HIGHER FANO MANIFOLDS

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ABSTRACT. In this paper we address Fano manifolds with positive higher Chern characters. They are expected to enjoy stronger versions of several of the nice properties of Fano manifolds. For instance, they should be covered by higher dimensional rational varieties, and families of higher Fano manifolds over higher dimensional bases should admit meromorphic sections (modulo the Brauer obstruction). Aiming at finding new examples of higher Fano manifolds, we investigate positivity of higher Chern characters of rational homogeneous spaces. We determine which rational homogeneous spaces of Picard rank 1 have positive second Chern character, and show that the only rational homogeneous spaces of Picard rank 1 having positive second and third Chern characters are projective spaces and quadric hypersurfaces. We also classify Fano manifolds of large index having positive second and third Chern characters. We conclude by discussing conjectural characterizations of projective spaces and complete intersections in terms of these higher Fano conditions.

CONTENTS

1. INTRODUCTION

Fano manifolds are complex projective manifolds having positive first Chern class $c_1(T_X)$. Examples of Fano manifolds include projective spaces, smooth complete intersections of low degree in projective spaces, and rational homogeneous spaces. The positivity condition on the first Chern class has far reaching geometric and arithmetic implications, making Fano manifolds a central subject in algebraic geometry. To illustrate the special role played by Fano manifolds, we briefly discuss some of their nice properties.

Hypersurfaces of low degree in projective spaces provide basic examples of Fano manifolds. The first Chern class of a smooth hypersurface $X_d \subset \mathbb{P}^{n+1}$ of degree $d \geq 1$ is given by

$$c_1(T_{X_d}) = c_1(\mathcal{O}_{X_d}(n+2-d)).$$

Therefore, $X_d \subset \mathbb{P}^{n+1}$ is Fano if and only if $d \leq n+1$. One can easily check that if $d \leq n$, then X_d is covered by lines, and if $d = n+1$, then X_d is covered by conics. In contrast, if $d \geq n+2$, then through a general point of X_d there are no rational curves. Hence, for smooth hypersurfaces in projective spaces:

$$X_d \subset \mathbb{P}^{n+1} \text{ is Fano} \iff d \leq n+1 \iff X_d \text{ is covered by rational curves.}$$

In the landmark paper [Mor79], Mori showed that any Fano manifold is covered by rational curves. Since then, rational curves have become a fundamental tool in the study of Fano manifolds. Later, it was proved in [KMM92] and [Cam92] that a much stronger property holds: any Fano manifold X is *rationally connected*, i.e., there are rational curves connecting any two points of X.

Fano hypersurfaces also satisfy special arithmetic properties, as illustrated by the classical Tsen's theorem. It states that a hypersurface $\mathcal{X} \subset \mathbb{P}_K^{n+1}$ of degree $d \leq n+1$ over the function field K of a curve always has a K-point. In geometric terms, Tsen's theorem says that families of hypersurfaces of degree $d \leq n+1$ in $\mathbb{P}^{n+1}$ over one dimensional bases always admit holomorphic sections. This result has been greatly generalized by Graber, Harris and Starr in [GHS03]. They showed that proper families of rationally connected varieties over curves always admit holomorphic sections.

In [Lan52], Lang provided a version of Tsen's theorem for function fields of higher dimensional varieties.

Theorem (Tsen-Lang Theorem). *Let K be the function field of a variety of dimension $r \geq 1$. If $\mathcal{X} \subset \mathbb{P}_K^{n+1}$ is a hypersurface of degree d with $d^r \leq n+1$, then $\mathcal{X}$ has a K-point.*

As before, this statement can be interpreted as saying that families of hypersurfaces of degree d in $\mathbb{P}^{n+1}$ over r-dimensional bases always admit meromorphic sections provided that $d^r \leq n+1$. A natural problem consists in finding natural geometric conditions on general fibers of a fibration $\pi : \mathcal{X} \to B$ over a higher dimensional base under which the conclusion on Tsen-Lang theorem holds. We note, however, that over higher dimensional bases, in addition to conditions on the fibers of the family, one must observe the existence of Brauer obstruction on the

base. We do not address the latter here, and refer to [Sta08] for a discussion on the Brauer obstruction in connection with the Tsen-Lang theorem.

In recent years, there has been great effort towards defining suitable higher analogues of the Fano condition. Higher Fano manifolds are expected to enjoy stronger versions of several of the nice properties of Fano manifolds.

Problem 1.1. *Find natural geometric conditions $\mathfrak{R}_r$ on a manifold X such that:*
- *for hypersurfaces $X_d \subset \mathbb{P}^{n+1}$, the condition $\mathfrak{R}_r$ is equivalent to $d^r \leq n+1$;*
- *if a complex projective manifold satisfies $\mathfrak{R}_r$, then X is covered by rational varieties of dimension r; and*
- *families of projective manifolds satisfying $\mathfrak{R}_r$ over r-dimensional bases always admit meromorphic sections (modulo Brauer obstruction).*

For $r = 1$, it follows from the previous discussion that the condition $\mathfrak{R}_1$ can be taken to be "X is rationally connected," or, more restrictively, "X is a Fano manifold."

In a series of papers, De Jong and Starr introduce and investigate possible candidates for the condition $\mathfrak{R}_2$ ([Sta06], [dJS06a], [dJS06b], [dJS07], [dJHS11]). They present some notions of *rationally simple connectedness*, inspired by the natural analogue of Problem 1.1 in topology. Namely, let $\pi : M \to B$ be a fibration of CW complexes with typical fiber F over an r-dimensional base B. If F is $(r-1)$-connected, then π admits a continuous section $s : B \to M$. If one draws a parallel between topology and algebraic geometry, interpreting a loop as a rational curve, one gets that the solution of the problem for $r = 1$ in topology, namely F is path-connected, translates precisely into the condition that the general fiber of the fibration $\pi : \mathcal{X} \to B$ is rationally connected. For $r > 1$, this analogy has limitations, and De Jong and Starr's notions of rationally simple connectedness are technical and hard to check in practice. Roughly speaking, and at the very least, one asks that a suitable irreducible component of the scheme of rational curves through two general points of X is itself rationally connected. This notion led to a version of the Tsen-Lang Theorem for fibrations by rationally simply connected manifolds over surfaces in [dJHS11]. More recently, other notions of rationally simple connectedness have been investigated, for example in [FGP19].

In [dJS06a], De Jong and Starr introduced 2-*Fano manifolds* as an alternative candidate to the condition $\mathfrak{R}_2$. A projective manifold X is said to be 2-*Fano* if it is Fano and the second Chern character $\mathrm{ch}_2(T_X) = \frac{1}{2}c_1(T_X)^2 - c_2(T_X)$ is positive, i.e., $\mathrm{ch}_2(T_X) \cdot S > 0$ for every surface S in X. This condition conjecturally implies rationally simple connectedness, but it is much easier to check. In [dJS07], it is shown that 2-Fano manifolds satisfying some mild assumptions are covered by rational surfaces.

In [AC12], Araujo and Castravet introduced a new approach to study 2-Fano manifolds, via *polarized minimal families of rational curves* (H_x, L_x). For a Fano manifold X and a point $x \in X$, let H_x be a proper irreducible component of the scheme $\mathrm{RatCurves}^n(X, x)$, parametrizing rational curves on X through x having

minimal anti-canonical degree. If x is a general point of X, then every such component is smooth and comes with a finite and birational morphism $\tau_x : H_x \to \mathbb{P}(T_x X^*)$, mapping a point parametrizing a smooth curve at x to its tangent direction at x. Consider the polarization $L_x := \tau_x^* \mathcal{O}(1)$ on H_x. Then [AC12, Proposition 1.3] gives a formula for all Chern characters of H_x in terms of the Chern characters of X and $c_1(L_x)$. As a consequence, if X is 2-Fano and $\dim(H_x) \geq 1$, then H_x is Fano. If moreover $\mathrm{ch}_3(X) > 0$ and $\dim(H_x) \geq 2$, then H_x is 2-Fano. This motivates the following definition.

Definition 1.2. *We say that a Fano manifold X satisfies the condition $\mathfrak{F}_r$ if its Chern characters $ch_i(X)$ are positive for all $1 \leq i \leq r$. This positivity condition means that $\mathrm{ch}_i(T_X) \cdot Z > 0$ for every effective i-cycle Z in X.*

Given a Fano manifold X satisfying condition $\mathfrak{F}_r$, and a polarized minimal family of rational curves (H_x, L_x) on it, one can ask whether, H_x satisfies condition $\mathfrak{F}_{r-1}$. We note the analogy with the higher notions of connectedness in topology: if a path-connected topological space M is r-connected, then its loop spaces (with the compact-open topology) are $(r-1)$-connected.

The inductive approach introduced in [AC12] was further explored in [Suz20] and [Nag19] to show that, under some extra assumptions, Fano manifolds satisfying condition $\mathfrak{F}_r$ are covered by rational r-folds.

Despite these recent developments, few examples of higher Fano manifolds are known. The strong restrictions on (H_x, L_x) imposed by the 2-Fano condition explain why examples are difficult to find, but they also suggest where to look for them. Following this trail, new examples of 2-Fano manifolds were described [AC12], including several homogeneous and quasi-homogeneous spaces. A few more examples were presented in [AC13], where Araujo and Castravet classified 2-Fano manifolds with large index.

