hilbert polynomial $et + 1$, and we denote by $R_e(X) \subset \text{Hilb}_{et+1}(X)$ the open subscheme parametrizing smooth rational curves of degree e on X. If $X = \mathbb{P}^n$, then $R_e(X)$ is a smooth irreducible rational variety of dimension $(e + 1)(n + 1) - 4$. But already in the case of hypersurfaces in $\mathbb{P}^n$, there are many basic questions concerning the geometry of $R_e(X)$ which are still open. In this paper we address and discuss some of these questions, focusing in particular on the dimension and irreducibility of $R_e(X)$, when X is a general complete intersection in $\mathbb{P}^n$.

To study the space of smooth rational curves on X, we consider the Kontsevich moduli space of stable maps $\overline{\mathcal{M}}_{0,0}(X, e)$ which compactifies $R_e(X)$ by allowing smooth rational curves to degenerate to morphisms from nodal curves. These have certain advantages over the Hilbert schemes for the problems studied here. We refer the reader to [1, 4, 8, 9] for detailed discussions of these moduli spaces and the comparison between them.

For every smooth hypersurface $X \subset \mathbb{P}^n$ of degree d, the dimension of every irreducible component of $\overline{\mathcal{M}}_{0,0}(X, e)$ is at least $e(n + 1 - d) + n - 4$, and if $d \leq n - 1$, then there is at least one irreducible component whose dimension is equal to $e(n+1-d)+n-4$ (see Section 2). The number $e(n+1-d)+n-4$ is referred to as *the expected dimension* of $\overline{\mathcal{M}}_{0,0}(X, e)$. If X is an arbitrary smooth hypersurface, $\overline{\mathcal{M}}_{0,0}(X, e)$ (or even $R_e(X)$) can be reducible and its dimension can be larger than expected (see [2, Section 1]). By a result of Harris, Roth, and Starr [9], if $d < \frac{n+1}{2}$ and X is a *general* hypersurface of degree d in $\mathbb{P}^n$, then for every $e \geq 1$, $\overline{\mathcal{M}}_{0,0}(X, e)$ is integral of the expected dimension and has only local complete intersection singularities. In this paper, we generalize this result to higher degree hypersurfaces.

Let $\overline{\mathcal{M}}_{0,1}(X,e)$ denote the moduli space of 1-pointed stable maps of degree e to X. In order to obtain the above mentioned result, Harris, Roth, and Starr show that if $d < \frac{n+1}{2}$ and X is a general hypersurface of degree d in $\mathbb{P}^n$, then the evaluation morphism

$$ev : \overline{\mathcal{M}}_{0,1}(X,e) \to X$$

is flat of relative dimension $e(n+1-d)-2$ for every $e \geq 1$ (see [9, Theorem 2.1 and Corollary 5.6]). It is conjectured that the same holds for any $d \leq n-1$:

Conjecture 1.1 (Coskun-Harris-Starr [2])**.** *If X is a general hypersurface of degree $d \leq n-1$ in $\mathbb{P}^n$, then the evaluation morphism*

$$ev : \overline{\mathcal{M}}_{0,1}(X,e) \to X$$

is flat of relative dimension $e(n+1-d)-2$ for every $e \geq 1$.

The above conjecture would imply the following:

Conjecture 1.2 (Coskun-Harris-Starr [2])**.** *If X is a general hypersurface of degree $d \leq n-1$ in $\mathbb{P}^n$, then for every $e \geq 1$, $\overline{\mathcal{M}}_{0,0}(X,e)$ has the expected dimension $e(n+1-d)+n-4$.*

Coskun and Starr [2] show that Conjecture 1.2 holds for $d \leq \frac{n+4}{2}$. When $d = n-1$ and $e \geq 2$, $\overline{\mathcal{M}}_{0,0}(X,e)$ is reducible for the following reason. By Lemma 2.1, $\overline{\mathcal{M}}_{0,0}(X,e)$ has at least one irreducible component of dimension $2e+n-4$ whose general point parametrizes an embedded smooth rational curve of degree e on X. On the other hand, the space of lines on X has dimension at least $n-2$, and therefore, the space of degree e covers of lines on X has dimension $\geq (n-2)+(2e-2)$, thus $\overline{\mathcal{M}}_{0,0}(X,e)$ has at least 2 irreducible components. It is expected that if X is general, then $\overline{\mathcal{M}}_{0,0}(X,e)$ is irreducible when $d \leq n-2$, and $R_e(X)$ is irreducible when $d \leq n-1$ (see [2, Conjecture 1.3]).

In this paper, we show:

Theorem 1.3. *Let X be a general hypersurface of degree $d \leq n-1$ in $\mathbb{P}^n$. If*

$$\binom{n-d}{2} > n-1,$$

then the evaluation morphism $ev : \overline{\mathcal{M}}_{0,1}(X,2) \to X$ is flat of relative dimension $2n-2d$.

A smooth rational curve on X is called *free* if its normal bundle in X is globally generated. The proof of the above theorem is based on an analysis of the space of non-free conics on X passing through an arbitrary point of X. It seems quite plausible that the same approach can be applied to the case of cubics or other higher degree rational curves to prove the flatness of ev when d and n satisfy the inequality of the theorem, but we have not carried out all the details, and we restrict the discussion here to the case of conics.

Theorem 1.3 along with the results of [9] gives the following:

Theorem 1.4. *If $X \subset \mathbb{P}^n$ is a general hypersurface of degree $d < \frac{2n+2}{3}$, $n \geq 23$, then for every $e \geq 1$, the following hold.*

(a) *The evaluation morphism $\overline{\mathcal{M}}_{0,1}(X,e) \to X$ is flat and of relative dimension $e(n+1-d)-2$.*

(b) *$\overline{\mathcal{M}}_{0,0}(X,e)$ is an integral local complete intersection stack of expected dimension $e(n+1-d)+(n-4)$.*

The results of Theorem 1.4 are generalized to the case of general complete intersections in Theorem 7.1. An important application of Theorem 1.4 is that it relates the genus zero Gromov-Witten invariants of a general hypersurface to its enumerative geometry when the degree of the hypersurface is in the range of the theorem. The Gromov-Witten invariants of any smooth hypersurface can be defined via integrals over moduli spaces of genus zero stable maps with marked points, but the enumerative significance of these invariants are, in general, not easily understood. Theorem 1.4 shows the following.

Corollary 1.5. *Let X be as in Theorem 1.4, and let $c_1, \ldots, c_k$ be a sequence of integers greater than 1 with $(c_1-1)+\cdots+(c_k-1) = e(n+1-d)+n-4$. Then for a general sequence $\Gamma_1, \ldots, \Gamma_k$ of linear subvarieties of $\mathbb{P}^n$ of codimensions $c_1, \ldots, c_k$, the subscheme of $R_e(X)$ parametrizing smooth rational curves of degree e on X which intersect each of $\Gamma_1, \ldots, \Gamma_k$ is a reduced 0-dimensional scheme whose total length is equal to the Gromov-Witten invariant $< H^{c_1}, \ldots, H^{c_k} >^X_{0,e[line]}$.*

Along the way to proving Theorem 1.3, we obtain the following result on the space of non-free lines on general complete intersections.

Theorem 1.6. *Let X be a general complete intersection of multidegree $(d_1, \ldots, d_m)$ in $\mathbb{P}^n$, and let p be an arbitrary point of X. If integers $k \geq 1$ and $a \geq 0$ are such that*

$$\binom{a+k+2}{k+1} > n-m,$$

then the family of lines L on X passing through p with $h^1(L, N_{L/X}(-1)) > m(k-1)$ has dimension $\leq a$.

In fact, we can modify the proof of the above theorem to say more in special cases. For example, it follows from the proof of the theorem that if $n \leq 5$ and X is a general hypersurface, then the space of non-free lines through any point of X is at most zero dimensional. The proof shows that if there is a 1-parameter family of non-free lines on X through p parametrized by $C \subset \mathbb{P}^{n-1}$, then $n \geq 2 + 2\dim \text{Linear Span}(C)$ (see Proposition 6.4). Of course, a general hypersurface of degree ≥ 2 in $\mathbb{P}^n$, $n \leq 5$, does not contain any 2-plane, so the dimension of the linear span of C is at least 2. Note that for a general hypersurface X of degree $3 \leq d \leq n-1$, the non-free lines on X sweep out a divisor in X (Proposition 6.2).

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2. Background and Summary

Fix positive integers $d_1 \leq \cdots \leq d_m$, and set $d := d_1 + \cdots + d_m$. Let X be a smooth complete intersection of multidegree $(d_1, \ldots, d_m)$ in $\mathbb{P}^n$. The Kontsevich moduli space $\overline{\mathcal{M}}_{0,r}(X, e)$ parametrizes isomorphism classes of tuples $(C, q_1, \ldots, q_r, f)$ where

(1) C is a proper, connected, at worst nodal curve of arithmetic genus 0.
(2) $q_1, \ldots, q_r$ are distinct smooth points of C.
(3) $f : C \to X$ is a morphism such that $f^*\mathcal{O}_X(1)$ has total degree e on C and f satisfies the following stability condition: any irreducible component of C which is mapped to a point by f has at least 3 points which are either marked or nodes.

