HYPERSURFACES WITH TOO MANY RATIONAL CURVES

ROYA BEHESHTI

ABSTRACT. We study smooth hypersurfaces of degree $d \geq n+1$ in $\mathbf{P}^n$ whose spaces of smooth rational curves of low degrees are larger than expected, and show that under certain conditions, the primitive part of the middle cohomology of such hypersurfaces have non-trivial Hodge substructures. As an application, we prove that the space of lines on any smooth Fano hypersurface of degree $d \leq 8$ in $\mathbf{P}^n$ has the expected dimension $2n-d-3$.

1. INTRODUCTION

For a smooth hypersurface $X \subset \mathbf{P}^n$ of degree d over $\mathbf{C}$ and a positive integer e, we let $R_e(X)$ denote the closure of the locus inside the Hilbert scheme of X which parametrizes smooth rational curves of degree e on X. The dimension of $R_e(X)$ is at least $e(n+1-d)+n-4$, and it is conjectured that when $d < n$ and X is general, $R_e(X)$ is irreducible of dimension $e(n+1-d)+n-4$ (for details see [5], [3], and [2]). We refer to the number $e(n+1-d)+n-4$ as the *expected dimension* of $R_e(X)$.

Recently, J. Ellenberg and A. Venkatesh have used the circle method to show that if X is *any* smooth hypersurface whose dimension is exponentially larger than its degree, then $R_e(X)$ is irreducible of the expected dimension for every $e \geq 1$ (unpublished; see also [10]). An interesting question which arises is: for a given degree e, what is the best upper bound on d such that for any smooth hypersurface X of degree d in $\mathbf{P}^n$, $R_e(X)$ has the expected dimension? An obvious upper bound comes from hypersurfaces which contain linear subvarieties of maximal dimensions: if, for example, X is the Fermat hypersurface of degree d in $\mathbf{P}^n$, then X contains a linear subvariety of dimension $\frac{n-1}{2}$ when n is odd and a 1-parameter family of linear subvarieties of dimension $\frac{n}{2}-1$ when n is even. Since $\dim R_e(\mathbf{P}^m) = (e+1)(m+1)-4$, we must have $d \leq \frac{(e+1)(n-1)}{2e}+1$ when e is odd, and $d \leq \frac{(e+1)n}{2e}$ when e is even, otherwise $R_e(X)$ can be larger than expected. For $e=1$, this is expected to be the correct bound [1].

In this paper, we study the above question for lines and conics by looking at Calabi-Yau hypersurfaces which are intersections of X with general linear subvarieties of $\mathbf{P}^n$. In Section 2, we study smooth hypersurfaces X of degree $\geq n+1$ in $\mathbf{P}^n$ for which $R_e(X)$ is larger than expected for small values of e, and prove that under certain conditions, the primitive part of $H^{n-1}(X, \mathbf{Q})$ has a nontrivial Hodge substructure (Proposition 2.1). In the case of conics, we use this to show that if X is a hypersurface of degree $\leq n$ in $\mathbf{P}^n$ and R is an irreducible component of $R_2(X)$ such that the conics parametrized by R sweep out a divisor in X, then R has the expected dimension (Theorem 3.2 (b)). In the case of lines we show:

Proposition 1.1. *Suppose that X is a smooth hypersurface of degree $n+1$ in $\mathbf{P}^n$ whose space of lines is larger than expected. If $n \leq 7$, then the primitive part of $H^{n-1}(X, \mathbf{Q})$ has a non-trivial Hodge substructure.*

As a corollary we get:

Theorem 1.2. *If X is any smooth Fano hypersurface of degree $d \leq 8$, then the space of lines on X has the expected dimension.*

Using a different technique, the case $d \leq 6$ of the above theorem was proved in [1]. Another proof was given later in [8]. It is an interesting question whether Proposition 1.1 holds for an arbitrary n. An affirmative answer to this question would imply that Theorem 1.2 holds for any smooth Fano hypersurface.

1.1. **Acknowledgments.** The author would like to thank Ravindra Girivaru and Matt Kerr for many helpful discussions.

2. non-Fano hypersurfaces

Let X be a smooth hypersurface of degree $d \geq n+1$ in $\mathbf{P}^n$, and let $H^{n-1}(X, \mathbf{Q})_{\text{prim}}$ denote the primitive part of $H^{n-1}(X, \mathbf{Q})$

$$H^{n-1}(X, \mathbf{Q})_{\text{prim}} = \text{Ker } (H^{n-1}(X, \mathbf{Q}) \to H^{n+1}(\mathbf{P}^n, \mathbf{Q})).$$

For $e \geq 1$, the expected dimension of $R_e(X)$ is $e(n+1-d)+n-4 < n-3$. In this section, we study the case when $\dim R_e(X) \geq n-3$ and prove the following:

Proposition 2.1. *Let X be a smooth hypersurface of degree $d \geq n+1$ in $\mathbf{P}^n$. In the following situations, $H^{n-1}(X, \mathbf{Q})_{\text{prim}}$ has a non-trivial Hodge substructure:*

(a) *If $e \leq 2d-n+1$ and the curves parametrized by $R_e(X)$ sweep out a subvariety of codimension 1 in X (note that since $d \geq n+1$, X is not swept out by the curves parametrized by $R_e(X)$).*

(b) *If $e = 1$ and there is an irreducible subvariety Y of dimension a in X such that the space of lines on Y has dimension $\geq n-3$ and*

$$n - 1 \geq \frac{(2a-n+2)(2a-n+3)}{2}.$$

Proposition 1.1 follows from the above proposition: Let X be a smooth hypersurface of degree $\geq n+1$ and let R be an irreducible subvariety of the space of lines on X such that $\dim R \geq n-3$. Let a denote the dimension of the subvariety of X swept out by the lines parametrized by R. Then $a \leq n-2$. If $a = n-2$, then part (a) of the above proposition shows that the primitive part of $H^{n-1}(X, \mathbf{Q})$ has a non-trivial Hodge substructure, and if $a \leq n-3$ and $n \leq 7$, then the inequality in the second part is satisfied, so we get the desired result.