The present paper addresses the problem of finding new examples of higher Fano manifolds. We start by investigating rational homogeneous spaces, for which the polarized minimal families of rational curves are well described in [LM03]. This search provides new examples of 2-Fano manifolds of exceptional type E and F (see Section 3 for the notation regarding rational homogeneous spaces), and yields the following classification of 2-Fano rational homogeneous spaces of Picard rank 1:

Theorem 1.3. *The following is the complete list of rational homogeneous spaces of Picard rank 1 satisfying the condition $\mathfrak{F}_2$:*
- *A_n/P^k, for $k = 1, n$ and for $n = 2k-1, 2k$ when $2 \leq k \leq \frac{n+1}{2}$;*
- *B_n/P^k, for $k = 1, n$ and for $2n = 3k+1$ when $2 \leq k \leq n-1$;*
- *C_n/P^k, for $k = 1, n$ and for $2n = 3k-2$ when $2 \leq k \leq n-1$;*
- *D_n/P^k, for $k = 1, n-1, n$ and for $2n = 3k+2$ when $2 \leq k < n-1$;*
- *E_n/P^k, for $n = 6, 7, 8$ and $k = 1, 2, n$;*
- *F_4/P^4;*
- *G_2/P^k, for $k = 1, 2$.*

However, we get no new examples of Fano manifolds satisfying $\mathfrak{F}_3$:

Theorem 1.4. *The only rational homogeneous spaces of Picard rank 1 satisfying $\mathfrak{F}_3$ are projective spaces $\mathbb{P}^n$, $n \geq 3$, and quadric hypersurfaces $Q^n \subset \mathbb{P}^{n+1}$, $n \geq 7$.*

Next we go through the list of 2-Fano manifolds with large index in [AC13] and check the $\mathfrak{F}_3$ condition for those. We obtain the following classification.

Theorem 1.5. *Let X be a Fano manifold of dimension $n \geq 3$ and index $i_X \geq n-2$. If X satisfies $\mathfrak{F}_3$, then X is isomorphic to one of the following.*

- $\mathbb{P}^n$.
- *Complete intersections in projective spaces:*
    - *Quadric hypersurfaces $Q^n \subset \mathbb{P}^{n+1}$ with $n > 6$;*
    - *Complete intersections of quadrics $X_{2 \cdot 2} \subset \mathbb{P}^{n+2}$ with $n > 13$;*
    - *Cubic hypersurfaces $X_3 \subset \mathbb{P}^{n+1}$ with $n > 25$;*
    - *Quartic hypersurfaces in $\mathbb{P}^{n+1}$ with $n > 62$;*
    - *Complete intersections $X_{2 \cdot 3} \subset \mathbb{P}^{n+2}$ with $n > 32$;*
    - *Complete intersections $X_{2 \cdot 2 \cdot 2} \subset \mathbb{P}^{n+3}$ with $n > 20$.*
- *Complete intersections in weighted projective spaces:*
    - *Degree 4 hypersurfaces in $\mathbb{P}(2, 1, \ldots, 1)$ with $n > 55$;*
    - *Degree 6 hypersurfaces in $\mathbb{P}(3, 2, 1, \ldots, 1)$ with $n > 181$;*
    - *Degree 6 hypersurfaces in $\mathbb{P}(3, 1, \ldots, 1)$ with $n > 188$;*
    - *Complete intersections of two quadrics in $\mathbb{P}(2, 1, \ldots, 1)$ with $n > 6$.*

So the following remains an open problem.

Problem 1.6. *Find examples of Fano manifolds satisfying condition $\mathfrak{F}_3$ other than complete intersections in weighted projective spaces.*

Our results show that conditions $\mathfrak{F}_r$ for $r \geq 3$ are extremely restrictive, and lead to conjectural characterizations of projective spaces and complete intersections in terms of positivity of Chern characters. It has already been asked in [AC12] whether the only n-dimensional Fano manifold satisfying $\mathfrak{F}_n$ is the projective space $\mathbb{P}^n$. We believe that a much stronger statement holds.

Conjecture 1.7. *If X is an n-dimensional Fano manifold satisfying condition $\mathfrak{F}_k$, with $k = \lceil \log_2(n+1) \rceil$, then $X \cong \mathbb{P}^n$.*

Problem 1.8. *For fixed n, find the smallest integer $k = k(n)$ such that the following holds. If X is an n-dimensional Fano manifold satisfying condition $\mathfrak{F}_k$, then X is a complete intersection in a weighted projective space. Can this integer k be chosen independently of n?*

Throughout this paper we work over $\mathbb{C}$.

The paper is organized as follows. In Section 2, we review the basic theory of minimal families of rational curves on Fano manifolds, and discuss their special properties when the ambient space satisfies higher Fano conditions. In Section 3, we provide a background on rational homogeneous spaces. In Sections 4 and 5, we go through the classification of rational homogeneous spaces of classical and exceptional types, and check conditions $\mathfrak{F}_2$ and $\mathfrak{F}_3$. This case by case analysis will prove Theorems 1.3 and 1.4. In Section 6, we prove Theorem 1.5.

2. Minimal rational curves on higher Fano manifolds

In this section, we discuss special properties of *minimal families of rational curves* on higher Fano manifolds. We start by reviewing the basic properties of minimal families of rational curves. We refer to [Kol96, Chapters I and II] for the basic theory of rational curves on complex projective varieties.

Let X be a Fano manifold and $x \in X$ a point. There is a scheme $\mathrm{RatCurves}^n(X, x)$ that parametrizes rational curves on X through x. It can be constructed as the normalization of a certain subscheme of the Chow variety $\mathrm{Chow}(X)$, which parametrizes effective 1-cycles on X (see [Kol96, II.2.11]). The superscript n in $\mathrm{RatCurves}^n(X, x)$ stands for the normalization. Let H_x be a *proper* irreducible component of $\mathrm{RatCurves}^n(X, x)$. We call it a *minimal family of rational curves through x*. For instance, one can take H_x to be an irreducible component of $\mathrm{RatCurves}^n(X, x)$ parametrizing rational curves through x, having minimal degree with respect to $-K_X$. If $x \in X$ is a general point, then H_x is smooth. From the universal properties of $\mathrm{Chow}(X)$, we get a universal family diagram:

$$
\begin{array}{ccc}
U_x & \xrightarrow{\;\mathrm{ev}\;} & X, \\
{\scriptstyle \pi}\big\downarrow & & \\
H_x & &
\end{array}
$$

where π is a $\mathbb{P}^1$-bundle.

The variety H_x comes with a natural finite morphism $\tau_x : H_x \to \mathbb{P}(T_x X^*)$ sending a curve that is smooth at x to its tangent direction at x (see [Keb02, Theorems 3.3 and 3.4]). The morphism τ_x is also birational by [HM04]. Set $L_x := \tau_x^* \mathcal{O}(1)$. The pair (H_x, L_x) is called a *polarized minimal family of rational curves through x*.

In [AC12], Araujo and Castravet computed all the Chern characters of H_x in terms of the Chern characters of X and $c_1(L_x)$. In order to state the result, first we define, for any $k \geq 1$:

$$
T := \pi_* ev^* : N^k(X)_{\mathbb{R}} \to N^{k-1}(H_x)_{\mathbb{R}}.
$$

By [AC12, Proposition 1.3],

$$\mathrm{ch}_k(H_x) = \sum_{j=0}^{k} A_j c_1(L_x)^j \cdot T(\mathrm{ch}_{k+1-j}(X)) - \frac{1}{k!}c_1(L_x)^k \qquad (2.1)$$

where $A_j = \frac{(-1)^j B_j}{j!}$ and the B_j's are the Bernoulli numbers.

Set $d := \dim(H_x)$ and $\eta := T(\mathrm{ch}_2(X)) - \frac{L_x}{2}$. Then, for $1 \le k \le 3$, (2.1) becomes:

$$c_1(H_x) = T(\mathrm{ch}_2(X)) + \frac{d}{2}c_1(L_x) \qquad (2.2)$$

$$\mathrm{ch}_2(H_x) = T(\mathrm{ch}_3(X)) + \frac{1}{2}\Big(c_1(H_x) - \frac{d}{2}c_1(L_x)\Big)L_x + \frac{d-4}{12}L_x^2 \qquad (2.3)$$

$$\mathrm{ch}_3(H_x) = T(\mathrm{ch}_4(X)) + \frac{1}{2}T(\mathrm{ch}_3(X)) \cdot L_x + \frac{1}{12}T(\mathrm{ch}_2(X)) \cdot L_x^2 - \frac{1}{6}L_x^3$$
$$= T(\mathrm{ch}_4(X)) + \frac{1}{2}T(\mathrm{ch}_3(X)) \cdot L_x + \frac{L_x^2}{12}\Big(\eta - \frac{3}{2}L_x\Big). \qquad (2.4)$$

Using the fact that the map $T : N^k(X)_{\mathbb{R}} \to N^{k-1}(H_x)_{\mathbb{R}}$ preserves positivity for $k - 1 \le d$ ([AC12, Lemma 2.7]), it is possible to show an inductive structure on the class of higher Fano manifolds.

Theorem 2.1. *Let X be a Fano manifold, and (H_x, L_x) a polarized minimal family of rational curves through a general point $x \in X$.*

  (1) ([AC12, Theorem 1.4 (2)]) If X satisfies $\mathfrak{F}_2$ and $d \ge 1$, then H_x satisfies $\mathfrak{F}_1$ (i.e., it is a Fano manifold).
  (2) ([AC12, Theorem 1.4 (3)]) If X satisfies $\mathfrak{F}_3$ and $d \ge 2$, then H_x satisfies $\mathfrak{F}_2$ and $\rho(H_x) = 1$.
  (3) If X satisfies $\mathfrak{F}_4$, $d \ge 3$ and $\eta \ge \frac{3}{2}L_x$, then H_x satisfies $\mathfrak{F}_3$ and $\rho(H_x) = 1$.

Proof of (3). Suppose that the Fano manifold X satisfies $\mathfrak{F}_4$, $d \ge 3$ and $\eta \ge \frac{3}{2}L_x$. We know from (2) that H_x is a Fano manifold satisfying $\mathfrak{F}_2$ and $\rho(H_x) = 1$. Using the fact that the map $T : N^k(X)_{\mathbb{R}} \to N^{k-1}(H_x)_{\mathbb{R}}$ preserves positivity for $k - 1 \le d$, it follows from (2.4) that $\mathrm{ch}_3(H_x) > 0$, and thus H_x satisfies $\mathfrak{F}_3$. $\qquad\square$

Remark 2.2. Let the assumptions be as in Theorem 2.1(3). By [AC12, Proof of Theorem 1.4, p.99], the class η is nef. Since both 2η and $2 \cdot \mathrm{ch}_2(X)$ are integral classes, the assumption $\eta \ge \frac{3}{2}L_x$ is satisfied, except if $\eta = 0, \frac{1}{2}L_x, L_x$.