The tuples $(C, q_1, \ldots, q_r, f)$ and $(C', q'_1, \ldots, q'_r, f')$ are isomorphic if there is an isomorphism $g : C \to C'$ taking q_i to q'_i, with $f' \circ g = f$. The moduli space $\overline{\mathcal{M}}_{0,r}(X, e)$ is a proper Deligne-Mumford stack, and the corresponding coarse moduli space $\overline{M}_{0,r}(X, e)$ is a projective scheme. There is an evaluation morphism

$$ev : \overline{\mathcal{M}}_{0,r}(X, e) \to X^r$$

sending a datum $(C, q_1, \ldots, q_r, f)$ to $(f(q_1), \ldots, f(q_r))$. We refer to [1] and [8] for constructions and basic properties of these moduli spaces.

The space of first order deformations of the map f with $(C, q_1, \ldots, q_r)$ fixed can be identified with $H^0(C, f^*T_X)$. If $H^1(C, f^*T_X) = 0$ at a point $(C, q_1, \ldots, q_r, f)$, then f is unobstructed and the moduli stack is smooth at that point. In particular, $\overline{\mathcal{M}}_{0,r}(\mathbb{P}^n, e)$ is smooth of dimension $(n+1)(e+1)+r-4$ (see for instance Appendix A of [16] for a brief discussion of the deformation theory of $\overline{\mathcal{M}}_{0,r}(X, e)$).

Denote by $\mathcal{C} \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, e)$ the universal curve and by $h : \mathcal{C} \to \mathbb{P}^n$ the universal map

$$\begin{tikzcd} \mathcal{C} \arrow[r, "h"] \arrow[d, "\pi"] & \mathbb{P}^n \\ \overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, e). & \end{tikzcd}$$

For any $d \geq 1$, the line bundle $h^*\mathcal{O}_{\mathbb{P}^n}(d)$ is the pullback of a globally generated line bundle and so it is globally generated. The first cohomology group of a globally generated line bundle over a nodal curve of genus zero vanishes, so by the theorem of cohomology and base change [10, Theorem 12.11], $E := \pi_* h^*(\oplus_{i=1}^m \mathcal{O}_{\mathbb{P}^n}(d_i))$ is a locally free sheaf of rank $de+m$. If $s \in \oplus_{i=1}^m H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d_i))$ is a section whose zero locus is X, then by [9, Lemma 4.5], $\pi_* h^* s$ is a section of E whose zero locus as a closed substack of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, e)$ is $\overline{\mathcal{M}}_{0,0}(X, e)$. The number

$$\dim \overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, e) - (de+m) = e(n+1-d) + n - m - 3$$

is called the *expected dimension* of $\overline{\mathcal{M}}_{0,0}(X, e)$.

It follows that the dimension of every component of $\overline{\mathcal{M}}_{0,0}(X, e)$ is at least the expected dimension, and if the equality holds, then $\overline{\mathcal{M}}_{0,0}(X, e)$ is a local complete intersection substack of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, e)$. Similarly, $\overline{\mathcal{M}}_{0,1}(\mathbb{P}^n, e)$ is a smooth stack of dimension $(e+1)(n+1)-3$, and $\overline{\mathcal{M}}_{0,1}(X, e)$ is the zero locus of a section of a locally free sheaf of rank $de+m$ on $\overline{\mathcal{M}}_{0,1}(\mathbb{P}^n, e)$. Therefore if $\dim \overline{\mathcal{M}}_{0,1}(X, e) = e(n+1-d)+n-m-2$, then it is a local complete intersection stack.

The number $e(n+1-d)+n-3-m$ can be also obtained as an Euler characteristic: if C is a smooth rational curve of degree e on X, and if $N_{C/X}$ denotes the normal bundle of C in X, then

$$\chi(N_{C/X}) = \chi(T_X|_C) - \chi(T_C) = e(n+1-d) + n - m - 3.$$

By [5, Lemma 4.4], when X is general and $d \leq n-1$, $\overline{\mathcal{M}}_{0,0}(X, e)$ has at least one irreducible component of the expected dimension. We prove this result here for the convenience of the reader. By a *non-free line* on a subvariety X of $\mathbb{P}^n$ we mean a line which is contained in the smooth locus of X and its normal bundle in X is not globally generated.

Lemma 2.1. *Let X be a complete intersection of multidegree $(d_1, \dots, d_m)$ in $\mathbb{P}^n$ such that $d_1 + \cdots + d_m \leq n-1$. Then the following hold.*

(a) *The non-free lines contained in the smooth locus of X do not cover a dense subset of X.*
(b) *If X is smooth, then $R_e(X)$ (and hence $\overline{\mathcal{M}}_{0,0,}(X, e)$) has at least one irreducible component of the expected dimension for every $e \geq 1$.*

Proof. (a) Let J be the locally closed subscheme (with the reduced induced structure) of $R_1(X)$ (the space of lines on X) parametrizing non-free lines contained in the smooth locus of X. Let $I \subset J \times X$ denote the incidence correspondence. We wish to show that the projection $I \to X$ is not dominant. Let $([L], p)$ be a general point of I. Then we have $T_{R_1(X),[L]} \cong H^0(L, N_{L/X})$, and since L is contained in the smooth locus of X, $T_{X,p} \to N_{L/X}|_p$ is surjective. We have a commutative diagram

$$\begin{tikzcd} T_{I,([L],p)} \arrow[rr] \arrow[d] & & T_{X,p} \arrow[d, two heads] \\ T_{J,[L]} \arrow[r] & T_{R_1(X),[L]} = H^0(L, N_{L/X}) \arrow[r] & N_{L/X}|_p. \end{tikzcd}$$

Since L is not free, the map $H^0(L, N_{L/X}) \to N_{L/X}|_p$ is not surjective, so $T_{I,([L],p)} \to T_{X,p}$ is not surjective and thus $I \to X$ is not dominant.

(b) We first show that smooth rational curves of degree e on X sweep out a dense subset of X. When $e = 1$, this is proved in [3, Proposition 2.13]. We just repeat the proof here. Let $p \in X$ and without loss of generality, assume that $p = (1 : 0 : \cdots : 0)$ and X is defined by $0 = f_i = a_{i1}x_0^{d_i-1} + \cdots + a_{id_i}$ for $1 \leq i \leq m$, a_{ij} homogeneous polynomials in $x_1, \ldots, x_n$. There are $\sum d_i$ of these a_{ij}s, and they have a common non-trivial zero in $\mathbb{P}^{n-1} = \{x_0 = 0\}$ since $n - 1 \geq \sum d_i$. The line joining p and this common zero in $\mathbb{P}^{n-1}$ is contained in X. Since p was arbitrary, we see that X is covered by lines.

By part (a), any line passing through a general point of X is free. Hence there is a chain of e free lines on X for every $e \geq 1$. By [12, Theorem 7.6], this chain of lines can be deformed to a smooth free rational curve C of degree e on X. Since C is free, its flat deformations in X sweep out X. If R is a component of $R_e(X)$ such that the curves parametrized by its points sweep out a dense subset of X, then the argument in part (a) shows that the normal bundle of a general curve C parametrized by R is globally generated. Since

$$\begin{aligned} \dim T_{R_e(X),[C]} &= h^0(C, N_{C/X}) \\ &= \chi(N_{C/X}) + h^1(C, N_{C/X}) \\ &= e(n+1-d) + (n-m-3), \end{aligned}$$

and since the dimension of R is at least the expected dimension $e(n+1-d) + n - m - 3$, R is of the expected dimension and smooth at $[C]$. $\square$

3. Deformations of rational curves

We fix a few notations for normal sheaves first. If Y is a closed subscheme of a smooth variety X, as usual we write $N_{Y/X}$ for the normal sheaf of Y in X. More generally, suppose that $f : Y \to X$ is a morphism between quasi-projective varieties and X is smooth. Denote by T_Y and T_X the tangent sheaves of Y and X, and denote by N_f the cokernel of the induced map $T_Y \to f^*T_X$. We refer to N_f as the *normal sheaf* of f. We may sometimes write $N_{f,X}$ instead to emphasize the range. If Y and X are both smooth and f is generically finite, then the exact sequence $T_Y \to f^*T_X \to N_f \to 0$ is exact on the left too. If X and Y are both smooth and g is a morphism from a quasi-projective variety Z to Y, then we get an exact sequence of normal sheaves on Z,

$$N_g \to N_{f\circ g} \to g^* N_f \to 0. \tag{1}$$

3.1. **Characteristic maps.** Let B and X be smooth quasi-projective varieties, and let $\pi : Y \to B$ be a smooth projective morphism. Denote by Y_b the fiber over $b \in B$. Let $F : Y \to B \times X$ be a morphism over B such that the restriction of F to every fiber of π is generically finite. Let $f = F|_{Y_b} : Y_b \to X$ and p_B (resp. p_X) be the projections from $B \times X$ to B (resp. X). Notice that $p_B \circ F = \pi$ and $T_{B\times X}$ is naturally isomorphic to $p_B^* T_B \oplus p_X^* T_X$. Thus, we have a natural map

$$\alpha : \pi^* T_B \to N_F. \tag{2}$$

Using the following diagram, we have $N_F|_{Y_b} = N_f$:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & T_{Y_b} \arrow[r] \arrow[d] & f^*T_X \arrow[r] \arrow[d] & N_f \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T_Y|_{Y_b} \arrow[r] \arrow[d] & F^*T_{B\times X}|_{Y_b} \arrow[r] \arrow[d] & N_F|_{Y_b} \arrow[r] & 0 \\
 & N_{Y_b/Y} \arrow[r, "="] \arrow[d] & \pi^*T_B|_{Y_b} \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Thus, restricting α, we get a map $\pi^* T_{B,b} \to N_f$, and since there is a natural map $T_{B,b} \to H^0(Y_b, \pi^* T_{B,b})$, we get a map

$$\alpha_b : T_{B,b} \to H^0(Y_b, N_f) \tag{3}$$

which we refer to as the *characteristic map* of F at b.