Proof of Proposition 2.1, part (a). Let R be an irreducible component of $R_e(X)$ such that the curves parametrized by R sweep out a subvariety of codimension

1 in X, and let $\tilde{R}$ be a desingularization of R. Denote by $\tilde{U}$ the pullback of the universal family to $\tilde{R}$:

$$\begin{tikzcd} \tilde{U} \arrow[r, "f"] \arrow[d, "\pi"] & X \\ \tilde{R} & \end{tikzcd}$$

Then $\tilde{U} \subset \tilde{R} \times X$ induces a morphism of Hodge structures

$$\alpha : H^{n-1}(X, \mathbf{Q}) \to H^{n-3}(\tilde{R}, \mathbf{Q})$$

of bidegree $(-1,-1)$ mapping $H^{n-1-i,i}(X)$ to $H^{n-2-i,i-1}(\tilde{R})$ for every $i \geq 0$. We show that the kernel of α is a non-trivial Hodge substructure of $H^{n-1}(X, \mathbf{Q})$. Since $d \geq n+1$, $H^{n-1,0}(X)$ is non-zero and it is mapped to zero under α, so to show the kernel of α is non-trivial, it is enough to show that restriction of α to $H^{n-2,1}(X)$ is non-zero. We prove this by showing that for a general rational curve C parametrized by $\tilde{R}$, if α_C denotes the composition of the maps

$$H^{n-2,1}(X) \to H^{n-3,0}(\tilde{R}) = H^0(\tilde{R}, \Omega^{n-3}_{\tilde{R}}) \to \Omega^{n-3}_{\tilde{R}}|_{[C]},$$

then α_C is non-zero.

By [4], α_C can be described as follows. Let $\alpha_1 : H^{n-2,1}(X) = H^1(X, \Omega^{n-2}_X) \to H^1(X, \Omega^{n-2}_X|_C)$ denote the restriction map. From the short exact sequence

$$0 \to T_C \to T_X|_C \to N_{C/X} \to 0,$$

we get an injective map $\bigwedge^{n-3} N_{C/X} \otimes T_C \to \bigwedge^{n-2} T_X|_C$, so we have an injective map

$$\alpha_2 : H^0(C, \bigwedge^{n-3} N_{C/X}) \to H^0(C, \bigwedge^{n-2} T_X|_C \otimes \omega_C).$$

Let $T_{\tilde{R},[C]}$ denote the Zariski tangent space to $\tilde{R}$ at $[C]$. Then there is a natural map $T_{\tilde{R},[C]} \to H^0(C, N_{C/X})$ whose dual induces a map

$$\alpha_3 : H^0(C, \bigwedge^{n-3} N_{C/X})^\vee \to \Omega^{n-3}_{\tilde{R}}|_{[C]}.$$

We get a commutative diagram

$$\begin{tikzcd}
H^1(X, \Omega^{n-2}_X) = H^{n-2,1}(X) \arrow[d, "\alpha_1"] \arrow[drrr, "\alpha_C"] & & & \\
H^1(C, \Omega^{n-2}_X|_C) \arrow[d, "="] & & & \\
H^0(C, \bigwedge^{n-2} T_X|_C \otimes \omega_C)^\vee \arrow[r, two heads, "\alpha_2^\vee"] & H^0(C, \bigwedge^{n-3} N_{C/X})^\vee \arrow[r] \arrow[rr, bend right, "\alpha_3"'] & \bigwedge^{n-3} H^0(C, N_{C/X})^\vee \arrow[r] & \Omega^{n-3}_{\tilde{R}}|_{[C]}
\end{tikzcd}$$

The map α_3 is non-zero since the curves parametrized by $\tilde{R}$ sweep out a subvariety of dimension $n-2$ in X, thus the assertion follows if we show α_1 is surjective. The short exact sequence

$$0 \to \mathcal{O}_X(-d) \to \Omega_{\mathbf{P}^n}|_X \to \Omega_X \to 0$$

gives the following short exact sequence

$$0 \to \Omega_X^{n-2} \to \Omega_{\mathbf{P}^n}^{n-1}(d)|_X \to \mathcal{O}_X(2d-n-1) \to 0.$$

Applying the long exact sequence of cohomology, it is enough to show that the restriction map $H^0(X, \mathcal{O}_X(2d-n-1)) \to H^0(C, \mathcal{O}_C(2d-n-1))$ is surjective and $H^1(C, \Omega_{\mathbf{P}^n}^{n-1}(d)|_C) = 0$. The latter is true because the Euler sequence gives a surjective map $\mathcal{O}_C(-n+d)^{n+1} \to \Omega_{\mathbf{P}^n}^{n-1}(d)|_C$ and $-n+d \geq 1$, and the former is true since every smooth rational curve of degree e in $\mathbf{P}^n$ is $(e-2)$-normal and by our assumption $2d-n-1 \geq e-2$.

$\square$

We next discuss part (b) of Proposition 2.1. Let Y be as in the proposition, and suppose that for a general point p of Y, the lines through p on Y span a cone of dimension k in Y. Then the space of lines on Y has dimension $a+k-2$, so by our assumption $k+a \geq n-1$. Considering the irreducible components of all such cones with their reduced induced structure, since our base field is uncountable, there is an irreducible component of the Hilbert scheme of k-dimensional subschemes of X and an integral subscheme G of this component such that a general point of G parametrizes an integral k-dimensional cone in X, and such that the subschemes of X parametrized by G sweep out a subvariety of dimension a in X. Let $\tilde{G}$ be a desingularization of G, and let $\tilde{U}$ be a desingularization of the pullback of the universal family to $\tilde{G}$, so we have a diagram

$$\begin{tikzcd} \tilde{U} \arrow[r, "F"] \arrow[d, "\pi"] \arrow[rr, bend left, "f"] & \tilde{G} \times X \arrow[r] & X \\ \tilde{G}. \end{tikzcd}$$

Then the universal family in $\tilde{G} \times X$ induces a morphim of Hodge structures

$$\alpha : H^{n-1}(X, \mathbf{Q}) \to H^{n-1-2k}(\tilde{G}, \mathbf{Q})$$

which sends $H^{n-k-1,k}(X)$ to $H^{n-1-2k,0}(\tilde{G})$. We show that when the inequality in part (b) is satisfied, the restriction of α to $H^{n-k-1,k}(X)$ is non-zero.