We end this section with the description of (H_x, L_x) when X is a Fano manifold satisfying $\mathfrak{F}_2$.

Theorem 2.3 ([AC12, Theorem 1.4 (2)])**.** *Let X be a Fano manifold, (H_x, L_x) a polarized minimal family of rational curves through a general point $x \in X$, and set $d = \dim H_x$.*

  (1) If X is 2-Fano, then H_x is a Fano manifold with $\mathrm{Pic}(H_x) = \mathbb{Z} \cdot [L_x]$, except if (H_x, L_x) is isomorphic to one of the following
     (a) $\big(\mathbb{P}^m \times \mathbb{P}^m, p_1^*\mathcal{O}(1) \otimes p_2^*\mathcal{O}(1)\big)$, with $d = 2m$,
     (b) $\big(\mathbb{P}^{m+1} \times \mathbb{P}^m, \ p_1^*\mathcal{O}(1) \otimes p_2^*\mathcal{O}(1)\big)$, with $d = 2m + 1$,

(c) $\left(\mathbb{P}_{\mathbb{P}^{m+1}}\left(\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus m}\right) , \mathcal{O}_{\mathbb{P}}(1)\right)$, with $d = 2m + 1$,

(d) $\left(\mathbb{P}^m \times Q^{m+1} , p_1^* \mathcal{O}(1) \otimes p_2^* \mathcal{O}(1)\right)$, with $d = 2m + 1$

(e) $\left(\mathbb{P}_{\mathbb{P}^{m+1}}(T_{\mathbb{P}^{m+1}}) , \mathcal{O}_{\mathbb{P}}(1)\right)$, with $d = 2m + 1$.

(f) $\left(\mathbb{P}^d, \mathcal{O}(2)\right)$, or

(g) $\left(\mathbb{P}^1, \mathcal{O}(3)\right)$.

(2) Suppose $b_4(X) = 1$. Then X is 2-Fano if and only if $-2K_{H_x} - dL_x$ is ample.

3. Background on homogeneous spaces

In order to investigate the conditions $\mathfrak{F}_2$ and $\mathfrak{F}_3$ on rational homogeneous spaces, we start with a brief recollection of definitions and properties from the theory of homogeneous varieties. We refer to [Bou68, FH91, Hum75, Mil13, Spr98] for the basics on algebraic groups and Lie algebras.

An algebraic variety is called *homogeneous* if there is an algebraic group acting transitively on it. A classical theorem by Borel-Remmert (see [Ak95, page 101]) asserts that if X is a projective homogeneous variety, then it is isomorphic to a product

$$X \cong \mathrm{Alb}(X) \times X',$$

where $\mathrm{Alb}(X)$ is the Albanese variety of X and X' is a projective rational homogeneous space. The latter can in turn be written as a product

$$X' \cong G_1/P_1 \times \cdots \times G_l/P_l$$

for some simple algebraic groups G_i and parabolic subgroups P_i. Next we explain how to encode the properties of G_i/P_i in a marked Dynkin diagram.

3.1. Semisimple algebraic groups and Dynkin diagrams.

Given a semisimple algebraic group G, we can choose a Borel subgroup $B \subset G$ (a maximal closed and connected solvable algebraic subgroup) and a maximal torus $T \subset B \subset G$. For example, when $G = \mathrm{SL}_n$ we can choose B to be the subgroup of lower-triangular matrices and $T \cong \mathbb{G}_m^{n-1}$ to be the group of all diagonal matrices. Define the *character lattice*

$$X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m) \cong \mathbb{Z}^l.$$

Let $\mathfrak{g}$ and $\mathfrak{t}$ denote the Lie algebras of the groups G and T, respectively. Base changing $X^*(T)$ to the complex numbers yields a complex vector space that is naturally isomorphic to the dual $\mathfrak{t}^\vee$, so we can consider the character lattice as a subset

$$X^*(T) \subset \mathfrak{t}^\vee.$$

By considering $\mathfrak{g}$ as a T-representation under the adjoint action, we can write its weight decomposition. Moreover, factoring out the subrepresentation $\mathfrak{t}$ whose weights are zero, we define a finite subset $\Phi \subset X^*(T) \setminus \{0\}$ of *roots*, so that we have the following form of the weight decomposition:

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$

For each α, the weight space $\mathfrak{g}_\alpha$ is 1-dimensional. Moreover, the set Φ forms a so-called *root system*. Taking a generic hyperplane

$$H \subset X^*(T)_{\mathbb{R}},$$

we can divide $X^*(T)_{\mathbb{R}}$ into two half-spaces: positive and negative. The roots in the positive half-space will be called *positive roots*, the set of which is denoted by Φ^+. Similarly we define the set of *negative roots* Φ^-. Inside Φ^+, we can choose a set of *simple roots*
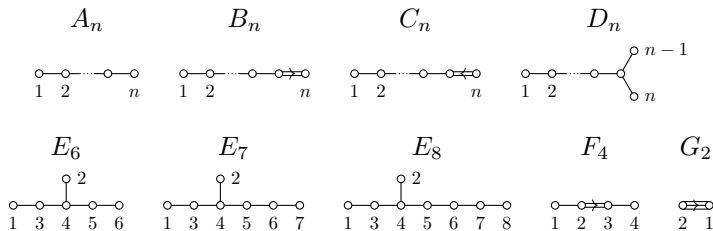
$$\{\alpha_1, \dots, \alpha_l\} \subset \Phi^+$$

that satisfy the following conditions: first, they form an $\mathbb{R}$-basis for $X^*(T)_{\mathbb{R}}$; and second, none of them is a nonnegative linear combination of other positive roots. It turns out that these conditions, together with the axioms of root systems, imply that the possible angles between the α_i (with respect to a scalar product invariant under the normalizer of the fixed maximal torus) are 90°, 120°, 135°, 150°. If we mark points on a plane labeled by the simple roots, we can connect them to each other with a number of lines that depends on the angle. For the angles listed above, we draw 0, 1, 2 and 3 lines, respectively. Moreover, for double and triple bonds we draw an arrow from the node marking the longer simple root to the shorter simple root. Nodes connected with just one line correspond to simple roots of the same length.

This procedure results in a *Dynkin diagram* corresponding to the root system. For example, for $G = \mathrm{SL}_3$, the torus is of rank two, the root system has six vectors pointing at vertices of a regular hexagon, and the two simple roots have angle 120°, so the corresponding Dynkin diagram is A_2:

$$\mathrm{SL}_3 \quad \circ\!\!-\!\!\circ \quad A_2.$$

We recall all possible connected Dynkin diagrams one can obtain; our convention is to use the ordering of roots as in Bourbaki [Bou68].

3.2. **Parabolic subgroups.** Let $B \subset G$ be a fixed Borel subgroup as above; we assume that its Lie algebra has nonpositive weights. A *parabolic subgroup* P of G is a connected subgroup that contains some Borel subgroup; up to conjugation, we may assume that P contains B. For example, B and G are trivially parabolic subgroups. To a fixed set of nodes in the Dynkin diagram, we can assign a parabolic subgroup of G as follows. Choose a subset $I \subset \{1, \dots, l\}$ of labels. Let Φ_I^+ denote

the set of positive roots generated (under taking nonnegative linear combinations) by

$$\{\alpha_i \mid i \in I\}.$$

Consider the following subspace $\mathfrak{p}_I \subset \mathfrak{g}$, which one can verify forms a Lie subalgebra:

$$\mathfrak{p}_I = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi^-} \mathfrak{g}_\alpha \bigoplus_{\alpha \in \Phi_I^+} \mathfrak{g}_\alpha.$$

Then the parabolic group P_I corresponding to I will be the subgroup of G that contains B and whose tangent space at the identity element is $\mathfrak{p}_I \subset \mathfrak{g}$.

The assignment of P_I to each I classifies parabolic subgroups of G up to conjugation, i.e., G-equivalence classes of parabolic subgroups of G are in bijection with subsets of nodes in the Dynkin diagram. If the subset I is empty, then $P_I = B$; if $I = \{1, \ldots, l\}$, then $P_I = G$. If the complement $\{1, \ldots, l\} \setminus I = \{k\}$ is of cardinality one, we write $P_I = P^k$ and this is a *maximal parabolic* subgroup, i.e. maximal among those not equal to the whole G.

3.3. Rational homogeneous spaces.

A *projective rational homogeneous space* is a quotient of a semisimple algebraic group by a parabolic subgroup. The quotient is a rational projective variety, and its geometry is heavily controlled by combinatorics. For instance, by Section 3.2 a rational homogeneous space G/P corresponds to the datum of a marked Dynkin diagram (possibly not connected).

The spaces G/P_I are Fano manifolds of dimension

$$\dim G/P_I = \dim G - \dim P_I = |\Phi^+ \setminus \Phi_I^+|,$$

computed by subtracting dimensions of tangent spaces at the identity element, and of Picard rank $l - |I|$ (see for example [Pasquier, Prop. 1.20], [Snow, Prop. 6.5, Thm. 9.5]). In particular, when G is simple, the quotients by maximal parabolic subgroups $X = G/P^k$ are Fano varieties of Picard rank one. In this case, the ample generator $\mathcal{O}_X(1)$ of the Picard group gives the smallest G-equivariant embedding of X in a projective space.