Let now $\pi' : Z \to B$ be another smooth projective morphism and suppose that there is a morphism $G : Z \to Y$ over B such that the restriction of $F \circ G$ to every fiber of π' is generically finite. The following lemma follows easily from the construction of the characteristic maps.

Lemma 3.1. *For $b \in B$, let α_b and β_b denote the characteristic maps corresponding to F and $F \circ G$ at b, respectively. If f and g denote the restriction of F and G to the fibers over b, then we have a commutative* *diagram*

$$\begin{tikzcd} T_{B,b} \arrow[r, "\alpha_b"] \arrow[d, "\beta_b"] & H^0(Y_b, N_f) \arrow[d] \\ H^0(Z_b, N_{f\circ g}) \arrow[r] & H^0(Z_b, g^*N_f). \end{tikzcd}$$

3.2. **Morphisms from $\mathbb{P}^1$ to general complete intersections.** Let $d_1 \leq \cdots \leq d_m$ be positive integers. For the rest of this section, we fix the following notation:

(1) $\mathcal{H}$ denotes the variety parametrizing smooth complete intersections in $\mathbb{P}^n$ which are of multidegree $(d_1, \ldots, d_m)$.
(2) $\mathcal{U} \subset \mathcal{H} \times \mathbb{P}^n$ denotes the universal family over $\mathcal{H}$.
(3) $\pi_1 : \mathcal{U} \to \mathcal{H}$ and $\pi_2 : \mathcal{U} \to \mathbb{P}^n$ denote the two projection maps.

Suppose that B is a smooth irreducible quasi-projective variety and $\psi : B \to \mathcal{H}$ is a dominant morphism. Let $\mathcal{U}_B \subset B \times \mathbb{P}^n$ be the fiber product. Let $\pi : Y \to B$ be a dominant smooth projective morphism whose fibers are connected curves, and let $F : Y \to \mathcal{U}_B$ be a morphism over B which is generically finite on each fiber of π. We have an exact sequence using the sequence (1),

$$0 \to N_{F,\mathcal{U}_B} \to N_{F,B\times\mathbb{P}^n} \to F^*N_{\mathcal{U}_B/B\times\mathbb{P}^n} \to 0. \tag{4}$$

Fix a point $b \in B$ and let $C \subset Y$ be the fiber of π over b. Let $f = F|_C : C \to \mathbb{P}^n$ and let X be the complete intersection parametrized by $\psi(b)$. Then the exact sequence (4) specializes to the following exact sequence

$$0 \to N_{f,X} \to N_{f,\mathbb{P}^n} \to f^*N_{X/\mathbb{P}^n} \to 0. \tag{5}$$

Proposition 3.2. *If ψ is smooth at b, then the image of the pullback map*
$$H^0(X, N_{X/\mathbb{P}^n}) \to H^0(C, f^*N_{X/\mathbb{P}^n})$$
is contained in the image of the map $H^0(C, N_{f,\mathbb{P}^n}) \to H^0(C, f^*N_{X/\mathbb{P}^n})$ obtained from the above short exact sequence.

Proof. Let $\alpha_b : T_{B,b} \to H^0(C, N_{f,\mathbb{P}^n})$ be the characteristic map of F at b. We have a diagram

$$\begin{tikzcd} T_{B,b} \arrow[r, "d\psi"] \arrow[d, "\alpha_b"] & T_{\mathcal{H},[X]} = H^0(X, N_{X/\mathbb{P}^n}) \arrow[d] \\ H^0(C, N_{f,\mathbb{P}^n}) \arrow[r] & H^0(C, f^*N_{X/\mathbb{P}^n}) \end{tikzcd}$$

and $d\psi$ is surjective since ψ is smooth at b by our assumption. Note that $d\psi$ is the characteristic map corresponding to the inclusion $\mathcal{U}_B \to B \times \mathbb{P}^n$ at b, so the diagram is commutative by Lemma 3.1, and the result follows. $\square$

Corollary 3.3. *Let $X \subset \mathbb{P}^n$ be a general complete intersection of multidegree $(d_1, \dots, d_m)$, $d_1 \le \cdots \le d_m$. If C is a smooth rational curve of degree e on X which is d_1-normal (that is, the restriction map $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \to H^0(C, \mathcal{O}_C(d))$ is surjective for $d \ge d_1$), then $H^1(C, N_{C/X}) = 0$. In particular, $R_e(X)$ is smooth at $[C]$.*

Proof. Consider the relative Hilbert scheme parametrizing pairs (X, C) such that X is a smooth complete intersection of multidegree $(d_1, \dots, d_m)$ in $\mathbb{P}^n$ and C is a rational curve of degree e on X. The locus B parametrizing pairs (X, C) such that C is a d_1-normal smooth rational curve and $H^1(C, N_{C/X}) \neq 0$ is a locally closed subscheme of this relative Hilbert scheme. If there is a dominant morphism $\psi : B \to \mathcal{H}$, then by generic smoothness, there is a smooth, irreducible subvariety B_0 of B such that the restriction of ψ to B_0 is dominant and smooth. Let Y denote the universal family of curves over B_0 and C the fiber of $Y \to B_0$ over a smooth point of ψ. Since $N_{X/\mathbb{P}^n} = \oplus \mathcal{O}_X(d_i)$, and since C is d_i-normal for every i, $1 \le i \le m$, by Proposition 3.2, the map $H^0(C, N_{C/\mathbb{P}^n}) \to H^0(C, N_{X/\mathbb{P}^n}|_C)$ is surjective. Applying the long exact sequence of cohomology to the sequence of normal sheaves

$$0 \to N_{C/X} \to N_{C/\mathbb{P}^n} \to N_{X/\mathbb{P}^n}|_C \to 0,$$

we get a contradiction since $H^1(C, N_{C/X}) \neq 0$ and $H^1(C, N_{C/\mathbb{P}^n}) = 0$. $\square$

Now assume that there exists a morphism $\phi : B \to \mathcal{U}$ so that the composition with the map $\pi_1 : \mathcal{U} \to \mathcal{H}$ is ψ. This is equivalent to saying that we are given a section σ for the map $\mathcal{U}_B \to B$. Assume further that $\sigma(B) \subset F(Y)$. Let b a point of B, and let C denote the fiber of π over b. Let $\phi(b) = ([X], p)$ and denote by $f : C \to X$ the restriction of F to C, so the image of f is a curve on X which passes through p. Let $D = (f^{-1}(p))_{\text{red}} \subset C$. If $\mathcal{I}_p$ denotes the ideal sheaf of p in $\mathbb{P}^n$, then the pullback of $H^0(X, N_{X/\mathbb{P}^n} \otimes \mathcal{I}_p)$ under f is contained in $H^0(C, f^* N_{X/\mathbb{P}^n}(-D))$. Consider the short exact sequence of normal sheaves (5) twisted with $\mathcal{O}_C(-D)$:

$$0 \to N_{f,X}(-D) \to N_{f,P^n}(-D) \to f^* N_{X/\mathbb{P}^n}(-D) \to 0.$$

Proposition 3.4. *For a general point $([X], p)$ in the image of ϕ, there is a subspace*

$$W_{X,p} \subset H^0(X, N_{X/\mathbb{P}^n}(-p))$$

of codimension at most $n - m$ and a non-empty open subset $U \subset \phi^{-1}([X], p)$ with the following property: for $b \in U$, if C, f, and D are as above, then for every $w \in W_{X,p}$, $f^* w$ can be lifted to a section of $N_{f,\mathbb{P}^n}(-D)$.

Proof. Let p be a general point in the image of $\pi_2 \circ \phi$. Set $B_p := (\pi_2 \circ \phi)^{-1}(p)$ with the reduced induced structure, and let $\mathcal{H}_p \subset \mathcal{H}$ be the closure of $\psi(B_p)$. Since p is general and ψ is dominant by our assumption, the codimension of $\mathcal{H}_p$ in $\mathcal{H}$ is at most n.

Let b be a point in B_p, and let $\phi(b) = ([X], p)$. Then $[X] \in \mathcal{H}_p$. We can identify $T_{\mathcal{H},[X]}$ with $H^0(X, N_{X/\mathbb{P}^n})$, and we set

$$W_{X,p} := T_{\mathcal{H}_p,[X]} \subset H^0(X, N_{X/\mathbb{P}^n})$$

under this identification. Since every complete intersection which is parametrized by $\mathcal{H}_p$ contains p, we have $W_{X,p} \subset H^0(X, N_{X/\mathbb{P}^n} \otimes \mathcal{I}_p)$. Since the codimension of $\mathcal{H}_p$ in $\mathcal{H}$ is at most n, $W_{X,p}$ is of codimension $\leq n$ in $H^0(X, N_{X/\mathbb{P}^n})$, and it is of codimension $\leq n - m$ in $H^0(X, N_{X/\mathbb{P}^n} \otimes \mathcal{I}_p)$.

Set

$$Y_p = \pi^{-1}(B_p),$$

and

$$Z_p = (F^{-1}(B_p \times \{p\}))_{\mathrm{red}} \subset Y_p.$$

By generic smoothness, there is a dense open subset $B_p^0 \subset B_p$ such that for every $b \in B_p^0$, the following hold.

(i) The induced map on Zariski tangent spaces $d\psi : T_{B_p,b} \to T_{\mathcal{H}_p,[X]} = W_{X,p}$ is surjective.