Let $\pi : Y \to \mathbf{P}^{a+1}$ be a general linear projection. Then π is a generically injective morphism onto a hypersurface $Y' \subset \mathbf{P}^{a+1}$. Since π is generically injective, the family of lines through a general point of Y' is $(k-1)$-dimensional. Suppose that p is a general point of Y which is mapped to p' in Y'. Then there are hypersurfaces $Y_i' \subset \mathbf{P}^{a+1}$ $(1 \leq i \leq \deg Y')$ such that $\deg Y_i' = i$ and the intersection $Y_1' \cap \cdots \cap Y_i'$ is a cone with vertex p' whose underlying subvariety is the subvariety of $\mathbf{P}^{a+1}$ swept out by all the lines in $\mathbf{P}^{a+1}$ which intersect Y' at p' with multiplicity $\geq i+1$ (If $p' = (1:0:\cdots:0)$, $Y' = \{g=0\}$, and we write

$$g = x_0^{d'-1} g_1 + \cdots + g_{d'}$$

where $d' = \deg g$ and g_i is homogeneous of degree i in $x_1, \ldots, x_{a+1}$ for $1 \leq i \leq d'-1$, then Y_i' is the hypersurface defined by g_i in $\mathbf{P}^{a+1}$). By [8, Theorem 2], if the codimension of an irreducible component of the cone $Y_1' \cap \cdots \cap Y_i'$ in $\mathbf{P}^{a+1}$ is smaller than i, then that component is contained in Y'. Since we are assuming that the cone of lines through p' has dimension k, the intersection $Y_1' \cap Y_2' \cap \cdots \cap Y_{a+1-k}'$ should be a complete intersection.

Denote by $Y_1 \subset \mathbf{P}^n$ the embedded tangent space to Y at p. Then $\pi : Y_1 \to Y_1'$ is an isomorphism. Denote the pre-image of $Y_1' \cap \cdots \cap Y_{a+1-k}'$ in Y_1 by Σ. Then Σ is a k-dimensional complete intersection of $n-a$ hyperplanes $H_1, \ldots, H_{n-a}$ and hypersurfaces $H_{n-a+1}, \ldots, H_{n-k}$ of degrees $2, \ldots, a-k+1$ in $\mathbf{P}^n$. Let Σ_X be the intersection of Σ with X. Then Σ_X is an almost complete intersection which may be reducible with some irreducible components of dimension $k-1$, but since every line in Y through p is contained in Σ_X, there will be at least one irreducible component of dimension k in Σ_X.

Proposition 2.2. *If a and k as above are such that $a+k \geq \frac{(a-k+1)(a-k+2)}{2}$, then the restriction map*

$$H^k(X, \Omega_X^{n-1-k}) \to H^k(\Sigma_X, \Omega_X^{n-1-k}|_{\Sigma_X})$$

is surjective.

Corollary 2.3. *Suppose that the inequality in the above proposition holds. If Z is a projective variety and $f : Z \to \Sigma_X$ is a generically injective morphism, then the pullback map*

$$H^k(X, \Omega_X^{n-1-k}) \to H^k(Z, f^*\Omega_X^{n-1-k})$$

is surjective.

Proof. Since f is generically injective, the support of the cokernel of the map $\Omega_X^{n-1-k}|_{\Sigma_X} \to f_*f^*\Omega_X^{n-1-k}$ is of dimension at most $k-1$. Thus it follows from Proposition 2.2 that the map $H^k(X, \Omega_X^{n-1-k}) \to H^k(\Sigma_X, f_*f^*\Omega_X^{n-1-k})$ is surjective. Note that for $0 \leq i \leq k-1$, $R_{f_*}^{k-i} f^*\Omega_X^{n-1-k}$ is supported along a subvariety of dimension $\leq i-1$, so $H^i(\Sigma_X, R_{f_*}^{k-i} f^*\Omega_X^{n-1-k}) = 0$. By the Leray spectral sequence

$$H^k(Z, f^*\Omega_X^{n-1-k}) = H^k(\Sigma_X, f_*f^*\Omega_X^{n-1-k}),$$

and the result follows. □

Proof of the Proposition 2.2. From the short exact sequence

$$0 \to \mathcal{O}_X(-d) \to \Omega_{\mathbf{P}^n}|_X \to \Omega_X \to 0$$

we get the following short exact sequence

$$0 \to \Omega_X^{n-k-1} \to \Omega_{\mathbf{P}^n}^{n-k}(d)|_X \to \Omega_X^{n-k}(d) \to 0.$$

Applying the long exact sequence of cohomology to the above exact sequence and its restriction to Σ_X, the assertion follows if we show that

(a) $H^k(\Sigma_X, \Omega^{n-k}_{\mathbf{P}^n}(d)|_{\Sigma_X}) = 0$,
(b) $H^{k-1}(X, \Omega^{n-k}_X(d)) \to H^{k-1}(\Sigma_X, \Omega^{n-k}_X(d)|_{\Sigma_X})$ is surjective.

To prove (a), it is enough to show that $H^k(\Sigma, \Omega^{n-k}_{\mathbf{P}^n}(d)|_\Sigma) = 0$ since there is a surjection $\Omega^{n-k}_{\mathbf{P}^n}(d)|_\Sigma \to \Omega^{n-k}_{\mathbf{P}^n}(d)|_{\Sigma_X}$ and the kernel is supported along a scheme of dimension at most k. From the restriction of the the Euler sequence to Σ we get a surjection

$$\bigwedge^{n-k+1} \mathcal{O}^{n+1}_\Sigma(-1) \to \Omega^{n-k}_{\mathbf{P}^n}|_\Sigma.$$

So it is enough to show that $H^k(\Sigma, \mathcal{O}_\Sigma(-n+k-1+d))$ vanishes which is by Serre duality equivalent to the vanishing of

$$H^0(\Sigma, \mathcal{O}_\Sigma(n-k+1-d)\otimes\omega_\Sigma) = H^0(\Sigma, \mathcal{O}_\Sigma(-d-k+n-a-1+(1+2+\cdots+a-k+1)).$$

Since $d \geq n+1$, this follows from the assumption of the Proposition.