The *Weyl group* $W = N_G(T)/Z_G(T)$ is defined as the quotient of the normalizer of the fixed maximal torus $T \subset G$ by its centralizer. It is a finite group that acts faithfully on $X^*(T)$ and permutes the roots Φ, see [Spr98, §7.1.4]. As in Section 3.1, fix an $N_G(T)$-invariant scalar product $(\,,\,)$ on $X^*(T)_{\mathbb{R}}$ – it will naturally be Weyl group invariant for the induced action of W. For a root $\alpha \in \Phi$, one can define a reflection:

$$s_\alpha(x) = x - 2 \cdot \frac{(x, \alpha)}{(\alpha, \alpha)} \cdot \alpha.$$

The Weyl group W is generated by s_i – reflections with respect to simple roots α_i. The length $l(w)$ of an element $w \in W$ is the smallest integer $r \geq 0$ such that w is a product of r simple reflections s_i; a reduced decomposition of w is a sequence $(s_{i_1}, \ldots, s_{i_r})$ such that $w = s_{i_1} \cdot \ldots \cdot s_{i_r}$ and with $r = l(w)$.

Let $I \subset \{1, \ldots, l\}$ and let $P = P_I$. We denote by W_P the subgroup of W generated by reflections s_i, for all the simple roots α_i with $i \in I$. In each coset of

W/W_P, there exists a unique representative w of minimal length [BGG73, Proposition 5.1(iii)]. We denote by $W^P \subseteq W$ the set of such representatives. We can identify the elements of W^P as the elements $w \in W$ for which any reduced decomposition $(s_{i_1}, \ldots, s_{i_r})$ of w has $i_r \notin I$ (see [BGG73, Proposition 5.1(iii)] and [Hum72, p. 50, last Corollary]). In particular, there is a bijection between W^P and the quotient group W/W_P.

The Chow group $A^*(G/P)$ is generated by the classes of *Schubert subvarieties* $X(w)$, for all $w \in W^P$; the classes of Schubert subvarieties form an additive basis for $A^*(G/P) \cong H^*(G/P; \mathbb{Z})$ (see for example [CMP08, Section 2.2], [Snow, Section 3]). For all $j \geq 0$, the Betti number $b_{2j}(G/P)$ equals the number of elements in W^P of length j since $\dim X(w) = 2l(w)$, see [BGG73, Proposition 5.1].

As an application of the above discussion, we get the following lemma:

Lemma 3.1. *When G is simple and $P = P^k$, we have $b_2(G/P^k) = 1$ and $b_4(G/P^k)$ equals the number of simple roots adjacent to α_k.*

Proof. Recall that $P^k = P_I$ for $I = \{1, \ldots, l\} \setminus \{k\}$. From the above description of W^P, the only length one element $w \in W^P$ is $w = s_k$, hence $b_2(G/P^k) = 1$. For $b_4(G/P^k)$, length two elements $w \in W^P$ are of the form $w = s_i \cdot s_k$ with $i \neq k$. Using the fact that $s_\alpha \cdot s_\beta = s_\beta \cdot s_\alpha$ for non-adjacent (hence orthogonal) simple roots α and β, we can see that for non-adjacent i and k the reduced decomposition $w = s_i \cdot s_k$ cannot be of minimal length in $[w] \in W/W_P$, because $[w] = [s_k \cdot s_i] = [s_k]$. So $b_4(G/P^k)$ is the number of simple roots adjacent to α_k. $\square$

For G simple and a parabolic subgroup $P = P^k$ corresponding to a non-short root α_k, i.e., a root such that no arrows in the diagram point in the direction of the corresponding node, Landsberg and Manivel describe the minimal family of rational curves H_x through a point of $X = G/P^k$:

Theorem 3.2 ([LM03, Theorem 4.8], see also [LM04, Theorem 2.5] and the subsequent paragraph). *Let $X = G/P^k$ be a rational homogeneous variety such that α_k is not short. Then H_x is homogeneous and the associated marked diagram is determined as follows: remove the node corresponding to k and mark the nodes adjacent to k. Moreover, the embedding of H_x in $\mathbb{P}(T_x X)$ is minimal if and only if the Dynkin diagram of G is simply laced, i.e., without multiple edges.*

$$X = E_8/P^6 \qquad H_x = \mathrm{Seg}(\mathrm{OG}_+(5,10) \times \mathbb{P}^2)$$

Example 3.3.

In Sections 4 and 5, we study rational homogeneous spaces of Picard rank one. We do a type by type analysis, which yields a proof of our main results Theorem 1.3 and Theorem 1.4.

4. Rational homogeneous varieties of classical type

In this section we consider rational homogeneous spaces constructed from Dynkin diagrams of type A, B, C and D as quotients by a maximal parabolic subgroup P^k. These give rise to the following varieties, see [Man20, §1.1] for a reference:

A_n			$A_n/P^k = \mathrm{Gr}(k, n+1)$
B_n		for $k < n$:	$B_n/P^k = \mathrm{OG}(k, 2n+1)$
		for $k = n$:	$B_n/P^n = \mathrm{OG}_+(n+1, 2(n+1))$
C_n			$C_n/P^k = \mathrm{SG}(k, 2n)$
D_n		for $k < n-1$:	$D_n/P^k = \mathrm{OG}(k, 2n)$
		for $k \in \{n-1, n\}$:	$D_n/P^{n-1} \cong D_n/P^n = \mathrm{OG}_+(n, 2n)$

We explain the notation in the table. We denote by $\mathrm{Gr}(k, n)$ the Grassmannian of k-dimensional linear subspaces of an n-dimensional vector space V. When $n \neq 2k$, $\mathrm{OG}(k, n)$ is the subvariety of the Grassmannian $\mathrm{Gr}(k, n)$ parametrizing linear subspaces of V that are isotropic with respect to a non-degenerate symmetric bilinear form Q. If $n = 2k$, $\mathrm{OG}(k, n)$ has two isomorphic disjoint components $\mathrm{OG}_\pm(k, 2k)$ such that $\mathrm{OG}_+(k, 2k) \cong \mathrm{OG}_-(k, 2k) \cong \mathrm{OG}(k-1, 2k-1)$. Finally, observe that for $n = 2k + 1$ we have $\mathrm{OG}(k, 2k+1) \cong \mathrm{OG}_+(k+1, 2(k+1))$. We denote by $\mathrm{SG}(k, n)$ the subvariety of the Grassmannian $\mathrm{Gr}(k, n)$ parameterizing linear subspaces of V that are isotropic with respect to a non-degenerate antisymmetric bilinear form ω. Whenever we refer to $\mathrm{SG}(k, n)$ we will assume n is even. In all cases except the case of the spinor variety $\mathrm{OG}_+(k, 2k)$ (i.e., B_{k-1}/P^{k-1}, D_k/P^{k-1}, D_k/P^k), the ample generator $\mathcal{O}_X(1)$ of the Picard group of $X = G/P$ is the restriction of the corresponding Plücker embedding polarization.

Remark 4.1. The integral cohomology of $\mathrm{Gr}(k, n)$ admits a basis of Schubert cycles indexed by Young diagrams that fit into a rectangle $k \times (n - k)$. Given a diagram λ, the codimension of the cycle σ_λ corresponding to λ is equal to the number of boxes. The intersection of two Schubert cycles is calculated by the Littlewood-Richardson rule, and for some special cycles this is also known as the Pieri rule. For further facts and proofs, we refer to Fulton's classical textbook [Ful97, §9.4].

We recall in a table, results from [AC12, §5] and [AC13, §6.2] on the condition $\mathfrak{F}_2$ for homogeneous spaces of classical type. Note that we have $\mathrm{Gr}(1, n) \cong \mathbb{P}^{n-1}$, $\mathrm{SG}(1, n) \cong \mathbb{P}^{n-1}$ (n even) and $\mathrm{OG}(1, n) = Q^{n-2}$.

		dim	$\mathfrak{F}_2$ is satisfied $\iff$
for $2 \le k \le \frac{n}{2}$	$\mathrm{Gr}(k,n)$	$k(n-k)$	$n \in \{2k, 2k+1\}$
for $2 \le k < \frac{n}{2} - 1$	$\mathrm{OG}(k,n)$	$\frac{k(2n-3k-1)}{2}$	$n = 3k+2$
	$\mathrm{OG}_+(k,2k)$	$\frac{k(k-1)}{2}$	$\forall k$
for $2 \le k \le \frac{n}{2}$	$\mathrm{SG}(k,n)$	$\frac{k(2n-3k+1)}{2}$	$n = 3k-2$
	$\mathrm{SG}(k,2k)$	$\frac{k(k+1)}{2}$	$\forall k$

We continue the previous table, collecting the results on (H_x, L_x) from [AC12, §5] and Theorem 3.2:

		(H_x, L_x)
for $2 \le k \le \frac{n}{2}$	$\mathrm{Gr}(k,n)$	$\left(\mathbb{P}^{k-1} \times \mathbb{P}^{n-k-1}, p_1^* \mathcal{O}(1) \otimes p_2^* \mathcal{O}(1)\right)$
for $2 \le k < \frac{n}{2} - 1$	$\mathrm{OG}(k,n)$	$\left(\mathbb{P}^{k-1} \times Q^{n-2k-2}, \; p_1^* \mathcal{O}(1) \otimes p_2^* \mathcal{O}(1)\right)$
	$\mathrm{OG}_+(k,2k)$	$(\mathrm{Gr}(2,k), H)$
for $2 \le k \le \frac{n}{2}$	$\mathrm{SG}(k,n)$	$\left(\mathbb{P}_{\mathbb{P}^{k-1}}\left(\mathcal{O}(2) \oplus \mathcal{O}(1)^{n-2k}\right), \; \mathcal{O}_{\mathbb{P}}(1)\right)$
	$\mathrm{SG}(k,2k)$	$(\mathbb{P}^{k-1}, \mathcal{O}(2))$

where H is a hyperplane class under the Plücker embedding.