(ii) The map $Z_p \to B_p$ is étale over b, so $Z_p|_b = D$.

Let $b \in B_p^0$, and let $\alpha_b : T_{B_p,b} \to H^0(C, N_{f,\mathbb{P}^n})$ be the characteristic map of $Y_p \to B_p \times \mathbb{P}^n$ at b. Let $g : D \to X$ be the restriction of f to D, and let $\beta_b : T_{B_p,b} \to H^0(D, N_{g,\mathbb{P}^n}) = f^*T_{\mathbb{P}^n}|_D$ be the characteristic map of $Z_p \to B_p \times \mathbb{P}^n$ at b. Then β_b is clearly the zero map. The following diagram is commutative by Lemma 3.1.

$$\begin{tikzcd}
f^*T_{\mathbb{P}^n}|_D = H^0(D, N_{g,\mathbb{P}^n}) \arrow[d] & T_{B_p,b} \arrow[l, "\beta_b=0"'] \arrow[r, two heads, "d\psi"] \arrow[d, "\alpha_b"] & W_{X,p} \subset H^0(X, N_{X/\mathbb{P}^n}) \arrow[d] \\
N_{f,\mathbb{P}^n}|_D & H^0(C, N_{f,\mathbb{P}^n}) \arrow[l] \arrow[r] & H^0(C, f^*N_{X/\mathbb{P}^n}),
\end{tikzcd}$$

so for any $w \in W_{X,p}$, f^*w can be a lifted to a section of $N_{f,\mathbb{P}^n}$ which vanishes over D. Thus for any $([X], p)$ in $\phi(B_p^0)$, the open set $U := B_p^0 \cap \phi^{-1}([X], p)$ has the desired property. $\square$

4. Conics in projective space

Let $\mathrm{Hilb}_{2t+1}(\mathbb{P}^n)$ denote the Hilbert scheme of subschemes of $\mathbb{P}^n$ with Hilbert polynomial $2t + 1$. In this section we prove the following:

Proposition 4.1. *Let p be a point of $\mathbb{P}^n$, and let $R \subset \mathrm{Hilb}_{2t+1}(\mathbb{P}^n)$ be an irreducible projective subscheme of dimension r such that every curve parametrized by R is a smooth conic through p. If $W \subset H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d) \otimes \mathcal{I}_p)$ is a subspace of codimension c with*

$$c < \min\left(\binom{r+1}{2}, (d-1)\left\lfloor \frac{r+1}{2} \right\rfloor + 1\right),$$

then for a general $[C] \in R$ *and every* $2 \le k \le 2d$*, the image of the restriction map*

$$W \to H^0(C, \mathcal{O}_C(d))$$

contains a section of $\mathcal{O}_C(d)$ *which has a zero of order* k *at* p*.*

We start with a lemma.

Lemma 4.2. *If* L *is a line in* $\mathbb{P}^n$ *through* p*, then there is no complete one-dimensional family of smooth conics through* p *all tangent to* L*.*

Proof. Assume to the contrary that there is such a family B. By passing to a desingularization we can assume that B is smooth. Let $Y \subset B \times \mathbb{P}^n$ be the universal family over B and $g : Y \to \mathbb{P}^n$ the projection map.

The point p gives a section σ_p of the family $Y \to B$. Fix a point $q \neq p$ on L. Then $B \times \{q\} \cap Y = \emptyset$. Thus the projection from q defines a morphism $g : Y \to B \times \mathbb{P}^{n-1}$. For any point $b \in B$, this map is a map from a conic to a line and thus g is a two-to-one map to its image. Let $R \subset Y$ be the ramification locus. Then, the map $R \to B$ is a double cover. But, $B \times \{p\} \subset R$ and the residual part is a section of $Y \to B$ which we denote by σ_q. Since if q_1 and q_2 are two distinct points of L, σ_{q_1} and σ_{q_2} are disjoint, $Y \simeq B \times \mathbb{P}^1$. Since the section σ_p is contracted by g, by the rigidity lemma [14], g factors through the projection map $B \times \mathbb{P}^1 \to \mathbb{P}^1$. Thus the image of g should be one-dimensional which is a contradiction.

□

Corollary 4.3. *Suppose that* $R \subset \mathrm{Hilb}_{2t+1}(\mathbb{P}^n)$ *is a closed subscheme such that every curve parametrized by* R *is a smooth conic through* p*. Then the following hold.*

(a) $\dim R \le n - 1$.
(b) *If the 2-planes spanned by the curves parametrized by* R *all pass through a point* $q \neq p$*, then* $\dim R \le 1$*.*

Proof. (a) If we associate to any $r \in R$ the tangent line through p to the conic corresponding to r, then this follows from Lemma 4.2 since the family of lines through p is $(n-1)$-dimensional.

(b) Set $r = \dim R$. Let L be the line through p and q, and let R' be the closed subscheme of R parametrizing conics tangent to L. By Lemma 4.2 if R' is not empty, it is zero dimensional. Since every conic parametrized by the complement of R' intersects L in a point other than p, there should be a point $q' \in L$, and a closed subvariety of dimension $r-1$ in R parametrizing conics passing through p and q'. By [9, Lemma 5.1], in any projective 1-dimensional family of conics passing through p and q, there are reducible conics, so $r \le 1$.

□

We fix a hyperplane Γ in $\mathbb{P}^n$ which does not pass through p and choose a homogeneous system of coordinates for $\mathbb{P}^n$ so that $p = (1 : 0 : \cdots : 0)$ and

$\Gamma = \{x_0 = 0\}$. With the same hypothesis as in Proposition 4.1, we fix the following notation.

Notation 4.4. For $[C] \in R$, we let q_C denote the intersection of Γ with the line tangent to C at p, L_C denote the line of intersection of Γ with the 2-plane spanned by C, and Y denote the subvariety of Γ swept out by the points q_C, $[C] \in R$.

Lemma 4.5. *For every section $f \in H^0(\Gamma, \mathcal{O}_\Gamma(i))$ whose restriction $f|_{L_C}$ vanishes to order j at q_C, the restriction $x_0^{d-i} f$ vanishes to order $i+j$ at p as a section of $H^0(C, \mathcal{O}_{\mathbb{P}^n}(d)|_C)$.*

Proof. Let P be the two plane spanned by C. Then the divisor of x_0 in this plane is just L_C. The divisor of $f|_{L_C}$ is $jq_C + E$ where E is an effective divisor of degree $i-j$ whose support does not contain q_C. Then the divisor in P of $x_0^{d-i}f$ is $(d-i)l_C + jT + E'$ where T is the tangent line of C at p, E' is a union of $(i-j)$ lines passing through p, none of them equal to T. Thus, the order of its restriction to C at p is just $2j + (i-j) = i + j$. □

Lemma 4.6. *Let Z be a k-dimensional irreducible subvariety of $\mathbb{P}^n$, and let $\mathcal{I}_Z$ denote the ideal sheaf of Z in $\mathbb{P}^n$.*

(a) *The codimension of $H^0(\mathbb{P}^n, \mathcal{I}_Z(t))$ in $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(t))$ is at least $\binom{t+k}{k}$.*
(b) *If $k \geq 1$, and if Z spans a linear subvariety of dimension s in $\mathbb{P}^n$, then the codimension of $H^0(\mathbb{P}^n, \mathcal{I}_Z(t))$ in $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(t))$ is at least $st + 1$.*

Proof. (a) If $\pi : Z \to \mathbb{P}^k$ is a general linear projection, then π is a finite map, so the induced map $\pi^* : H^0(\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(t)) \to H^0(Z, \mathcal{O}_Z(t))$ is injective. We have a commutative diagram

$$\begin{tikzcd} H^0(\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(t)) \arrow[r, hook] \arrow[d, hook, "\pi^*"] & H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(t)) \arrow[dl] \\ H^0(Z, \mathcal{O}_Z(t)) & \end{tikzcd}$$

so the codimension of the kernel of the restriction map $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(t)) \to H^0(Z, \mathcal{O}_Z(t))$ is $\geq \dim H^0(\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(t)) = \binom{t+k}{k}$.

(b) Let $C \subset Z$ be an irreducible curve whose span is equal to the span of Z, which can always be achieved by taking an irreducible curve passing through finitely many linearly independent points on Z. Since $\mathcal{I}_Z \subset \mathcal{I}_C$, it is clear that we need to prove the lemma only for C and thus we may assume $k = 1$.

We can assume that the codimension 2 subvariety of $\mathbb{P}^n$ defined by $\{x_0 = x_1 = 0\}$ does not intersect Z. Then the surjective map

$$\mathcal{O}_Z^{\oplus t} \xrightarrow{(x_0^{t-1}, x_0^{t-2}x_1, \ldots, x_1^{t-1})} \mathcal{O}_Z(t-1)$$

gives a short exact sequence

$$0 \to \mathcal{O}_Z^{\oplus t-1} \to \mathcal{O}_Z(1)^{\oplus t} \to \mathcal{O}_Z(t) \to 0.$$

Since Z spans a linear subvariety of dimension s, the image of the restriction map $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus t}) \to H^0(Z, \mathcal{O}_Z(1)^{\oplus t})$ has dimension at least $(s+1)t$, so the image of the map $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus t}) \to H^0(Z, \mathcal{O}_Z(t))$ has dimension at least $(s+1)t - (t-1) = st+1$. Therefore the image of the restriction map $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(t)) \to H^0(Z, \mathcal{O}_Z(t))$ is at least $(st+1)$-dimensional. $\square$

We are now ready to prove Proposition 4.1.