To prove (b), let $V_2, \ldots, V_{n-a}$ be general hyperplanes in $\mathbf{P}^n$ which contain Σ_X, and for $n-a+1 \leq i \leq n-k$, let V_i be a general hypersurface of degree equal to $\deg H_i$ which contains Σ_X. Set

$$V := V_2 \cap \cdots \cap V_{n-k} \cap X.$$

Lemma 2.4. *V is a complete intersection of dimension k containing Σ_X.*

Proof. We show by decreasing induction on i that the intersection $V_{i+1}\cap\cdots\cap V_{n-k}\cap X$ is a complete intersection for every $1 \leq i \leq n-k-1$. If $i = n-k-1$, this is certainly true. Suppose that the statement holds for i. Note that if q is a point which is contained in every hypersurface whose degree is equal to $\deg H_i$ and contains Σ_X, then since $\deg H_i \geq \deg H_j$ for every $1 \leq j \leq i$, q is contained in $H_1 \cap H_2 \cap \cdots \cap H_i$. If the intersection $V_i \cap V_{i+1} \cap \cdots \cap V_{n-k} \cap X$ $(i \geq 2)$ is not a proper intersection, then there should be a component of $V_{i+1} \cap \cdots \cap V_{n-k} \cap X$ which is contained in $H_1 \cap \cdots \cap H_i$. This is true for general $V_{i+1}, \ldots, V_{n-k}$ containing Σ_X, in particular, there should be an irreducible component of $H_{i+1}\cap\cdots\cap H_{n-k}\cap X$ which is contained in $H_1 \cap \cdots \cap H_i$. But this is not possible since the dimension of every irreducible component of $H_{i+1} \cap \cdots \cap H_{n-k} \cap X$ is at least $k+i-1 > k$, and

$$\dim H_1 \cap \cdots \cap H_{n-k} \cap X = k.$$

Hence the statement holds for $i-1$. □

If $\mathcal{I}_{\Sigma_X/V}$ denotes the ideal sheaf of Σ_X in V, then we have a short exact sequence

$$0 \to \Omega^{n-k}_X(d) \otimes \mathcal{I}_{\Sigma_X/V} \to \Omega^{n-k}_X(d)|_V \to \Omega^{n-k}_X(d)|_{\Sigma_X} \to 0.$$

So (b) is proved if we show that

(c) $H^{k-1}(X, \Omega^{n-k}_X(d)) \to H^{k-1}(V, \Omega^{n-k}_X(d)|_V)$ is surjective.
(d) $H^k(V, \Omega^{n-k}_X(d) \otimes \mathcal{I}_{\Sigma_X/V}) = 0$.

Part (c) holds by the first part of the next lemma. To prove part (d), note that there is a surjective map $\mathcal{O}_V(-1) \to \mathcal{I}_{\Sigma_X/V}$, so it is enough to show $H^k(V, \Omega_X^{n-k}(d-1)|_V) = 0$ which holds by the second part of the next lemma.

□

Lemma 2.5. *Suppose that X is a smooth hypersurface of degree $d \geq 2$ in $\mathbf{P}^n$ and $V \subset X$ is a k-dimensional ($k \geq 2$) complete intersection of type $(d_1, \dots, d_{n-k})$, $d_1 = d$. If $l > -k + 2 + \sum_{i=2}^{n-k} d_i$, then the following hold.*

1. *The restriction map $H^{k-1}(X, \Omega_X^{n-k}(l)) \to H^{k-1}(V, \Omega_X^{n-k}(l)|_V)$ is surjective.*
2. *We have $H^k(V, \Omega_X^{n-k}(l-1)|_V) = 0$.*

Proof. We show that for every $1 \leq m \leq k$,

(i) $H^{m-1}(X, \Omega_X^{n-m}(l+d(k-m))) \to H^{m-1}(V, \Omega_X^{n-m}(l+d(k-m)|_V)$ is surjective if $l > -k + 2 + \sum_2^{n-k} d_i$.

(ii) $H^m(V, \Omega_X^{n-m}(l + d(k-m))|_V) = 0$ if $l > -k + 1 + \sum_2^{n-k} d_i$.

We proceed by induction on m. Since V is a complete intersection, the statements clearly hold when $m = 1$. Assume (i) and (ii) hold for m, and consider the exact sequence

$$0 \to \Omega_X^{n-1-m}(l+d(k-m-1))|_V \to \Omega_{\mathbf{P}^n}^{n-m}(l+d(k-m))|_V \to \Omega_X^{n-m}(l+d(k-m))|_V \to 0,$$

Applying the long exact sequence of cohomology, to show the statements hold for $m+1$ ($m \leq k-1$), it is enough to show that H^m of the middle term vanishes if l satisfies the inequality in part (i) and H^{m+1} of the middle term vanishes if l satisfies the inequality in part (ii). Both of these vanishing statements follow if we show that for $t, j \geq 1$ and $b > j - t + k - n + \sum_1^{n-k} d_i$, we have $H^t(V, \Omega_{\mathbf{P}^n}^j(b)|_V) = 0$. There is a resolution:

$$0 \to \mathcal{O}_{\mathbf{P}^n}(-(n+1)+b)|_V \to \cdots \to \oplus \mathcal{O}_{\mathbf{P}^n}(-(j+1)+b)|_V \to \Omega_{\mathbf{P}^n}^j(b)|_V \to 0.$$

To show $H^t(V, \Omega^j(b)|_V) = 0$, it is enough to show $H^{t+r}(V, \mathcal{O}_V(-(j+r+1)+b) = 0$ for every $r \geq 0$. This is clearly true if $t + r \neq k$ as V is a complete intersection of dimension k. If $t + r = k$, we have

$$H^k(V, \mathcal{O}_V(-(j+k-t+1)+b) = H^0(V, \mathcal{O}_V(j+k-t-b-n-\sum_1^{n-k} d_i) = 0$$

□

Proof of Proposition 2.1, part (b). Let k be the dimension of the cone of lines on Y though a general point, and let $\tilde{G}$, $\tilde{U}$ and $\alpha : H^{n-1}(X, \mathbf{Q}) \to H^{n-1-2k}(\tilde{G}, \mathbf{Q})$ be defined as before. We show the restriction of α to $H^{n-k-1,k}(X)$ is non-zero. Since $H^{n-1,0}(X)$ is non-zero and clearly mapped to zero under α, we conclude that the kernel of α is a non-trivial Hodge substructure of $H^{n-1}(X, \mathbf{Q})$.