4.1. **Type A: Grassmannians.**

Proposition 4.2. *For $2 \le k \le n-2$, $\mathrm{Gr}(k,n)$ does not satisfy the condition $\mathfrak{F}_3$.*

Proof. Let $X = \mathrm{Gr}(k,n)$. As $H_x \cong \mathbb{P}^{k-1} \times \mathbb{P}^{n-k-1}$ is a product, by [dJS06a, §3.3] it does not satisfy $\mathfrak{F}_2$. By Theorem 2.1 it follows that X does not satisfy $\mathfrak{F}_3$. $\square$

4.2. **Types B and D: orthogonal Grassmannians.**

Proposition 4.3. *For $2 \le k < \frac{n}{2} - 1$, $\mathrm{OG}(k,n)$ does not satisfy the condition $\mathfrak{F}_3$.*

Proof. Let $X = \mathrm{OG}(k,n)$. As H_x is a product, by [dJS06a, §3.3] it does not satisfy $\mathfrak{F}_2$. By Theorem 2.1 it follows that X does not satisfy $\mathfrak{F}_3$. $\square$

Now we consider $\mathrm{OG}_+(k,2k)$, whose family of minimal rational curves through a general point is $H_x \cong \mathrm{Gr}(2,k)$, by Theorem 3.2. We have that $\mathrm{Gr}(2,k)$ satisfies $\mathfrak{F}_2$ if and only if $4 \le k \le 5$ by [AC12, §5.2]. Thus, by Theorem 2.1 the condition $k \in \{4,5\}$ is necessary for $\mathrm{OG}_+(k,2k)$ to satisfy the condition $\mathfrak{F}_3$. However, we show that $\mathrm{OG}_+(k,2k)$ does not satisfy $\mathfrak{F}_3$ by considering a 3-cycle whose intersection with $\mathrm{ch}_3(\mathrm{OG}_+(k,2k))$ is non-positive. To this end, we start by recalling a result by Coskun about restriction of the Schubert cycles from $\mathrm{Gr}(k,n)$ to $\mathrm{OG}_+(k,n)$.

Proposition 4.4 ([Cos11, Proposition 6.2]). *Let $j : \mathrm{OG}(k,n) \hookrightarrow \mathrm{Gr}(k,n)$ be the natural inclusion, and $\sigma_{\lambda_1,\dots,\lambda_k}$ a Schubert cycle in $\mathrm{Gr}(k,n)$. Then*

(1) *$j^* \sigma_{\lambda_1,\dots,\lambda_k} = 0$ unless $n - k - i \ge \lambda_i$ for all i with $1 \le i \le k$.*
(2) *Suppose that $n - k - i > \lambda_i$ for all $1 \le i \le k$. Then $j^* \sigma_{\lambda_1,\dots,\lambda_k}$ is effective and nonzero.*
(3) *Suppose that $n = 2k$ and $k - i = \lambda_i$ for all $1 \le i \le k$. Then $j^* \sigma_{\lambda_1,\dots,\lambda_k}$ is 2^{k-1} times the Poincaré dual of a point.*

We recall from [AC13, §6.2] the expression of the third Chern character in terms of restrictions of Schubert cycles from $\mathrm{Gr}(k, 2k)$:

$$\mathrm{ch}_3(\mathrm{OG}_+(k, 2k)) = -\frac{k+7}{6} j^* \sigma_3 + \frac{k+4}{6} j^* \sigma_{2,1} - \frac{k+1}{6} j^* \sigma_{1,1,1}.$$

Lemma 4.5. *For $X = \mathrm{OG}_+(k, 2k)$ with $k \geq 3$, we have $j^* \sigma_3 = j^* \sigma_{1,1,1}$ and*

$$\mathrm{ch}_3(X) = \frac{k+4}{2} \Big(\frac{1}{6} j^* \sigma_1^3 - j^* \sigma_3 \Big).$$

Proof. By [AC13, Claim 33] the equality $j^* \sigma_2 = j^* \sigma_{1,1} = \frac{1}{2} j^* \sigma_1^2$ holds on X. Applying Pieri's formula we have

$$\frac{1}{2} j^* \sigma_1^3 = j^* (\sigma_1 \cdot \sigma_2) \overset{\mathrm{Pieri}}{=} j^* \sigma_{2,1} + j^* \sigma_3$$

$$\frac{1}{2} j^* \sigma_1^3 = j^* (\sigma_1 \cdot \sigma_{1,1}) \overset{\mathrm{Pieri}}{=} j^* \sigma_{2,1} + j^* \sigma_{1,1,1}.$$

This implies that $j^* \sigma_3 = j^* \sigma_{1,1,1}$ and

$$\mathrm{ch}_3(X) = -\frac{k+7}{6} j^* \sigma_3 + \frac{k+4}{6} \Big(\frac{1}{2} j^* \sigma_1^3 - j^* \sigma_3 \Big) - \frac{k+1}{6} j^* \sigma_3 = \frac{k+4}{2} \Big(\frac{1}{6} j^* \sigma_1^3 - j^* \sigma_3 \Big).$$

$\square$

Proposition 4.6. *For $k = 4$ or 5, $\mathrm{OG}_+(k, 2k)$ does not satisfy the condition $\mathfrak{F}_3$.*

Proof. Consider the codimension 3 Schubert cycle $\sigma_{2,1}$ on $\mathrm{Gr}(4, 8)$, and respectively the codimension 7 Schubert cycle $\sigma_{3,2,1,1}$ on $\mathrm{Gr}(5, 10)$. We have

$$\frac{6}{8} \mathrm{ch}_3(\mathrm{OG}_+(4, 8)) \cdot j^* \sigma_{2,1} \overset{\mathrm{Lemma\ 4.5}}{=} j^* \Big(\frac{1}{2} \sigma_1^3 \cdot \sigma_{2,1} - 3\, \sigma_3 \cdot \sigma_{2,1} \Big)$$

$$\overset{\substack{\mathrm{Pieri\ +} \\ \mathrm{Proposition\ 4.4}}}{=} j^* \Big(\frac{1}{2} 6\, \sigma_{3,2,1} - 3\, \sigma_{3,2,1} \Big) = 0$$

$$\frac{6}{9} \mathrm{ch}_3(\mathrm{OG}_+(5, 10)) \cdot j^* \sigma_{3,2,1,1} \overset{\mathrm{Lemma\ 4.5}}{=} j^* \Big(\frac{1}{2} \sigma_1^3 \cdot \sigma_{3,2,1,1} - 3\, \sigma_3 \cdot \sigma_{3,2,1,1} \Big)$$

$$\overset{\substack{\mathrm{Pieri\ +} \\ \mathrm{Proposition\ 4.4}}}{=} j^* \Big(\frac{1}{2} 6\, \sigma_{4,3,2,1} - 3\, \sigma_{4,3,2,1} \Big) = 0.$$

By Proposition 4.4, $j^* \sigma_{2,1}$ and $j^* \sigma_{3,2,1,1}$ are nonzero effective 3-cycles which intersect non-positively with ch_3, giving that $\mathrm{OG}_+(k, 2k)$ does not satisfy $\mathfrak{F}_3$ for $k = 4, 5$. $\square$

4.3. Linear sections in orthogonal Grassmannians.

We consider a linear section X of $\mathrm{OG}_+(k, 2k)$ of codimension c, i.e.

$$X = \mathrm{OG}_+(k, 2k) \cap H_1 \cap \ldots \cap H_c,$$

where $H_i \sim \frac{\sigma_1}{2}$ is a hyperplane section of the half-spinor embedding of $\mathrm{OG}_+(k, 2k)$. By [AC13, Proposition 34], X satisfies $\mathfrak{F}_2$ if and only if $c < 4$. By Theorem 3.2 the family of minimal rational curves on $\mathrm{OG}_+(k, 2k)$ is $\mathrm{Gr}(2, k)$. Since these minimal

rational curves are lines under the half-spinor embedding, the family of minimal rational curves on X is

$$H_x = \mathrm{Gr}(2,k) \cap L^c$$

where L is a hyperplane section under the Plücker embedding.

Proposition 4.7. *For $k = 5$ and $c < 4$, $\mathrm{OG}_+(5,10) \cap H^c$ does not satisfy the condition $\mathfrak{F}_3$.*

Proof. This follows from Theorem 2.1 as $H_x = \mathrm{Gr}(2,5) \cap L^c$ does not satisfy $\mathfrak{F}_2$ by [AC13, Proposition 32 (iv)]. □

4.4. Type C: symplectic Grassmannian.

Proposition 4.8. *For $2 \leq k < \frac{n}{2}$, $\mathrm{SG}(k,n)$ does not satisfy the condition $\mathfrak{F}_3$.*

Proof. Let $X = \mathrm{SG}(k,n)$. By Theorem 2.1 X does not satisfy $\mathfrak{F}_3$ as $\rho(H_x) > 1$. □

Now we consider $\mathrm{SG}(k,2k)$ and show that in fact this also fails to satisfy the condition $\mathfrak{F}_3$. First, we consider the following result by Coskun.

Proposition 4.9 ([Cos13, Corollary 3.38]). *Let $i : \mathrm{SG}(k,2k) \hookrightarrow \mathrm{Gr}(k,2k)$ be the natural inclusion, and $\sigma_{\lambda_1,\ldots,\lambda_j}$ a Schubert cycle in $\mathrm{Gr}(k,2k)$. Then $i^*\sigma_{\lambda_1,\ldots,\lambda_j} = 0$ unless $k + 1 - j \geq \lambda_j$ for all j with $1 \leq j \leq k$.*

Corollary 4.10. $i^*\sigma_{k,k-1,\ldots,2,1} \neq 0$.