Proof of Proposition 4.1. Fix a hyperplane Γ in $\mathbb{P}^n$ which does not pass through p. We will follow Notation 4.4. Set $r = \dim R$, and notice that by Lemma 4.2, $\dim Y = r$.

For $1 \le i \le d$, multiplication by x_0^{d-i} identifies $H^0(\Gamma, \mathcal{O}_\Gamma(i))$ with a subspace of $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d) \otimes \mathcal{I}_p)$. Let $W_i \subset H^0(\Gamma, \mathcal{O}_\Gamma(i))$ be the intersection of W with $H^0(\Gamma, \mathcal{O}_\Gamma(i))$ under this identification. The codimension of W in $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d) \otimes \mathcal{I}_p)$ is c, so the codimension of W_i in $H^0(\Gamma, \mathcal{O}_\Gamma(i))$ is $\le c$. There are three cases:

(1) $2 \le k \le d$: Since $\dim Y = r$, by part (a) of Lemma 4.6, the codimension of the space of sections of $\mathcal{O}_\Gamma(k)$ which vanish on Y is at least $\binom{r+2}{2}$. Since the codimension of W_k in $H^0(\Gamma, \mathcal{O}_\Gamma(k))$ is smaller than $\binom{r+2}{2}$, there is $g_k \in W_k$ which does not vanish on Y. Hence $x_0^{d-k} g_k$ is in W and its restriction to a general curve parametrized by R has a zero of order k at p by Lemma 4.5.

(2) $d+1 \le k \le 2d-2$: Let $\Gamma^\vee$ denote the space of hyperplanes in Γ, and consider the incidence correspondence $\{([C],[H]) |\ [C] \in R,\ [H] \in \Gamma^\vee,\ L_C \subset H\}$. Projection to the first component shows that the dimension of the incidence correspondence is $r+n-3$. Since $\dim \Gamma^\vee = n-1$, for a general hyperplane H, there is either no C with $L_C \subset H$, or the locus of curves C with $L_C \subset H$ is of dimension $r-2$. Fix now a general hyperplane H in Γ, and set $Y' = Y \cap H$. Let R' be the locus in R parametrizing conics C for which $q_C \in Y'$. Then $\dim R' = \dim Y' = r-1$ by Lemma 4.2, and for a general C parametrized by R', L_C does not lie on H.

Choose a system of homogeneous coordinates for $\mathbb{P}^n$ so that $p = (1 : 0 : \cdots : 0)$, Γ is given by $x_0 = 0$, and H is given by $x_0 = x_1 = 0$. Consider the vector space U of all polynomials of the form $x_1^{k-d} f \in H^0(\Gamma, \mathcal{O}_\Gamma(d))$ where f is a homogeneous polynomial of degree $2d - k \ge 2$ in $x_2, \ldots, x_n$, and let U_0 be the subspace of U consisting of those $x_1^{k-d} f$ such that f vanishes on Y'. Then the codimension of U_0 in U is $\ge \binom{(r-1)+2}{2}$ by Lemma 4.6 (a). Since $\binom{r+1}{2}$ is greater than the codimension of W_d in $H^0(\Gamma, \mathcal{O}_\Gamma(d))$, there is an element of the form $x_1^{k-d} f$ in W_d such that f does not vanish on Y'. So for a general C parametrized by R', f does not vanish on q_C. And L_C

does not lie on H, so x_1 does not vanish on L_C, and by Lemma 4.5, the order of vanishing of $x_1^{k-d} f|_C$ at p is k.

(3) $k = 2d-1$ or $2d$: We first show that if $s = \lfloor \frac{r+1}{2} \rfloor$ and $C_1, \ldots, C_s$ are general conics parametrized by R, then $L_{C_1}, \ldots, L_{C_s}$ are linearly independent, i.e. they span a linear subvariety of dimension $2s-1$. Let s' be the largest number for which there are conics $C_1, \ldots, C_{s'}$ parametrized by R such that $L_{C_1}, \ldots, L_{C_{s'}}$ are linearly independent. Let Σ be the linear span of $L_{C_1}, \ldots, L_{C_{s'}}$. Then for any curve C parametrized by R, L_C intersects Σ. By Corollary 4.3, for every point q in Σ, there is at most a 1-dimensional subscheme of R parametrizing conics C such that L_C passes through q. Therefore, $\dim R \le \dim \Sigma + 1 = 2s'$, and so $s = \lfloor \frac{r+1}{2} \rfloor \le \lfloor \frac{2s'+1}{2} \rfloor = s'$. Hence there are s conics parametrized by R whose corresponding L_C's are linearly independent, and so the same is true for s general conics parametrized by R.

Let first $k = 2d$. If $r = 1$, then $c = 0$ by our assumption and there is nothing to prove. So assume $r \ge 2$, and put $s = \lfloor \frac{r+1}{2} \rfloor$. Let $[C_1], \ldots, [C_s]$ be general points of R. By the above argument, $L_{C_1}, \ldots, L_{C_s}$ are linearly independent. Choose points $q'_{C_i} \ne q_{C_i}$ on L_{C_i}. Denote by H' the $(s-1)$-dimensional linear subvariety spanned by the points q'_{C_i}, $1 \le i \le s$, and let H be a general linear subvariety of Γ of codimension s containing $q_{C_1}, \ldots, q_{C_s}$ (note that $n-1-s \ge s-1$ by Corollary 4.3, so such H exists). Then H and H' are disjoint. Since $C_1, \ldots, C_s$ and H are general and $r - s \ge 1$, by Bertini's theorem the locus R' in R parametrizing curves C such that $q_C \in H$ is irreducible.

For $[C] \in R'$, let Σ_C be the linear subvariety spanned by H and L_C. Since for each $1 \le i \le s$, L_{C_i} does not lie on H, for a general $[C] \in R'$, L_C does not lie on H, and so Σ_C is of codimension $s-1$ in Γ and intersects H' at a point q'_C. Since R' is irreducible, the points q'_C span an irreducible quasi-projective subvariety Z of H'. Since Z is irreducible and contains $q'_{C_1}, \ldots, q'_{C_s}$, it is non-degenerate in H' and has dimension at least 1.

Set $U := H^0(H', \mathcal{O}_{H'}(d))$ which can be considered as a subspace of $H^0(\Gamma, \mathcal{O}_\Gamma(d))$. By Lemma 4.6 (b), forms of degree d on H' which vanish on Z form a subspace of codimension at least $d(s-1)+1$ in U. Therefore if $d(s-1)+1 > c$, there is a form $f \in W_d \cap U$ which does not vanish at the generic point of Z. If $[C]$ is such that f does not vanish at q'_C, then it does not vanish at any point of Σ_C which is not on H, so f cannot be identically zero on L_C. Hence $f \in W_d$ and $f|_{L_C}$ has a zero of order d at q_C, so $f|_C$ has a zero of order $2d$ at p.

If $k = 2d-1$, then repeating the above argument with d replaced by $d-1$ and choosing a form h of degree 1 on Γ which does not vanish on H, we see that if $(d-1)(s-1)+1 > c$, then there is a form g of degree $d-1$ in $H^0(H', \mathcal{O}_{H'}(d-1))$ such that $gh \in W_d$ and $gh|_{L_C}$ has a zero of order $d-1$ at q_C. This completes the proof of Proposition 4.1.

$\square$

5. Proof of Theorem 1.3

Let X be a general hypersurface of degree d in $\mathbb{P}^n$. In this section we show that the evaluation map

$$ev : \overline{\mathcal{M}}_{0,1}(X, 2) \to X$$

is flat of constant fiber dimension $2(n-d)$ if d is in the range of Theorem 1.3. Recall from Section 2 that $\overline{\mathcal{M}}_{0,1}(\mathbb{P}^n, 2)$ is a smooth stack of dimension $3n$ and that $\overline{\mathcal{M}}_{0,1}(X, 2)$ is the zero locus of a section of a locally free sheaf of rank $2d+1$ over $\overline{\mathcal{M}}_{0,1}(\mathbb{P}^n, 2)$. If the fibers of ev are of expected dimension $2(n-d)$, then $\overline{\mathcal{M}}_{0,1}(X, 2)$ has dimension $2(n-d)+n-1 = 3n-(2d+1)$, so it is a local complete intersection and in particular a Cohen-Macaulay substack of $\overline{\mathcal{M}}_{0,1}(\mathbb{P}^n, 2)$. Since a map from a Cohen-Macaulay scheme to a smooth scheme is flat if and only if it has constant fiber dimension ([13, Theorem 23.1]), to prove the theorem, it is enough to show that ev has constant fiber dimension $2(n-d)$. Note that $\dim \overline{\mathcal{M}}_{0,1}(X, 2)$ is at least $3n-(2d+1)$, and ev is surjective, so every irreducible component of any fiber of ev has dimension at least $2(n-d)$.