Let q be a general point of $\tilde{G}$, and denote by $\tilde{U}_q$, the fiber of π over q. Then $\tilde{U}_q$ is smooth of dimension k. We denote the restriction of $f : \tilde{U} \to X$ to $\tilde{U}_q$ by f as well. The image of $\tilde{U}_q$ in X is a cone of lines Y_p through a point p of Y and the map $f : \tilde{U}_q \to Y_p$ is generically injective. Since we assume

$$n - 1 \geq \frac{(2a - n + 2)(2a - n + 3)}{2},$$

and since $a+k \geq n-1$, the inequality in Corollary 2.3 is satisfied, so we can conclude the map

$$\alpha_1 : H^k(X, \Omega_X^{n-k-1}) \to H^k(\tilde{U}_q, f^*\Omega_X^{n-k-1})$$

is surjective.

Since F is generically injective, there are injective maps $T_{\tilde{U}_q} \to f^*T_X$ and $T_{\tilde{U}} \to F^*T_{\tilde{G}\times X} = \pi^*T_{\tilde{G}} \oplus f^*T_X$. Denote the cokernels of these maps by N_q and N respectively. The exact sequence

$$0 \to T_{\tilde{U}_q} \to f^*T_X \to N \to 0$$

gives an injective map

$$(\bigwedge^{n-2k-1} N)/\mathrm{tor} \otimes \bigwedge^{k} T_{\tilde{U}_q} \to \bigwedge^{n-k-1} f^*T_X. \tag{1}$$

From the following diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & T_{\tilde{U}_q} \arrow[r] \arrow[d] & f^*T_X \arrow[r] \arrow[d] & N_q \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & T_{\tilde{U}}|_{\tilde{U}_q} \arrow[r] \arrow[d] & F^*T_{\tilde{G}\times X}|_{\tilde{U}_q} \arrow[r] \arrow[d] & N|_{\tilde{U}_q} \arrow[r] & 0 \\
 & N_{\tilde{U}_q/\tilde{U}} \arrow[r, "="] \arrow[d] & \pi^*T_{\tilde{G}}|_{\tilde{U}_q} \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

we get $N|_{\tilde{U}_q} = N_q$. Also, there is a natural map $T_{\tilde{G},q} \to H^0(\tilde{U}_q, \pi^*T_{\tilde{G},q}) = H^0(\tilde{U}_q, \pi^*T_{\tilde{G}}|_{U_q})$. The composition of the maps

$$\pi^*T_{\tilde{G}}|_{\tilde{U}_q} \to \left(\pi^*T_{\tilde{G}} \oplus f^*T_X\right)|_{\tilde{U}_q} = F^*T_{\tilde{G}\times X}|_{\tilde{U}_q} \to N|_{\tilde{U}_q} = N_q$$

hence gives a map

$$T_{\tilde{G},q} \to H^0(\tilde{U}_q, N_q). \tag{2}$$

Denote by α_q the composition of the maps

$$H^{n-k-1,k}(X) \to H^{n-2k-1,0}(\tilde{G}) = H^0(\tilde{G}, \Omega_{\tilde{G}}^{n-2k-1}) \to \Omega_{\tilde{G}}^{n-2k-1}|_q.$$

By [4], α_q factors through the following maps:

- the surjective pullback map

$$\alpha_1 : H^{n-k-1,k}(X) = H^k(X, \Omega_X^{n-k-1}) \to H^k(\tilde{U}_q, f^*\Omega_X^{n-k-1}) = H^0(\tilde{U}_q, f^* \bigwedge^{n-k-1} T_X \otimes \omega_{\tilde{U}_q})^\vee.$$

- the surjective map

$$\alpha_2 : H^0(\tilde{U}_q, f^* \bigwedge^{n-k-1} T_X \otimes \omega_{\tilde{U}_q})^\vee \to H^0(\tilde{U}_q, \bigwedge^{n-2k-1} N/\mathrm{tor})^\vee$$

  which comes from the injective map (1)
- the natural map

$$\alpha_3 : H^0(\tilde{U}_q, \bigwedge^{n-2k-1} N/\mathrm{tor})^\vee \to (\bigwedge^{n-2k-1} H^0(\tilde{U}_q, N))^\vee.$$

- The map

$$\alpha_4 : (\bigwedge^{n-2k-1} H^0(\tilde{U}_q, N))^\vee \to (\bigwedge^{n-2k-1} T_{\tilde{G},q})^\vee = \Omega_{\tilde{G}}^{n-2k-1}|_q$$

  which comes from the dual of the map $T_{\tilde{G},q} \to H^0(\tilde{U}_q, N_q)$ in (2).