Proof. By Proposition 4.9, the condition for a Schubert cycle to be nonzero on $\mathrm{SG}(k,2k)$ is that $k \geq \lambda_1, k-1 \geq \lambda_2, \ldots, 2 \geq \lambda_{k-1}, 1 \geq \lambda_k$. Therefore the only codimension $\frac{k(k+1)}{2}$ Schubert cycle whose restriction could be nonzero is $\sigma_{k,k-1,\ldots,2,1}$. By non-degeneracy of the intersection pairing in cohomology on $\mathrm{Gr}(k,2k)$, we have necessarily that $i^*\sigma_{k,k-1,\ldots,2,1} \neq 0$. □

We recall from [AC13, §6.3] the expression of the third Chern character in terms of restrictions of Schubert cycles from $\mathrm{Gr}(k,2k)$:

$$\mathrm{ch}_3(\mathrm{SG}(k,2k)) = \frac{-k+1}{6}i^*\sigma_3 - \frac{-k+4}{6}i^*\sigma_{2,1} + \frac{-k+7}{6}i^*\sigma_{1,1,1}.$$

Lemma 4.11. *For $X = \mathrm{SG}(k,2k)$ with $k \geq 3$, we have $i^*\sigma_3 = i^*\sigma_{1,1,1}$ and*

$$\mathrm{ch}_3(X) = \frac{-k+4}{2}i^*\Big(\sigma_3 - \frac{1}{6}\sigma_1^3\Big).$$

Proof. By [AC13, Claim 35] the equality $i^*\sigma_2 = i^*\sigma_{1,1} = i^*\frac{1}{2}\sigma_1^2$ holds on X. Applying Pieri's formula we have

$$\frac{1}{2}i^*\sigma_1^3 = i^*(\sigma_1 \cdot \sigma_2) \overset{\mathrm{Pieri}}{=} i^*\sigma_{2,1} + i^*\sigma_3$$

$$\frac{1}{2}i^*\sigma_1^3 = i^*(\sigma_1 \cdot \sigma_{1,1}) \overset{\mathrm{Pieri}}{=} i^*\sigma_{2,1} + i^*\sigma_{1,1,1}.$$

This implies that $i^*\sigma_3 = i^*\sigma_{1,1,1}$ and

$$\mathrm{ch}_3(X) = \frac{-k+1}{6}i^*\sigma_3 - \frac{-k+4}{6}i^*\Big(\frac{1}{2}\sigma_1^3 - \sigma_3\Big) + \frac{-k+7}{6}i^*\sigma_3 = \frac{-k+4}{2}i^*\Big(\sigma_3 - \frac{1}{6}\sigma_1^3\Big).$$

□

Proposition 4.12. $\mathrm{SG}(k, 2k)$ *does not satisfy the condition* $\mathfrak{F}_3$.

Proof. We consider the cycle $\rho := \sigma_{k-1,k-2,k-3,k-3,k-4,\ldots,2,1}$ on $\mathrm{Gr}(k, 2k)$: it has codimension $\frac{k(k+1)}{2} - 3$ and its restriction $i^*\rho$ to $\mathrm{SG}(k, 2k)$ is effective as it is a non-negative linear combination of effective cycles on $\mathrm{SG}(k, 2k)$ by [Pra00, §1]. By the Pieri rule and Proposition 4.9 we have

$$i^*(\sigma_3 \cdot \rho) = i^*\sigma_{k,k-1,\ldots,2,1}, \qquad i^*(\sigma_1^3 \cdot \rho) = 6 \cdot i^*\sigma_{k,k-1,\ldots,2,1}.$$
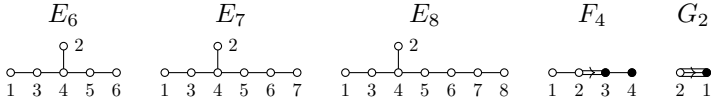
It follows that $i^*\rho \neq 0$ as $i^*\sigma_{k,k-1,\ldots,2,1} \neq 0$ by Corollary 4.10, and

$$\mathrm{ch}_3(\mathrm{SG}(k, 2k)) \cdot i^*\rho \overset{\text{Lemma 4.11}}{=} \frac{-k+4}{2}\left(\sigma_3 - \frac{1}{6}\sigma_1^3\right) \cdot i^*\rho$$
$$= \frac{-k+4}{2} i^*\left(\sigma_{k,k-1,\ldots,2,1} - \frac{1}{6}6 \cdot \sigma_{k,k-1,\ldots,2,1}\right) = 0.$$

Therefore $\mathrm{SG}(k, 2k)$ does not satisfy $\mathfrak{F}_3$. $\qquad\square$

5. Rational homogeneous varieties of exceptional type

In this section, we show that none of the exceptional groups, when quotiented by a maximal parabolic subgroup, satisfies the condition $\mathfrak{F}_3$, and we study when the condition $\mathfrak{F}_2$ is satisfied. We recall the Dynkin diagrams of exceptional type and mark in black the short roots, i.e. the roots such that there is an arrow in the diagram pointing in its direction.

Proposition 5.1. E_n/P^α *with* $\alpha \neq 1, 2, n$ *and* F_4/P^2 *do not satisfy* $\mathfrak{F}_2$.

Proof. Let X be one of the homogeneous spaces in the statement. By Theorem 3.2, H_x is a product and thus has Picard rank > 1. To show that X does not satisfy $\mathfrak{F}_2$, we check that the polarized variety (H_x, L_x) is not isomorphic to any of the exceptional pairs $(a) - (e)$ from the list in Theorem 2.3:

- For $X = E_n/P^\alpha$ and $\alpha = 3, 5$, we have $H_x \cong \mathrm{Gr}(2, k) \times \mathbb{P}^l$, for some $k = 5, 6, 7$ and $l = 1, 2, 3$.
- For $X = E_n/P^4$, we have $H_x \cong \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^{n-4}$.
- For $X = E_n/P^6$, $n = 7, 8$, we have $H_x \cong D_5/P^5 \times \mathbb{P}^{n-6} \cong \mathrm{OG}_+(5, 10) \times \mathbb{P}^{n-6}$.
- For $X = E_8/P^7$, we have $H_x \cong E_6/P^6 \times \mathbb{P}^1$.
- For $X = F_4/P^2$, we have $H_x \cong \mathbb{P}^1 \times \mathbb{P}^2$. By Theorem 3.2, the embedding of H_x in $\mathbb{P}(T_xX)$ is not minimal, and thus $L_x \neq \mathcal{O}(1, 1)$.

Thus, X does not satisfy $\mathfrak{F}_2$. $\qquad\square$

5.1. Type E.

5.1.1. *Parabolic groups P^1, P^2.*

Proposition 5.2. E_n/P^1 *satisfies the condition $\mathfrak{F}_2$ but not $\mathfrak{F}_3$ for $n = 6, 7, 8$.*

Proof. Let $X = E_n/P^1$. By Theorem 3.2 we have $H_x = D_{n-1}/P^{n-1} = \mathrm{OG}_+(n - 1, 2(n-1))$ and L_x is a generator of $\mathrm{Pic}(H_x)$. Write $k = n - 1$. We have $d = \dim(H_x) = \frac{k(k-1)}{2}$ and $\frac{\sigma_1}{2} \sim L_x$, as $\mathrm{Pic}(H_x) = \mathbb{Z}[\frac{\sigma_1}{2}]$, where we denote by the same symbol the Schubert cycle σ_1 in $\mathrm{Gr}(k, 2k)$ and its restriction to $\mathrm{OG}_+(k, 2k)$.

As $b_4(X) = 1$ by Lemma 3.1, we apply Theorem 2.3 and conclude that X satisfies $\mathfrak{F}_2$ as

$$-2K_{H_x} - dL_x = 2 \cdot 2(k-1)L_x - \frac{k(k-1)}{2}L_x = \frac{(k-1)(8-k)}{2}L_x$$

is ample. We compute

$$T(\mathrm{ch}_3(X)) \overset{(2.3)}{=} \mathrm{ch}_2(H_x) - \frac{1}{2}\Big(c_1(H_x) - \frac{d}{2}c_1(L_x)\Big)L_x - \frac{d-4}{12}L_x^2$$

$$= 2L_x^2 - \frac{1}{2}\Big(2(k-1) - \frac{k(k-1)}{4}\Big)L_x^2 - \frac{k(k-1)-8}{24}L_x^2$$

$$= \frac{(k-5)(k-8)}{12}L_x^2.$$

which implies that $\mathrm{ch}_3(X)$ is not positive, so X does not satisfy $\mathfrak{F}_3$. $\qquad\square$

Remark 5.3. As $E_6/P^6 \cong E_6/P^1$, Proposition 5.2 holds for E_6/P^6 as well.

Proposition 5.4. E_n/P^2 *satisfies the condition $\mathfrak{F}_2$ but not $\mathfrak{F}_3$ for $n = 6, 7, 8$.*

Proof. Let $X = E_n/P^2$. By Theorem 3.2 we have $H_x = A_{n-1}/P^3 = \mathrm{Gr}(3, n)$ and and L_x is a generator of $\mathrm{Pic}(H_x)$, therefore $d = \dim(H_x) = 3(n - 3)$ and $\sigma_1 \sim L_x$. As $b_4(X) = 1$ by Lemma 3.1, we apply Theorem 2.3 and conclude that X satisfies $\mathfrak{F}_2$ as $-2K_{H_x} - dL_x = 2nL_x - 3(n-3)L_x = (9-n)L_x$ is ample.

For $n = 8$, H_x does not satisfy $\mathfrak{F}_2$, hence X does not satisfy $\mathfrak{F}_3$ by Theorem 2.1. From now on we assume that $n = 6, 7$. We have

$$T(\mathrm{ch}_3(X)) \overset{(2.3)}{=} \mathrm{ch}_2(H_x) - \frac{1}{2}\Big(c_1(H_x) - \frac{d}{2}c_1(L_x)\Big)L_x - \frac{d-4}{12}L_x^2$$

$$= \frac{n-4}{2}\sigma_2 - \frac{n-8}{2}\sigma_{1,1} - \frac{1}{2}\Big(n - \frac{3(n-3)}{2}\Big)\sigma_1^2 - \frac{3n-13}{12}\sigma_1^2$$

$$\overset{\mathrm{Pieri}}{=} \Big(\frac{n-4}{2} - \frac{n}{2} + \frac{3(n-3)}{4} - \frac{3n-13}{12}\Big)\sigma_1^2 - \Big(\frac{n-8}{2} + \frac{n-4}{2}\Big)\sigma_{1,1}$$

$$= \frac{-19+3n}{6}\sigma_1^2 - (n-6)\sigma_{1,1}.$$

If $n = 6$ we have $T(\mathrm{ch}_3(X)) < 0$, and if $n = 7$ we have $T(\mathrm{ch}_3(X)) \cdot \sigma_{4,3,3} = -\frac{2}{3} < 0$. In both cases, this implies that $\mathrm{ch}_3(X)$ is not positive, hence X does not satisfy $\mathfrak{F}_3$. $\qquad\square$

5.1.2. *Freudenthal variety.* The homogeneous variety E_7/P^7 is also known as the Freudenthal variety $G(\mathbb{O}^3, \mathbb{O}^6)$; it has dimension 27 and index 18. We refer for instance to [CMP08, §2.1, §2.3] for more details on the geometry of E_7/P^7.