Let p be a point in X and $\mathcal{M}$ an irreducible component of $ev^{-1}(p)$. We claim that if there is a reducible conic parametrized by $\mathcal{M}$, then $\mathcal{M}$ has the expected dimension. Let $\mathcal{M}'$ be the substack of $\mathcal{M}$ parametrizing stable maps with reducible domains. Since X is general, by [9, Theorem 2.1], the space of lines through every point of X has dimension $n-d-1$, so $\dim \mathcal{M}' \leq 2n-2d-1$. Since $\overline{\mathcal{M}}_{0,1}(\mathbb{P}^n, 2)$ is smooth of dimension $3n$, and since the fibers of the evaluation map

$$\widetilde{ev} : \overline{\mathcal{M}}_{0,1}(\mathbb{P}^n, 2) \to \mathbb{P}^n$$

are all isomorphic, $\widetilde{ev}^{-1}(p)$ is smooth of dimension $2n$. Since the space of lines through every point of $\mathbb{P}^n$ has dimension $n-1$, the stable maps with reducible domains form a divisor in $\widetilde{ev}^{-1}(p)$. Therefore, since $\mathcal{M}'$ is non-empty, $\dim \mathcal{M}' = \dim \mathcal{M} - 1$ or $\dim \mathcal{M}' = \dim \mathcal{M}$, so $\dim \mathcal{M} \leq 2n-2d$. This proves the claim.

Now assume to the contrary that for a general hypersurface of degree d, there is a point p such that $ev^{-1}(p)$ is larger than expected. If $1 \leq d \leq \frac{n+4}{2}$, then by [9, Theorem 1.2 and Corollary 5.5], ev is flat of relative dimension $2n-2d$, hence $\frac{n+4}{2} < d \leq n-1$. Let $\mathcal{H}$ be the space of smooth hypersurfaces of degree d in $\mathbb{P}^n$, and let B be the subscheme of $\mathcal{H} \times \mathbb{P}^n \times \mathrm{Hilb}(\mathbb{P}^n)$ parametrizing triples (X, p, C) such that C is a smooth conic on X which passes through p and $[C]$ belongs to a larger than expected component of $ev^{-1}(p)$. Let $\psi : B \to \mathcal{H}$ and $\phi : B \to \mathcal{U}$ be the projection maps. By our assumption, ψ is dominant, and by passing to the irreducible components of B, we can assume that B is irreducible. By Proposition 3.4, for a general point $([X], p)$ in the image of ϕ, there is a subspace $W_{X,p} \subset H^0(X, N_{X/\mathbb{P}^n} \otimes \mathcal{I}_p)$ of codimension at most $n-1$ and an open subset $U \subset \phi^{-1}([X], p)$ such that for every $([X], p, [C]) \in U$ and every $w \in W_{X,p}$, $w|_C$ can be lifted to a section of $N_{C/\mathbb{P}^n}(-p)$ under the map

$$\rho : H^0(C, N_{C/\mathbb{P}^n}(-p)) \to H^0(C, N_{X/\mathbb{P}^n}|_C(-p)).$$

We show this implies that for a general $([X], p, [C])$ in U, the map ρ as above is surjective.

Let $\mathcal{M}$ be a larger than expected irreducible component of $ev^{-1}(p)$ which contains $[C]$. Any map parametrized by $\mathcal{M}$ should be either an isomorphism onto a smooth conic through p or a double cover of a line through p. For any line $L \subset X$ through p, there is a 2-parameter family of degree 2 covers of L (determined by the 2 branch points), and by [9, Theorem 2.1], the family of lines through p on X has dimension $n-d-1$. So the substack of $\mathcal{M}$ parametrizing double covers of lines has dimension at most $n-d+1$. Therefore, there is an irreducible closed subscheme R of dimension $n-d-1$ in $\mathcal{M}$ which contains $[C]$ and parametrizes only smooth embedded conics through p on X. Since $d > \frac{n+4}{2}$, we have

$$\binom{n-d}{2} \le (d-1)\left\lfloor \frac{n-d}{2} \right\rfloor + 1.$$

So by Proposition 4.1, if $n \le \binom{n-d}{2}$, then for every $2 \le i \le 2d$, there exists $s_i \in H^0(C, N_{X/\mathbb{P}^n}|_C) = H^0(C, \mathcal{O}_C(d))$ which has a zero of order i at p and can be lifted to a section of $N_{C/\mathbb{P}^n}(-p)$. The next lemma shows that there is a section of $N_{X/\mathbb{P}^n}|_C$ which has a zero of order 1 at p and can be lifted to a section of $N_{C/\mathbb{P}^n}(-p)$. Hence ρ is surjective.

Lemma 5.1. *Let X be a smooth complete intersection of multidegree $(d_1, \ldots, d_m)$ in $\mathbb{P}^n$. Let C be a smooth curve C on X and p a point on C. For every j, $1 \le j \le m$, there exists a section α of $N_{X/\mathbb{P}^n}|_C = \oplus \mathcal{O}_C(d_i)$ such that the j-th component of α has a zero of order 1 at p, the i-th component of α has a zero of order ≥ 2 at p for $i \ne j$, and α can be lifted to a section of $N_{C/\mathbb{P}^n}(-p)$ under the map*

$$N_{C/\mathbb{P}^n} \to N_{X/\mathbb{P}^n}|_C.$$

Proof. Suppose that X is given by $f_1 = \cdots = f_m = 0$ in $\mathbb{P}^n$, and consider the commutative diagram

$$\begin{tikzcd} \mathcal{O}_C^{n+1} \arrow[r, two heads, "A"] \arrow[d] & N_{X/\mathbb{P}^n}|_C(-p) = \oplus_{i=1}^m \mathcal{O}_C(d_i - 1) \\ T_{\mathbb{P}^n}|_C(-p) \arrow[r, two heads] & N_{C/\mathbb{P}^n}(-p) \arrow[u, two heads] \end{tikzcd}$$

where A is the m by $n+1$ matrix whose ij-th entry is $\frac{\partial f_i}{\partial x_j}$. Since X is smooth at p, A has rank m at p. Hence for every j with $1 \leq j \leq m$, there is an element in the image of $H^0(X, \mathcal{O}_C(1)^{n+1}) \to H^0(C, \oplus \mathcal{O}_C(d_i))$ whose i-th component vanishes to order 1 at p if $i = j$ and vanishes to order ≥ 2 at p if $i \neq j$. Such a section can be lifted to a section of $N_{C/\mathbb{P}^n}(-p)$. □

Applying the long exact sequence of cohomology to the short exact sequence

$$0 \to N_{C/X}(-p) \to N_{C/\mathbb{P}^n}(-p) \to \mathcal{O}_C(d)(-p) \to 0,$$

we get $H^1(C, N_{C/X}(-p)) = 0$, thus

$$h^0(C, N_{C/X}(-p)) = \chi(N_{C/X}(-p)) = \chi(T_X|_C(-p)) - \chi(T_C(-p)) = 2(n-d).$$

On the other hand, the Zariski tangent space to $ev^{-1}(p)$ at $[C]$ is isomorphic to $H^0(C, N_{C/X}(-p))$, thus $\dim \mathcal{M}$ should be at most $2(n-d)$, which is a contradiction.

6. Non-free lines on complete intersections

In this section we prove Theorem 1.6, and when $k = 1$ and X is a hypersurface, we prove a result which sometimes gives a stronger bound (Proposition 6.4).

Proof of Theorem 1.6. Assume to the contrary that every smooth complete intersection of multidegree $(d_1, \ldots, d_m)$ has a family of lines of dimension $a+1$ passing through one point such that for every line L in the family, $h^1(L, N_{L/X}(-1)) \geq k$. Let $\mathcal{H}$ be the space of smooth complete intersections of multidegree $(d_1, \ldots, d_m)$ in $\mathbb{P}^n$ and $\mathcal{U}$ the universal family over $\mathcal{H}$. Let $F_{p,k}(X)$ be the subvariety of the Grassmannian of lines on X passing though p parametrizing lines L with $h^1(L, N_{L/X}(-1)) \geq k$, and let B be the closed subvariety of $\mathcal{U} \times \mathrm{Gr}(1,n)$ parametrizing triples (X, p, L) such that $\dim F_{p,k}(X)$ at $[L]$ is larger than a.

Denote by $\phi : B \to \mathcal{U}$ and $\pi_1 : \mathcal{U} \to \mathcal{H}$ the projection maps. By our assumption $\psi = \pi_1 \circ \phi$ is dominant. We replace B by an irreducible component of B for which ψ is still dominant. By Proposition 3.4, for a general point $([X], p)$ in the image of ϕ, there is a non-empty open subset U of $\phi^{-1}([X], p)$ and a subspace

$$W_{X,p} \subset H^0(X, N_{X/\mathbb{P}^n} \otimes \mathcal{I}_p)$$

of codimension at most $n - m$ such that for every $b = ([X], p, [L]) \in U$ and $w \in W_{X,p}$, $w|_L$ can be lifted to a section of $N_{L/\mathbb{P}^n}(-p)$ under the map

$$\rho : H^0(L, N_{L/\mathbb{P}^n}(-p)) \to H^0(L, N_{X/\mathbb{P}^n}|_L(-p))$$

Let $([X], p, [L])$ be a point of U, and let Γ be a hyperplane in $\mathbb{P}^n$ which does not pass through p. Choose a system of coordinates for $\mathbb{P}^n$ so that $p = (1 : 0 : \cdots : 0)$ and Γ is given by $x_0 = 0$. Let F be an irreducible component of $F_{p,k}(X)$ containing $[L]$ whose dimension is larger than a. The cone of lines parametrized by F intersects Γ along a subvariety Y of dimension $\geq a+1$.