Lemma 2.6. *The map $\alpha_4 \circ \alpha_3$ is non-zero.*

Proof. Let u be a general point of $\tilde{U}_q$. We have a commutative diagram

$$\begin{tikzcd} T_{\tilde{U},u} \arrow[rr] \arrow[d] & & f^*T_X|_u \arrow[d] \\ T_{\tilde{G},q} \arrow[r] & H^0(\tilde{U}_q, N_q) \arrow[r] & N_q|_u \end{tikzcd}$$

Since we assume the image of $\tilde{U} \to X$ is of dimension a, and since u is a general point of $\tilde{U}_q$ and hence a general point of $\tilde{U}$, the image of the induced map on Zariski tangent spaces $T_{\tilde{U},u} \to T_{X,f(u)} = f^*T_X|_u$ is of dimension a as well. So the image of the composition map $T_{\tilde{U},u} \to f^*T_X|_u \to N_q|_u$ is of dimension at least $a-k$, and the same is true for the image of the map

$$T_{\tilde{G},q} \to N_q|_u.$$

By our assumption $a + k \geq n-1$, so $a - k \geq n - 2k - 1$, hence the map

$$\bigwedge^{n-2k-1} T_{\tilde{G},q} \to H^0(\tilde{U}_q, \bigwedge^{n-2k-1} N/\mathrm{tor}),$$

is non-zero, so its dual is also non-zero. □

Putting all these together we conclude that α is a non-zero map. □

We end this section with an example which shows that for every $n \geq 4$, there is a smooth hypersurface of degree $n+1$ in $\mathbf{P}^n$ which has a larger than expected family of lines such that a general line in the family is not contained in any cone of lines in X. This is based on the example of Albano-Katz [6] for $n = 4$. Let $X_0 = \{\sum_{i=0}^n x_i^{n+1} = 0\}$ be the Fermat hypersurface of degree $n+1$ in $\mathbf{P}^n$, and consider the family of hypersurfaces $X_t := \{\sum_{i=0}^n x_i^{n+1} - tx_0 \dots x_n = 0\}$ over $\mathbf{P}^1$.

Proposition 2.7. *For a general t in $\mathbf{P}^n$, there is a line L in X_t which belong to a larger than expected family of lines on X_t and*

$$N_{L/X_t} = \mathcal{O}_L^{n-3} \oplus \mathcal{O}_L(-2).$$

The expected dimension of the space of lines on a hypersurface of degree $n+1$ in $\mathbf{P}^n$ is $n-4$. Assuming the proposition, if we let R be an irreducible family of dimension $\geq n-3$ of lines on X_t and L a line parametrized by R whose normal bundle has the above splitting type, then since the Zariski tangent space to the space of lines on X passing through a point p of L is isomorphic to $H^0(L, N_{L/X}(-p))$, there are only finitely many lines on X though any point of L. So L is not contained in any cone, and the same holds for a general line parametrized by R.

Proof of Proposition 2.7. We give the proof for n even; the proof for the odd case is similar. We first show that there is a family of dimension at least $n-2$ of lines in $\mathbf{P}^n$ such that each line in this family lies on X_t for some t. Set $Z = X_0 \cap \{x_0 \dots x_n = 0\}$, and let $Z_i \subset Z$ be the intersection of X_0 with the coordinate hyperplane $x_i = 0$. If L is a line which intersects Z_0 in $p = (a_0 : a_1 : \cdots : a_n)$ and Z_1 in $q = (b_0 : b_1 : \cdots : b_n)$, and if p and q are distinct points which do not belong to any other coordinate hyperplane, then the point $(a_0 b_j - a_j b_0 : \cdots : a_n b_j - a_j b_n)$ is on $L \cap \{x_j = 0\}$ for every $j \geq 0$. Since

$$\sum_i \sum_j (a_i b_j - a_j b_i)^{n+1} = 0,$$

if L intersects $Z_0, \dots, Z_{n-1}$, then it intersects Z_n as well. Since the family of lines intersecting any subvariety of codimension 2 in $\mathbf{P}^n$ is an ample divisor in the Grassmannian of lines in $\mathbf{P}^n$, the family of lines intersecting $Z_0, \dots, Z_{n-1}$ has dimension $\geq n-2$. Any such line which intersects the coordinate hyperplanes in distinct points is a $(n+1)$-secant line of every X_t, and is therefore contained in X_t for some $t \in \mathbf{P}^1$.

Set $g = x_0^{n+1} + \cdots + x_n^{n+1} - (n+1)x_0 \dots x_n$ and let X_{n+1} be the hypersurface defined by g. Let L be the line which passes through the points $p = (1 : \cdots : 1)$ and $q = (1 : \omega : \cdots : \omega^n)$ where w is a primitive $(n+1)$-th root of unity. Then L lies on X_{n+1}: for every $(s : t) \in \mathbf{P}^1$,

$$\begin{aligned} g(tp + sq) &= \sum_{0 \leq i \leq n} (t + w^i s)^{n+1} - (n+1) \prod_{0 \leq i \leq n} (t + w^i s) \\ &= \sum_{0 \leq k \leq n+1} \left(\binom{n+1}{k} \sum_{0 \leq i \leq n} w^{ik} - (n+1) \sum_{i_1 < \cdots < i_k} w^{i_1} \dots w^{i_k} \right) t^{n+1-k} s^k \\ &= 0. \end{aligned}$$

(note that $\sum_{i_1 < \cdots < i_k} w^{i_1} \dots w^{i_k}$ is 0 if $1 \leq k \leq n$ and 1 if $k = n+1$). We show that

$$N_{L/X_{n+1}} = \mathcal{O}_L^{n-3} \oplus \mathcal{O}_L(-2).$$

Since w is a primitive root, L intersects the coordinate hyperplanes in distinct points. By the argument of the previous paragraph, there is an irreducible family G of dimension $\geq n-2$ of lines in $\mathbf{P}^n$ containing $[L]$ such that each line in the family is contained in some X_t. We have $\dim G \ \cap \ \dim R_1(X_{n+1}) \geq n-3$ hence

$h^0(L, N_{L/X_{n+1}}) \geq n-3$. If $N_{L/X_{n+1}} = \oplus_{i=1}^{n-2}\mathcal{O}_L(a_i)$, then $\sum a_i = -2$. So to show the normal bundle has the given form, it is enough to show that $h^1(L, N_{L/X_{n+1}}(-1)) = 2$.