Proposition 5.5. E_7/P^7 *satisfies the condition* $\mathfrak{F}_2$ *but not* $\mathfrak{F}_3$.

Proof. Let $X = E_7/P^7$. By Theorem 3.2 we have $H_x = E_6/P^6$ and L_x is a generator of $\mathrm{Pic}(H_x)$, therefore $d = \dim(H_x) = 16$ and $L_x \sim H$, as the hyperplane section H in H_x is a generator of $\mathrm{Pic}(H_x)$ by [IM05, Proposition 5.1]. By Lemma 3.1 $b_4(X) = 1$, hence we apply Theorem 2.3 and obtain that X satisfies $\mathfrak{F}_2$ as

$$2T(\mathrm{ch}_2(H_x)) = -2K_{H_x} - dL_x = 24H - 16H = 8H$$

is ample. We apply Eq. (2.1) to compute:

$$T(\mathrm{ch}_3(X)) \overset{(2.3)}{=} \mathrm{ch}_2(H_x) - \frac{1}{2}\Big(c_1(H_x) - \frac{d}{2}c_1(L_x)\Big) \cdot L_x - \frac{d-4}{12}L_x^2$$

$$\overset{\text{Lemma } 5.6}{=} 3H^2 - \frac{1}{2}4H^2 - H^2 = 0.$$

We conclude that X does not satisfy $\mathfrak{F}_3$. $\qquad\square$

Lemma 5.6. $\mathrm{ch}_2(E_6/P^6) = 3H^2$ *and* $\mathrm{ch}_3(E_6/P^6) = 0$.

Proof. Let $X = E_6/P^1 = E_6/P^6$; as described for instance in [IM05], X admits an embedding in $\mathbb{P}^{26}$. Denote by $\mathcal{N}$ the normal bundle of X in $\mathbb{P}V \cong \mathbb{P}^{26}$; by [IM05, Proposition 7.1] we have $c_1(\mathcal{N}) = 15H$, $c_2(\mathcal{N}) = 102H^2$, and $c_3(\mathcal{N}) = 414H^3$, where H denotes a hyperplane section. Then we obtain

$$\mathrm{ch}_2(X) = \mathrm{ch}_2(\mathbb{P}^{26})_{|X} - \mathrm{ch}_2(\mathcal{N}) = 3H^2,$$

$$\mathrm{ch}_3(X) = \mathrm{ch}_3(\mathbb{P}^{26})_{|X} - \mathrm{ch}_3(\mathcal{N}) = 0.$$

$\qquad\square$

Proposition 5.7. E_8/P^8 *satisfies the condition* $\mathfrak{F}_2$ *but not* $\mathfrak{F}_3$.

Proof. Let $X = E_8/P^8$. By Theorem 3.2 we have $H_1 \coloneqq H_x = E_7/P^7$ the Freundenthal variety, whose corresponding minimal family of rational curves through a general point is $H_2 = E_6/P^6$. Thus, we have $d = \dim(H_1) = 27$, $c_1(H_1) = 18L_1$ and by [CMP08, §2.3] $\mathrm{Pic}(H_i) = \mathbb{Z}[h_i]$ where h_i denote a hyperplane class on H_i; by Theorem 3.2 $h_i \sim L_i$.

We have $b_4(X) = 1$ by Lemma 3.1, hence we apply Theorem 2.3 and obtain that X satisfies $\mathfrak{F}_2$ as $-2K_{H_1} - dL_1 = 36L_1 - 27L_1 = 9L_1$ is ample. We compute

$$T(\mathrm{ch}_3(X)) \overset{(2.3)}{=} \mathrm{ch}_2(H_1) - \frac{1}{2}\Big(c_1(H_1) - \frac{d}{2}c_1(L_1)\Big)L_1 - \frac{d-4}{12}L_1^2$$

$$= \mathrm{ch}_2(H_1) - \frac{9}{4}L_1^2 - \frac{23}{12}L_1^2 = \mathrm{ch}_2(H_1) - \frac{25}{6}L_1^2$$

$$T \circ T(\mathrm{ch}_3(X)) = T(\mathrm{ch}_2(H_1)) - \frac{25}{6}T(L_1^2) \overset{\text{Prop. } (5.5)}{=} 4L_2 - \frac{25}{6}T(L_1^2)$$

$$\leq 4L_2 - \frac{25}{6}L_2 = -\frac{1}{6}L_2$$

where the last inequality holds by [AC12, Lemma 2.7 (1)]. We conclude that X does not satisfy $\mathfrak{F}_3$. □

5.2. Type F.

Proposition 5.8. F_4/P^1 *does not satisfy the condition* $\mathfrak{F}_2$.

Proof. Let $X = F_4/P^1$. By Theorem 3.2 we have $H_x = C_3/P^3 = \mathrm{SG}(3,6)$, and the embedding of H_x in $\mathbb{P}(T_xX)$ is not minimal. Therefore $\mathrm{Pic}(H_x)$ is not generated by $[L_x]$. Since this pair (H_x, L_x) is not in the exceptional list of Theorem 2.3, we conclude that X does not satisfy $\mathfrak{F}_2$. □

Proposition 5.9. F_4/P^3 *does not satisfy the condition* $\mathfrak{F}_2$.

Proof. Let $X = F_4/P^3$. By [LM03, Proposition 6.9] H_x is a nontrivial Q^4-bundle over $\mathbb{P}^1$, in particular $\rho(H_x) \neq 1$. Note that none of the exceptional pairs (H_x, L_x) in Theorem 2.3 admit a nontrivial Q^4-fibration. We conclude that X does not satisfy $\mathfrak{F}_2$. □

Proposition 5.10. F_4/P^4 *satisfies the condition* $\mathfrak{F}_2$ *but not* $\mathfrak{F}_3$.

Proof. By [LM03, proof of Proposition 6.5] F_4/P^4 is the generic hyperplane section of E_6/P^6. We write $X = E_6/P^6 \cap H$, $Y = E_6/P^6$ and recall that $\mathrm{Pic}(X) = \mathbb{Z}[H]$. From Lemma 5.6 and (6.1), we obtain

$$\mathrm{ch}_2(X) = \left(\mathrm{ch}_2(Y) - \frac{1}{2}H^2 \right)_{|Y} = \left(3H^2 - \frac{1}{2}H^2 \right)_{|Y} = \frac{5}{2}H^2_{|x}$$

$$\mathrm{ch}_3(X) = \left(\mathrm{ch}_3(Y) - \frac{1}{6}H^3 \right)_{|Y} = -\frac{1}{6}H^3_{|X}.$$

This implies that $X = F_4/P^4$ satisfies $\mathfrak{F}_2$ but not $\mathfrak{F}_3$. □

5.3. Type G.

Proposition 5.11. G_2/P^1 *satisfies the condition* $\mathfrak{F}_2$ *but not* $\mathfrak{F}_3$.

Proof. By [LM03, §6.1], we have that $G_2/P^1 = Q^5 \subset \mathbb{P}^6$. Applying Theorem 6.1 and Eq. (6.1), we conclude that G_2/P^1 satisfies $\mathfrak{F}_2$ but not $\mathfrak{F}_3$. □

Proposition 5.12. G_2/P^2 *satisfies the condition* $\mathfrak{F}_2$ *but not* $\mathfrak{F}_3$.

Proof. Let $X = G_2/P^2$. Then X satisfies $\mathfrak{F}_2$ by Theorem 6.1. Moreover, we have

$$6 \cdot \mathrm{ch}_3(X) \cdot c_1^2 = (c_1^3 - 3\,c_1\,c_2 + 3\,c_3) \cdot c_1^2 = c_1^5 - 3\,c_1^3\,c_2 + 3\,c_3\,c_1^2$$
$$= 4374 - 3 \cdot 2106 + 3 \cdot 594 = -162$$

where the Chern numbers are computed in [KT20, Table 1]. As $\frac{c_1^2}{27}$ is an integral class by [KT20, §2], we obtain $\mathrm{ch}_3(X) \cdot \frac{c_1^2}{27} = -1$, so that X does not satisfy $\mathfrak{F}_3$. □

6. Higher Fano manifolds with high index

In [AC13] Araujo and Castravet classify 2-Fano manifolds of high index. We note that there are inaccuracies in the dimension bounds in the original statement of [AC13, Theorem 3]. We state their classification with the correct bounds.

Theorem 6.1 ([AC13, Theorem 3]). *Let X be a Fano manifold of dimension $n \geq 3$ and index $i_X \geq n - 2$. If X satisfies $\mathfrak{F}_2$, then X is isomorphic to one of the following.*

- $\mathbb{P}^n$.
- *Complete intersections in projective spaces:*
    - *Quadric hypersurfaces $Q^n \subset \mathbb{P}^{n+1}$ with $n > 2$;*
    - *Complete intersections of quadrics $X_{2 \cdot 2} \subset \mathbb{P}^{n+2}$ with $n > 5$;*
    - *Cubic hypersurfaces $X_3 \subset \mathbb{P}^{n+1}$ with $n > 7$;*
    - *Quartic hypersurfaces in $\mathbb{P}^{n+1}$ with $n > 14$;*
    - *Complete intersections $X_{2 \cdot 3} \subset \mathbb{P}^{n+2}$ with $n > 10$;*
    - *Complete intersections $X_{2 \cdot 2 \cdot 2} \subset \mathbb{P}^{n+3}$ with $n > 8$.*
- *Complete intersections in weighted projective spaces:*
    - *Degree 4 hypersurfaces in $\mathbb{P}(2, 1, \ldots, 1)$ with $n > 11$;*
    - *Degree 6 hypersurfaces in $\mathbb{P}(3, 2, 1, \ldots, 1)$ with $n > 23$;*
    - *Degree 6 hypersurfaces in $\mathbb{P}(3, 1, \ldots, 1)$ with $n > 26$;*
    - *Complete intersections of two quadrics in $\mathbb{P}(2, 1, \ldots, 1)$ with $n > 2$.*
- $\mathrm{Gr}(2, 5)$.
- $\mathrm{OG}_+(5, 10)$ *and its linear sections of codimension $c < 4$.*
- $\mathrm{SG}(3, 6)$.
- G_2/P^2.