For positive integers i and j with $1 \leq j \leq m$ and $1 \leq i \leq d_j$, multiplication by $x_0^{d_j-i}$ identifies $H^0(\Gamma, \mathcal{O}_\Gamma(i))$ with a subspace of $H^0(X, \mathcal{O}_X(d_j) \otimes \mathcal{I}_p)$ which is itself a direct summand of $H^0(X, N_{X/\mathbb{P}^n} \otimes \mathcal{I}_p)$. Set

$$W_{i,j} := W_{X,p} \cap \ x_0^{d_j-i} H^0(\Gamma, \mathcal{O}_\Gamma(i)).$$

Since the codimension of $W_{X,p}$ in $H^0(X, N_{X/\mathbb{P}^n} \otimes \mathcal{I}_p)$ is at most $n-m$, the codimension of $W_{i,j}$ in $x_0^{d_j-i} H^0(\Gamma, \mathcal{O}_\Gamma(i))$ is at most $n-m$. For every i with $i \geq k+1$, we have

$$\binom{a+1+i}{i} > n-m.$$

Since $\dim Y \geq a+1$, by Lemma 4.6 (a), for every i and j with $1 \leq j \leq m$, $k+1 \leq i \leq d_j$, there is $f_{i,j} = x_0^{d_j-i} g_{i,j} \in W_{i,j}$ such that $g_{i,j}$ does not vanish on Y. So $f_{i,j}|_L$ is a section of $N_{X/\mathbb{P}^n}|_L = \oplus_{i=1}^m \mathcal{O}_L(d_i)$ with the following properties:

(1) The l-th component of $f_{i,j}$ is zero for $l \neq j$.
(2) The j-th component of $f_{i,j}|_L$ has a zero of order equal to i at p.
(3) The element $f_{i,j}|_L$ can be lifted to a section of $N_{L/\mathbb{P}^n}(-p)$.

Combining this with Lemma 5.1, we see that the dimension of the image of ρ is at least $d_1 + \cdots + d_m - m(k-1)$. Applying the long exact sequence of cohomology to

$$0 \to N_{L/X}(-p) \to N_{L/\mathbb{P}^n}(-p) \to N_{X/\mathbb{P}^n}|_L \otimes \mathcal{O}_L(-p) \to 0.$$

we get $h^1(L, N_{L/X}(-p)) \leq m(k-1)$, which is a contradiction. □

For a general complete intersection $X \subset \mathbb{P}^n$ of multidegree $(d_1, \ldots, d_m)$, it is likely that a similar strategy as in the proof of [9, Theorem 2.1] could be applied to show that if $\sum_i d_i \leq n-1$, then the space of lines on X through every point of X has dimension equal to $n - \sum_i d_i - 1$, but in the first part of the following theorem, we get a weaker result as an immediate corollary to Theorem 1.6.

Theorem 6.1. *Let $X \subset \mathbb{P}^n$ be a general complete intersection of multidegree $(d_1, \ldots, d_m)$, and set $d = d_1 + \cdots + d_m$.*

(a) *If $\binom{n-d+2}{2} > n-m$, then the evaluation map $ev : \overline{\mathcal{M}}_{0,1}(X,1) \to X$ is flat of relative dimension $n-d-1$.*
(b) *If $\binom{n-d}{2} > n-m$, then the evaluation map $ev : \overline{\mathcal{M}}_{0,1}(X,2) \to X$ is flat of relative dimension $2n-2d$.*

Proof. (a) As was shown in the proof of Theorem 1.3, to prove the flatness of ev, it suffices to show that the fibers of ev have constant dimension $n-d-1$. But the fibers of ev have dimension at least $n-d-1$, and if there is an irreducible component $\mathcal{M}$ of $ev^{-1}(p)$ whose dimension is larger than $n-d-1$, then every line parametrized by $\mathcal{M}$ should be non-free. This is not possible by case $k=1, a=n-d-1$ of Proposition 1.6.

(b) This follows easily from the proofs of Theorem 1.3 and Proposition 1.6. ☐

For the rest of this section, we consider general hypersurfaces in $\mathbb{P}^n$.

Proposition 6.2. *If X is a general hypersurface of degree $3 \leq d \leq n-1$ in $\mathbb{P}^n$, then the non-free lines on X sweep out a divisor.*

Remark 6.3. If $d = 1$, clearly this set is empty. Same is true if $d = 2$. We have already seen that the codimension of this set is at least one in lemma 2.1.

Proof. Let $\mathcal{H}$ be the projective space of hypersurfaces of degree d in $\mathbb{P}^n$. Consider the subvariety $I \subset \mathbb{P}^n \times \mathcal{H}$ consisting of pairs $(p, [X])$ such that there is either a non-free line on X through p or a line on X through p which intersects the singular locus of X. Denote by π_1 and π_2 the projection maps from I to $\mathbb{P}^n$ and $\mathcal{H}$. We show that the dimension of the fibers of π_1 is equal to $\dim \mathcal{H} + n - 2$.

Since all the fibers of π_1 are isomorphic, we can assume $p = (1 : 0, \cdots : 0)$. A hypersurface X which contains p is given by an equation of the form $x_0^{d-1} f_1 + \cdots + f_d = 0$ where f_i is homogenous of degree i in $x_1, \dots, x_n$ for $1 \leq i \leq d$. The space of lines through p on X, which we denote by $F_p(X)$, is isomorphic to the scheme $V(f_1, \dots, f_d)$ in $\mathbb{P}^{n-1}$, so $\dim F_p(X) \geq n - d - 1$. If $F_p(X)$ is singular at $[L]$, then we see that $([X], p) \in \pi_1^{-1}(p)$ as follows. Since

$$T_{F_p(X),[L]} \cong H^0(L, N_{L/X}(-p)),$$

if $F_p(X)$ is singular at $[L]$, then

$$h^0(L, N_{L/X}(-p)) > \dim F_p(X) \geq n - d - 1.$$

So either L is not contained in the smooth locus of X, or it is contained in the smooth locus of X and

$$\begin{aligned} h^1(L, N_{L/X}(-p)) &= h^0(l, N_{L/X}(-p)) - \chi(N_{L/X}(-p)) \\ &= h^0(L, N_{L/X}(-p)) - (n - d - 1) \\ &> 0, \end{aligned}$$

so L is non-free.

If $2 \leq d \leq n - 1$, and if f_i is a general homogeneous polynomial of degree i in $x_1, \dots, x_n$ for $2 \leq i \leq d$, then the intersection $Y := V(f_2, \dots, f_d)$ is a smooth complete intersection subvariety of $\mathbb{P}^{n-1}$ of dimension $n - d \geq 1$. By [7, Proposition 3.1], the dual variety of Y in $\mathbb{P}^{n-1\vee}$ is a hypersurface, so there is a codimension 1 subvariety of $H^0(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$ consisting of forms f_1 such that $Y \cap \{f_1 = 0\}$ is singular. This shows that the space of tuples $(f_1, \dots, f_d)$ for which the scheme $V(f_1, \dots, f_d)$ is singular is of codimension 1 in the space of all tuples $(f_1, \dots, f_d)$. So the fibers of π_1 over p form a subvariety of codimension at most 1 in the space of all hypersurfaces which contain p, and $\dim I \geq \dim \mathcal{H} + n - 2$.

Consider now the map $\pi_2 : I \to \mathcal{H}$. Since the fibers of π_2 have dimension at most $n-1$, either π_2 is dominant or its image is of codimension 1 in $\mathcal{H}$. We show the latter cannot happen. For any hypersurface X, the space of lines which are contained in the smooth locus of X and are not free cannot sweep out a dense subset in X by Lemma 2.1, so if $\dim \pi_2^{-1}([X]) = n-1$, then the lines passing through the singular points of X should sweep out X. The locus of hypersurfaces which are singular at least along a curve is of codimension greater than 1 in $\mathcal{H}$, and so is the locus of hypersurfaces which are cones over hypersurfaces in $\mathbb{P}^{n-1}$ when $d \geq 3$. Therefore π_2 is dominant, and $\dim I = \dim \mathcal{H} + n - 2$, so a general fiber of π_2 has dimension $n-2$. □

The proof of Theorem 1.6 yields stronger results if we know the dimension of the linear span of Y defined in the proof of the theorem. The next proposition gives such a result when $k = 1$.

Proposition 6.4. *Suppose that X is a general hypersurface of degree d in $\mathbb{P}^n$ and Σ is a cone of lines on X over a curve $Y \subset \mathbb{P}^{n-1}$. If the linear span of Y has dimension $> (n-2)/2$, then a general line parametrized by Y is free.*

Proof. The proof is similar to that of Theorem 1.6 except that we apply part (b) of Lemma 4.6 to Y. We follow the proof of Theorem 1.6. Let s be the dimension of the linear span of Y, and let p be the vertex of the cone over Y. Since X is general, there is a subspace $W_{X,p} \subset H^0(X, \mathcal{O}_X(d) \otimes \mathcal{I}_p)$ of codimension at most $n-1$ such that for every $w \in W$ and a general line L parametrized by Y, $w|_L$ can be lifted to a section of $N_{L/\mathbb{P}^n}(-p)$. By Lemma 4.6 (b), if $2s+1 > n-1$, then for every $2 \leq i \leq d$, there is a section $f_i = x_0^{d-i} g_i \in W_{X,p}$ such that g_i does not vanish on Y. So f_i has a zero of order i at p and is contained in the image of ρ. The image of ρ also contains a section which has a simple zero at p by Lemma 5.1, so ρ is surjective and $H^1(L, N_{L/X}(-p)) = 0$. □