Twisting the exact sequence of normal bundles with $\mathcal{O}_L(-1)$, we get the following short exact sequence

$$0 \to N_{L/X_{n+1}}(-1) \to N_{L/\mathbf{P}^n}(-1) \to \mathcal{O}_L(n) \to 0,$$

If we show the image of the map $H^0(L, N_{L/\mathbf{P}^n}(-1)) \to H^0(L, \mathcal{O}_L(n))$ has dimension $n-1$, then we can conclude that $h^1(L, N_{L/X_{n+1}}(-1)) = 2$. We have a diagram

$$\begin{tikzcd} H^0(L, \mathcal{O}_L^{n+1}) \arrow[r, two heads] \arrow[rrr, bend right, "\phi"'] & H^0(L, T_{\mathbf{P}^n}(-1)|_L) \arrow[r, two heads] & H^0(L, N_{L/\mathbf{P}^n}(-1)) \arrow[r] & H^0(L, \mathcal{O}_L(n)) \end{tikzcd}$$

where the map ϕ is given by $(\frac{\partial g}{\partial x_0}, \dots, \frac{\partial g}{\partial x_n})$, so

$$\frac{\partial g}{\partial x_i}|_L = (n+1)(x_i^n - x_0 \dots \widehat{x_i} \dots x_n)|_L = (n+1) \sum_{1 \leq k \leq n-1} \left(\binom{n}{k} + (-1)^{k+1}\right) w^{ik} t^{n-k} s^k$$

The Vandermonde determinant shows that the image of ϕ is $(n-1)$-dimensional, so the result follows.

Since the dimension of $G \cap R_1(X_{n+1})$ at $[L]$ is $\geq n-3$, and since the Zariski tangent space to the space of lines on X_{n+1} at $[L]$ is isomorphic to $H^0(L, N_{L/X_n+1})$, the dimension of $G \cap R_1(X_{n+1})$ at $[L]$ is equal to $n-3$. So for a general t, $\dim G \cap R_1(X_t) = n-3$ as well. Therefore, by upper semicontinuity, for a general t and for a general $[L]$ in any irreducible component of $G \cap R_1(X_t)$, N_{L/X_t} has the given splitting type.

$\square$

3. Proof of the main theorem

To prove Theorem 1.2, we need the following result:

Theorem 3.1. *Suppose that X is a smooth hypersurface of degree d in $\mathbf{P}^n$, and $d \geq 4$, or $d = 3$ and $n \geq 5$. If X' is a very general hyperplane section of X, then $H^{n-2}(X', \mathbf{Q})_{\text{prim}}$ does not have any non-trivial Hodge substructure.*

Peters and Steenbrink [9, Corollary 10.23] prove that the primitive part of the middle cohomology of a very general hypersurface of degree d in $\mathbf{P}^n$ has no non-trivial Hodge substructure if $d \geq 4$, or if $d = 3$ and $n \geq 4$. The same proof works in the situation of the above theorem and the reasoning goes roughly as follows. Let $U \subset (\mathbf{P}^n)^\vee$ denote the open set parametrizing smooth hyperplane sections of X, and let $u_0 \in U$ corresponds to X'. Then there is a monodromy action

$$\pi_1(U, u_0) \to \text{Aut } H^{n-2}(X', \mathbf{C})$$

which leaves $H^{n-2}(X', \mathbf{C})_{\text{prim}}$ stable and acts irreducibly on it. Let

$$\rho : \pi_1(U, u_0) \to \text{Aut } H^{n-2}(X', \mathbf{C})_{\text{prim}}$$

be the restriction of the monodromy action. Let Q denote the intersection paring on $H^{n-2}(X', \mathbf{C})_{\text{prim}}$ if n is even and $(-1)^{\frac{n-1}{2}}$ times the intersection paring if n is odd, so Q is a non-degenerate bilinear form which is either symmetric or anti-symmetric. Then any automorphism in the image of ρ respects Q, and if $d \geq 4$, or $d = 3$ and $n \geq 5$, then the Zariski closure of the image of ρ is exactly the set of automorphisms of $H^{n-2}(X', \mathbf{C})_{\text{prim}}$ which respect Q [9, Theorem 22], hence it is either the full orthogonal subgroup or the full symplectic subgroup of the group of automorphisms. Finally by [9, Theorem 20], if $\mathbf{V}$ is a rational variation of Hodge structure on a connected complex manifold S, and if for a point $s_0 \in S$ the identity connected component of the Zariski closure of the monodromy representation

$$\pi_1(S, s_0) \to \mathrm{GL}(\mathbf{V}_{\mathbf{s_0}})$$

acts irreducibly on $\mathbf{V}_{s_0} \otimes \mathbf{C}$, then for a very general $s \in S$, $\mathbf{V}_s$ has no non-trivial rational Hodge substructure.

Theorem 3.2. *Let X be a smooth hypersurface of degree $d \leq n$ in $\mathbf{P}^n$. Then the following hold.*

(a) *If R is an irreducible component of $R_1(X)$ such that the lines parametrized by R sweep out a subvariety of dimension a in X, and if*
$$d - 2 \geq \frac{(2a + d - 2n + 1)(2a + d - 2n + 2)}{2},$$
then R has the expected dimension $2n - d - 3$.

(b) *If R is an irreducible component of $R_e(X)$, $e = 1$ or 2, and if the curves parametrized by R sweep out a subvariety of codimension at most 1 in X, then R has the expected dimension $e(n + 1 - d) + n - 4$.*

Proof. (a) Assume to the contrary that $\dim R > 2n - d - 3$. Cutting X with a general linear subvariety of dimension $d - 1$ in $\mathbf{P}^n$, we get a smooth hypersurface X' of degree d in $\mathbf{P}^{n':=d-1}$. If we consider $R_1(X')$ as a subscheme of $R_1(X)$ and set $R' = R \cap R_1(X')$, then we have

$$\dim R' = \dim R - 2(n + 1 - d) \geq d - 4 = n' - 3.$$

If Y' denotes the subvariety of X' swept out by the lines parametrized by R' and if $a' = \dim Y'$, then $a' = a - (n + 1 - d)$. The inequality in the statement of the theorem now gives

$$n' - 1 \geq \frac{(2a' - n' + 2)(2a' - n' + 3)}{2},$$

so by Proposition 2.1, $H^{n'-1}(X', \mathbf{Q})$ has a non-trivial Hodge substructure which is a contradiction by Theorem 3.1.