We recall that by [AC13, Lemma 8], if Y is a smooth variety and X is a smooth complete intersection of divisors $D_1, \ldots, D_c$ in Y, then

$$\mathrm{ch}_k(X) = \left(\mathrm{ch}_k(Y) - \frac{1}{k!} \sum_{i=1}^{c} D_i^k \right)_{|X}. \tag{6.1}$$

In particular, let X be a smooth complete intersection of hypersurfaces of degrees $d_1, \ldots, d_c$ in $\mathbb{P}^n$ and denote by $h := c_1(\mathcal{O}_{\mathbb{P}^n}(1))$ the hyperplane class in $\mathbb{P}^n$. Then

$$\mathrm{ch}_k(X) = \frac{1}{k!} ((n+1) - \sum d_i^k) \, h_{|X}^k,$$

hence X satisfies $\mathfrak{F}_k$ if and only if $\sum d_i^k \leq n$.

Let $\mathbb{P}(a_0, \ldots, a_n)$ denote the weighted projective space with $\gcd(a_0, \ldots, a_n) = 1$, and H be the effective generator of its class group. Let X be a smooth complete intersection of hypersurfaces in $\mathbb{P}(a_0, \ldots, a_n)$ with classes $d_1 H, \ldots, d_c H$. Assume that X is smooth, and contained in the smooth locus of $\mathbb{P}(a_0, \ldots, a_n)$. Then

$$\mathrm{ch}_k(X) = \frac{a_0^k + \ldots + a_n^k - \sum d_i^k}{k!} c_1(H_{|X})^k;$$

it follows that X satisfies $\mathfrak{F}_k$ if and only if $\sum_{i=1}^{c} d_i^k < \sum_{j=0}^{n} a_j^k$.

Proof of Theorem 1.5. The list in the theorem is obtained starting from Theorem 6.1 and studying the positivity of the third Chern character case by case. For complete intersections we apply Eq. (6.1) and we determine sharp bounds on n. The remaining 2-Fano manifolds do not satisfy the condition $\mathfrak{F}_3$ by Propositions 4.2, 4.6, 4.7, 4.12 and 5.12. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

7. HIGHER PICARD RANK

Proposition 7.1. *Let X be a smooth uniruled projective variety and let Y be a smooth quadric bundle over X of relative dimension m, i.e., Y is smooth and there exist a morphism $q : Y \to X$ such that each fiber is a smooth quadric of dimension m. If $m \in \{8, 9, 10\}$, then Y is not 2-Fano.*

Proof. We have that $Y \subset \mathbb{P}(E)$, where E is a rank $m + 2$ vector bundle on X, and the map q is induced by the projection $\pi : \mathbb{P}(E) \to X$ (this result is attributed to Beauville in [Nag, Prop. 2.1.4]). We denote $\xi := c_1(\mathcal{O}_\pi(1))$. Here $\mathbb{P}(E)$ stand for $\mathrm{Proj}(\mathrm{Sym}(E))$. We may assume without loss of generality that the dual vector bundle $E^\vee$ is globally generated. By [AC13, Lemma 12], we have

$$\mathrm{ch}_2(\mathbb{P}(E)) = \pi^*(\,\mathrm{ch}_2(X) + \mathrm{ch}_2(E^\vee)) + \pi^*(c_1(E^\vee)) \cdot \xi + \frac{m+2}{2}\xi^2.$$

As Y is a quadric bundle, we have that Y belongs to a linear system $|\mathcal{O}_\pi(2) \otimes \pi^*(L)|$, where L is a line bundle on X. Hence, we have

$$\mathrm{ch}_2(Y) = (\,\mathrm{ch}_2(\mathbb{P}(E)) - \frac{1}{2}Y^2)_{|Y} =$$

$$= (\pi^*(\,\mathrm{ch}_2(X) + \mathrm{ch}_2(E^\vee) - \frac{1}{2}c_1(L)^2) + \pi^*(c_1(E^\vee) - 2c_1(L)) \cdot \xi + \frac{m-2}{2}\xi^2)_{|Y}.$$

As X is uniruled, there exists a minimal family of rational curves passing through a general point of X. Let $l \cong \mathbb{P}^1$ be a curve in the family. We denote

$$L_{|l} = \mathcal{O}(d), \quad E_{|l} = \oplus_{i=1}^{m+2}\mathcal{O}(a_i), \quad a_1 \geq a_2 \geq \ldots \geq a_{m+2},$$

Since $E^\vee$ is globally generated, it follows that $a_1 \leq 0$. In addition,

$$H^0(\mathbb{P}(E), \mathcal{O}_\pi(2) \otimes \pi^*L) = H^0(X, \mathrm{Sym}^2(E) \otimes L) \neq 0.$$

It follows that $H^0(l, (\,\mathrm{Sym}^2(E) \otimes L)_{|l}) \neq 0$; hence, $d + 2a_1 \geq 0$. In particular, $d \geq 0$.

Consider now the surface $S_l := Y \cap \mathbb{P}(F_l)$, where $\rho : \mathbb{P}(F_l) \to l$ is the $\mathbb{P}^2$-bundle given by the quotient $F_l := \mathcal{O}(a_m) \oplus \mathcal{O}(a_{m+1}) \oplus \mathcal{O}(a_{m+2})$ of $E_{|l}$. We denote by $\xi_{F_l} := c_1(\mathcal{O}_{\mathbb{P}(F_l)}(1))$. Note that S_l is a conic bundle over $\mathbb{P}^1$. We compute:

$$\mathrm{ch}_2(Y) \cdot S_l = \pi^*(c_1(E^\vee) - 2c_1(L)) \cdot \xi \cdot S_l + \frac{m-2}{2}\xi^2 \cdot S_l.$$

where the intersection products are on $\mathbb{P}(E)$. Note, $S_l \subset \mathbb{P}(F_l)$ belongs to the linear system $|\mathcal{O}_\rho(2) \otimes \rho^*\mathcal{O}(d)|$. Both products can be computed on $\mathbb{P}(F_l)$ as follows:

$$\pi^*(c_1(E^\vee) - 2c_1(L)) \cdot \xi \cdot S_l = \rho^*(c_1(E^\vee_{|l}) - 2c_1(L_{|l})) \cdot \xi_{F_l} \cdot (2\xi_{F_l} + \rho^*c_1(\mathcal{O}(d))) =$$

$$= \rho^*(c_1(E^\vee_{|l}) - 2c_1(L_{|l})) \cdot (2\xi_{F_l}^2) = -2\sum_{i=1}^{m+2}a_i - 4d,$$

$$\xi^2 \cdot S_l = \xi_{F_l}^2 \cdot (2\xi_{F_l} + \rho^* c_1(\mathcal{O}(d))) = 2\xi_{F_l}^3 + d.$$

As F_l is a rank 3 vector bundle on $\mathbb{P}^1$, we have $\xi_{F_l}^3 - \rho^*(c_1(F_l)) \cdot \xi_{F_l}^2 = 0$, i.e., $\xi_{F_l}^3 = a_m + a_{m+1} + a_{m+2}$. It follows that

$$\mathrm{ch}_2(Y) \cdot S_l = \frac{m-10}{2}d + (m-4)(a_m + a_{m+1} + a_{m+2}) - 2\sum_{i=1}^{m-1} a_i.$$

If $m = 8, 9, 10$, then $\mathrm{ch}_2(Y) \cdot S_l \le 0$ follows from $a_1 \ge \ldots \ge a_{m+2}$ and $d + 2a_1 \ge 0$. If $m = 10$:

$$\mathrm{ch}_2(Y) \cdot S_l = 6(a_{10} + a_{11} + a_{12}) - 2\sum_{i=1}^{9} a_i.$$

There are 18 terms appearing with a $+$ sign and 18 terms appearing with a $-$ sign. Each term with a $+$ sign is $\le$ than the terms with the $-$ sign.

If $m = 9$:

$$\mathrm{ch}_2(Y) \cdot S_l = -\frac{1}{2}d + 5(a_9 + a_{10} + a_{11}) - 2\sum_{i=1}^{8} a_i = -\frac{1}{2}(d + 2a_1) + 5(a_9 + a_{10} + a_{11}) - a_1 - 2\sum_{i=2}^{8} a_i.$$

Note that $d + 2a_1 \ge 0$. There are 15 terms in $5(a_9 + a_{10} + a_{11})$ and 15 terms in $-a_1 - 2\sum_{i=2}^{8} a_i$. Again, each term with a $+$ sign is $\le$ than the terms with the $-$ sign.

If $m = 8$:

$$\mathrm{ch}_2(Y) \cdot S_l = -d + 4(a_8 + a_9 + a_{10}) - 2\sum_{i=1}^{7} a_i = -(d + 2a_1) + 4(a_8 + a_9 + a_{10}) - 2\sum_{i=2}^{7} a_i.$$

Similarly, There are 12 terms in $4(a_8 + a_9 + a_{10})$ and 12 terms in $-2\sum_{i=2}^{7} a_i$ and each term with a $+$ sign is $\le$ than the terms with the $-$ sign.

Note that if $a_1 = \ldots = a_{m+2}$, this number is ≥ 0 if $m \ge 10$. $\qquad\square$

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