7. Dimension and irreducibility of $\overline{\mathcal{M}}_{0,0}(X, e)$

For a complete intersection X of multidegree $(d_1, \ldots, d_m)$ in $\mathbb{P}^n$ with $\sum d_i \leq n$, the *threshold degree* of X is defined to be

$$E(X) := \left\lfloor \frac{n-m+2}{n+1-\sum d_i} \right\rfloor .$$

Theorem 7.1. *Let $X \subset \mathbb{P}^n$ be a general complete intersection of multidegree $(d_1, \ldots, d_m)$. If n is sufficiently large and $d < (2n+m+1)/3$, then the evaluation map $ev : \overline{\mathcal{M}}_{0,1}(X, e) \to X$ is flat and has relative dimension $e(n+1-d)-2$ for every $e \geq 1$.*

Proof. By [9, Corollary 5.5], if the evaluation map $ev : \overline{\mathcal{M}}_{0,1}(X, e) \to X$ is flat of relative dimension $e(n+1-d)-2$ for every $1 \leq e \leq E(X)$, then it is flat of relative dimension $e(n+1-d)-2$ for every $e \geq 1$. If $d < \frac{2n+m+1}{3}$, then

$E(X) \leq 2$, so to prove the statement it is enough to prove it for $e = 1, 2$. If n is large compared to m, and if $d < \frac{2n+m+1}{3}$, then $\binom{n-d}{2} > n - m$ (if $m = 1$ for example, it is enough to have $n \geq 23$). Thus by Theorem 6.1, ev is flat of the expected relative dimension for $e = 1, 2$. □

Remark 7.2. It is proven in [2] that when $m = 1$, the threshold degree could be improved to

$$E(X) = \left\lfloor \frac{n-m}{n+1-\sum d_i} \right\rfloor.$$

The same proof works for an arbitrary m. This improved bound shows that the statement of the above corollary is true for $d < \frac{2n+m+3}{3}$ and n sufficiently large.

Theorem 7.3. *With the same assumptions as in Theorem 7.1, $\overline{\mathcal{M}}_{0,0}(X, e)$ is an integral complete intersection stack of dimension $e(n+1-d)+n-m-3$ for every $e \geq 1$.*

Proof. By Theorem 7.1,

$$\dim \overline{\mathcal{M}}_{0,0}(X, e) = \dim \overline{\mathcal{M}}_{0,1}(X, e) - 1 = e(n + 1 - d) + n - m - 3.$$

The stack $\overline{\mathcal{M}}_{0,0}(X, e)$ is the zero locus of a section of a locally free sheaf of rank $de + m$ over the smooth stack $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, e)$ (see Section 2). Since $\dim \overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, e) = (e+1)(n+1) - 4$, and since $\overline{\mathcal{M}}_{0,0}(X, e)$ has the expected dimension $= \dim \overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, e) - (de + m)$, it is a local complete intersection stack.

Next we prove that $\overline{\mathcal{M}}_{0,0}(X, e)$ is irreducible. By [9, Corollary 6.9], if X is a smooth complete intersection, then $\overline{\mathcal{M}}_{0,0}(X, e)$ is irreducible for every $e \geq 1$ if all the following hold.

(i) The evaluation map $ev : \overline{\mathcal{M}}_{0,1}(X, e) \to X$ is flat of relative dimension $e(n + 1 - d) - 2$ for every $e \geq 1$.
(ii) General fibers of ev are irreducible.
(iii) There is a free line on X.
(iv) $\mathcal{M}_{0,0}(X, e)$ is irreducible for every $1 \leq e \leq E(X)$ where $\mathcal{M}_{0,0}(X, e)$ denotes the stack of stable maps of degree e with irreducible domains.

By Theorem 7.1 the first property is satisfied, and property (iii) holds for every smooth complete intersection which is covered by lines, that is every smooth complete intersection with $d \leq n-1$. By Corollary 3.3, for every line L on X, $H^1(L, N_{L/X}) = 0$, so $\overline{\mathcal{M}}_{0,0}(X, 1)$ and hence $\overline{\mathcal{M}}_{0,1}(X, 1)$ are smooth. Therefore, by generic smoothness, a general fiber of $ev : \overline{\mathcal{M}}_{0,1}(X, 1) \to X$ is smooth. Since every fiber of this map has the expected dimension $n - d - 1$, it is a complete intersection of dimension ≥ 1 in $\mathbb{P}^{n-1}$, so it is also connected and therefore irreducible.

To show property (iv) holds, we need to show that $\mathcal{M}_{0,0}(X, e)$ is irreducible for $e = 1, 2$. When X is a smooth hypersurface of degree $\leq 2n - 4$ in $\mathbb{P}^n$, $n \geq 4$, $\mathcal{M}_{0,0}(X, 1)$ is irreducible by [12, V.4.3]. The same proof can be generalized to the case of complete intersections. The irreducibility of the space of lines on general complete intersections with $d \leq n-1$ is also proved in [5, Corollary 4.5].

By [15, Corollary 7.6], $\mathcal{M}_{0,0}(X,2)$ is irreducible for a general hypersurface X of degree $\leq n-2$ in $\mathbb{P}^n$ (see also [6]). Let us explain how one can generalizes the same argument to the case of general complete intersections with $d \leq n-2$. Since the dimension of $\mathcal{M}_{0,0}(X,2)$ is $3n-2d-m-1$, and since the space of double covers of lines on X has dimension

$$\dim \mathcal{M}_{0,0}(X,1)+2 = 2n-d-m < 3n-2d-m-1,$$

every irreducible component of $\mathcal{M}_{0,0}(X,2)$ contains an open subscheme parametrizing smooth embedded conics on X. Therefore, to prove $\mathcal{M}_{0,0}(X,2)$ is irreducible, it is enough to show that $\mathrm{Hilb}_{2t+1}(X)$ is irreducible. To this end, let $I \subset \mathrm{Hilb}_{2t+1}(\mathbb{P}^n) \times \mathcal{H}$ be the incidence correspondence parametrizing pairs (C,X) such that C is a conic on X, and let $\pi_1 : I \to \mathrm{Hilb}_{2t+1}(\mathbb{P}^n)$ and $\pi_2 : I \to \mathcal{H}$ denote the two projection maps. Since $\mathrm{Hilb}_{2t+1}(\mathbb{P}^n)$ is smooth and irreducible, and since the fibers of π_1 are product of projective spaces, I is smooth and irreducible.

Let J be the closed subscheme of I parametrizing pairs (C,X) such that C is a non-reduced conic, so the support of C is a line on X. Then J is smooth and irreducible since J maps to the Grassmannian of lines in $\mathbb{P}^n$ and the fibers are smooth and irreducible. Let $\pi_2' : J \to \mathcal{H}$ be the projection map. Note that for any smooth complete intersection X and $L \subset X$, the space of non-reduced conics on X whose support is L can be identified with $\mathbb{P}(H^0(L, N_{L/X}(-1)))$. If $[X] \in \mathcal{H}$ is general, then the space of lines on X is irreducible, thus the fiber of π_2' over $[X]$ is connected. By generic smoothness, $\pi_2'^{-1}([X])$ is smooth and therefore irreducible.

By [5, Lemma 3.2], if $i : N \to M$ and $e : M \to Y$ are morphisms of irreducible schemes and i maps the generic point of N to a normal point of M, then e has irreducible general fibers provided that $e \circ i$ is dominant with irreducible general fibers. We apply this result to $N = J$, $M = I$, $Y = \mathcal{H}$, $i =$ the inclusion map, and $e = \pi_2$. Since $d \leq n-2, h^0(L, N_{L/X}(-1)) \geq 1$ for any smooth X parametrized by $\mathcal{H}$ and any line $L \subset X$, so $e \circ i = \pi_2'$ is dominant and we have shown its general fibers are irreducible. Since I is smooth, a general fiber of π_2 is irreducible.

Finally, since $\overline{\mathcal{M}}_{0,0}(X,e)$ is a local complete intersection irreducible stack, and since $\overline{\mathcal{M}}_{0,0}(X,e)$ is smooth and hence reduced over the dense open substack parametrizing embedded smooth free rational curves of degree e, it is everywhere reduced.

□

Proof of Corollary 1.5. Let $ev = (ev_1, \ldots, ev_k) : \overline{M}_{0,k}(X,e) \to X^k$ denote the evaluation map. By Theorem 1.4, $\overline{M}_{0,0}(X,e)$ and hence $\overline{M}_{0,k}(X,e)$, are irreducible of the expected dimension, so by definition,

$$< H^{c_1}, \dots, H^{c_k} >^X_{0,e[\text{line}]} = \int_{\overline{M}_{0,k}(X,e)} ev_1^*\Gamma_1 \cup \dots \cup ev_k^*\Gamma_k.$$

There is a smooth dense open subscheme U of $\overline{M}_{0,k}(X,e)$ such that every stable map parametrized by U is a smooth embedded rational curve of degree e on X. Applying the Kleiman-Bertini Theorem [11] to the diagram

$$\begin{tikzcd} & U \arrow[d, "ev"] \\ \Gamma_1 \times \dots \times \Gamma_k \arrow[r, hook] & (\mathbb{P}^n)^k, \end{tikzcd}$$

we see that for general $\Gamma_1, \dots, \Gamma_k$, $ev^{-1}(\Gamma_1 \times \dots \times \Gamma_k)$ is reduced of dimension equal to $\dim U + \dim \Gamma_1 + \dots + \dim \Gamma_k - \dim(\mathbb{P}^n)^k = 0$. Another use of the Kleiman-Bertini Theorem shows $ev^{-1}(\Gamma_1 \times \dots \times \Gamma_k)$ does not intersect the complement of U in $\overline{M}_{0,k}(X,e)$, so we get the desired result. $\square$

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