(b) Assume to the contrary that there is an irreducible component R of $R_e(X)$ which is larger than expected, and the curves parametrized by R sweep out a divisor Y in X. Then the dimension of the family of curves parametrized by R which pass through an arbitrary point of Y is $\geq e(n + 1 - d)$. Let X' be a general hyperplane section of X, and let $R' = R_e(X') \cap R$. We show that R' is larger than expected (so its dimension is at least $e(n - d) + n - 4$) and that the curves parametrized by R' sweep out a divisor in X'. Once we prove this, an induction will show that the intersection of X with a general linear subvariety of dimension $d-1$ in $\mathbf{P}^n$ is a hypersurface of degree d in $\mathbf{P}^{d-1}$ in which the smooth curves of degree e sweep out a divisor. Applying Proposition 2.1 and Theorem 3.1 we get a contradiction.

To show that R' has the mentioned properties, it is enough to show that the curves parametrized by R' through a general point of $Y \cap X'$ has dimension $\geq e(n-d)$. If p is a point of Y, and if I is the incidence correspondence $\{(C,H)|\ [C] \in R, H \in (\mathbf{P}^n)^\vee, C \subset H, p \in C\}$, then projection to the first factor shows that $\dim I \geq e(n+1-d)+n-(e+1)$, so it is enough to show that the projection map from I to the space of hyperplanes which contain p is dominant. This is clear if $e=1$, so from now on we assume $e=2$.

Let p be a general point of Y, and let H be a general hyperplane which contains p. Let H' be a hyperplane which does not contain p. Then every conic C through p spans a 2-plan Γ_C which intersects H' alone a line L_C. For every family of lines in $\mathbf{P}^n$ which do not all pass through the same point, there is a subfamily of codimension at most 2 parametrizing lines which lie on H. So there are three possibilities:

(i) There is a curve C for which L_C lies on H.
(ii) The lines L_C form a family of dimension at most 1.
(iii) The lines L_C all pass though the same point.

If (i) holds, then we are done. If (ii) holds, then the family of conics though p would be of dimension at most 5. Since the family of conics though every point of Y has dimension at least $2(n-d+1)$, we have $d=n-1$ or n. And since $2(n+1-d) \geq 2$, there should be a 1-parameter family of conics through p which all lie on the same plane, so Y is covered by 2-planes. This is not possible because by [1, Theorem 2.1], when $d \geq n-1$ and X is smooth, a subvariety of $R_1(X)$ cannot be uniruled if the lines parametrized by it sweep out a divisor in X.

Assume now that (iii) holds, and let q be the point on H' through which all the lines L_C pass. We show that we can assume the line L through p and q is contained in Y. If L is not in Y, then by [5, Lemma 5.1], there is a family of dimension at least $2(n+1-d)-1$ of reducible conics through p and q on Y, so there is a family of dimension $\geq 2(n+1-d)-1$ of lines on Y through p. Since p is a general point of Y, the family of lines on Y is of dimension $\geq 2(n+1-d)-1+\dim Y-1 \geq 2n-d-2$ and we are reduced to the case $e=1$ of the theorem. So we can assume that L is contained in Y. This shows that for a general conic C parametrized by R and for a general point p on C, there is a line though p in Y whose intersection with C has length 2, therefore, the 2-plane spanned by C is in Y. By [1, Theorem 2.1], this is not possible if $d \geq n-1$. If $d \leq n-2$, then the space of lines in Y though a general point should be of dimension $\geq 2(n+1-d)-3$ which is larger than expected, so we are again reduced to the case of lines.

□

Corollary 3.3. *If $X \subset \mathbf{P}^n$ is a smooth Fano hypersurface of degree $d \leq 8$, then the space of lines on X has dimension $2n-d-3$.*

Proof. Intersecting X with general linear subvarieties of dimension d in $\mathbf{P}^n$, it is enough to consider the case when $d=n \leq 8$. Let R be an irreducible component of $R_1(X)$, and suppose that the lines parametrized by R sweep out a subvariety Y of dimension a in X. If $a = n-1$, then there is nothing to prove since in this case the normal bundle of a general line L parametrized by R is globally generated and therefore $h^0(L, N_{L/X}) = h^0(L, N_{L/\mathbf{P}^n}) - (d+1) = 2n-d-3$. If $a = n-2$, part (b) of Theorem 3.2 shows that F has the expected dimension. If $d \leq 8$ and $a \leq n-3$, then the inequality in part (a) of Theorem 3.2 is satisfied, so R has the expected dimension. □

References

[1] R. Beheshti, Lines on projective hypersurfaces. J. Reine Angew. Math. 592 (2006), 1–21.

[2] R. Beheshti and Mohan Kumar, Spaces of rational curves on complete intersections, preprint.

[3] I. Coskun and J. Starr, Rational curves on smooth cubic hypersurfaces. Int. Math. Res. Not. 2009, no. 24, 4626–4641.

[4] P. Griffiths, Periods of integrals on algebraic manifolds. II. Local study of the period mapping. Amer. J. Math. 90 1968 805–865.

[5] J.Harris, M. Roth, and J. Starr, Rational curves on hypersurfaces of low degree. J. Reine Angew. Math. 571 (2004), 73–106.

[6] A. Albano and S. Katz, Van Geemen's families of lines on special quintic threefolds. Manuscripta Math. 70 (1991), no. 2, 183–188.

[7] J. M. Landsberg, Lines on projective varieties. J. Reine Angew. Math. 562 (2003), 1–3.

[8] J.M. Landsberg and C. Robles, Fubini's theorem in codimension two. J. Reine Angew. Math. 631 (2009), 221–235.

[9] C. Peters and J. Steenbrink, Mixed Hodge structures, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, Berlin, 2008.

[10] T. Pugin, An Algebraic Circle Method, Ph.D. Thesis, Columbia University, 2011.

Department of Mathematics, Washington University in St. Louis, St. Louis, Missouri, 63130

E-mail address: beheshti@wustl.edu