Lines on Projective Hypersurfaces

Roya Beheshti

Abstract

We study the Hilbert scheme of lines on hypersurfaces in the projective space. The main result is that for a smooth Fano hypersurface of degree at most 6 over an algebraically closed field of characteristic zero, the Hilbert scheme of lines has always the expected dimension.

1 Introduction

Let $\mathbf{k}$ be an algebraically closed field and $X \subset \mathbf{P}^n_{\mathbf{k}}$ a projective variety. Denote by $F(X)$ the Hilbert scheme of lines on X. It is a subscheme of the Grassmannian of lines in $\mathbf{P}^n_{\mathbf{k}}$ and is called the Fano variety of lines on X. We are interested in studying $F(X)$ when X is a hypersurface. For general hypersurfaces, these schemes have been studied classically, but little is known when X is not general.

It is known that for a general hypersurface $X \subset \mathbf{P}^n_{\mathbf{k}}$ of degree d, $F(X)$ is smooth and has the expected dimension $2n-d-3$. It is also known that $F(X)$ may be reducible or non-reduced for particular hypersurfaces, even smooth ones. What can be said about dimension of $F(X)$ when X is an arbitrary smooth hypersurface? O. Debarre and J. de Jong, independently, asked the following question in this regard.

Question 1.1. *Let $\mathbf{k}$ be an algebraically closed field of characteristic p, and let $X \subset \mathbf{P}^n_{\mathbf{k}}$ be a hypersurface of degree d. Assume $d \leq n$, and if $p > 0$, assume furthermore that $d \leq p$. Does $F(X)$ have the expected dimension $2n-d-3$ whenever X is smooth?*

When p is positive and $d \geq p+1$, $F(X)$ does not always have the expected dimension. It is easy to see that the family of lines contained in the Fermat hypersurface of degree $p+1$ in $\mathbf{P}^n_{\mathbf{k}}$ with equation $\sum_{i=0}^n x_i^{p+1} = 0$ has dimension at least $2n-6$, which is larger than the expected dimension when p is odd ([4], 2.15). Also, it can be shown that if $\mathrm{char}(\mathbf{k}) = 0$ and X is the Fermat hypersurface of degree $d \geq n$ in $\mathbf{P}^n_{\mathbf{k}}$, then $\dim F(X) = n-3$, which is larger than the expected dimension for $d > n$ ([4], Exercise 2.5). Hence the assumptions in Question 1.1 are necessary.

In [7], Harris et al. gave a positive answer to the above question when $p = 0$ and d is very small with respect to n. The main result of this paper is the following.

Theorem 4.2. *Assume $\mathbf{k}$ has characteristic zero and X is any smooth hypersurface of degree $d \leq 6$ in $\mathbf{P}^n_{\mathbf{k}}$. Then $F(X)$ has the expected dimension when $d \leq n$.*

The case $d = 3$ of the above theorem is elementary. The case $d = 4$ is due to A. Collino; he proved in [3] that Question 1.1 holds true for all smooth quartic hypersurfaces when the characteristic of the base field is not 2 or 3. Also, the case $d = 5$ of the above theorem was proved by O. Debarre before, but our approach here is different from the previous ones and allows us to treat all cases $d \leq 6$ in a unified way.

This paper is organized as follows. The proof of the above theorem is given in Section 4. The results of Sections 2 and 3 are necessary for the proof of this theorem. In Section 2, we show that a subvariety of $F(X)$ that sweeps out a divisor in X cannot be uniruled when $d \geq n-1$. In Section 3, we discuss a theorem of J. Landsberg on the dimension of the family of lines having contact to a specific order with a hypersurface at a general point, and we use the same method used in the proof of his theorem to prove a proposition on the singularities of the second fundamental forms of hypersurfaces.

1.1 Conventions.

1. All schemes are considered over a fixed algebraically closed field of characteristic zero, and all points are closed points unless otherwise stated.
2. For any projective variety $X \subset \mathbf{P}^n$, $F(X)$ always denotes the Fano variety of lines on X.
3. For a scheme X and a sheaf $\mathscr{F}$ of $\mathscr{O}_X$-modules on X, we denote by $\mathscr{F}/\mathrm{tor}$ the sheaf obtained from dividing out $\mathscr{F}$ by its torsion.
4. If X is a scheme and Y is a closed subscheme of X with ideal sheaf $\mathscr{I}$, then we denote by $N_{Y/X}$ the normal sheaf of Y in X:

$$N_{Y/X} = \underline{\mathrm{Hom}}(\mathscr{I}/\mathscr{I}^2, \mathscr{O}_Y).$$

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2 Rational curves on the Fano variety of lines on complete intersections

A projective variety Y of dimension m is called uniruled if there exist a variety Z of dimension $m-1$ and a dominant rational map $\mathbf{P}^1 \times Z \dashrightarrow Y$.

Let $X \subset \mathbf{P}^n$ be a smooth complete intersection of type $(d_1, \ldots, d_l)$, and let $d = \sum_{i=1}^{l} d_i$. The purpose of this section is to prove the following theorem.

Theorem 2.1. *Let $\mathcal{Y}$ be an irreducible closed subvariety of $F(X)$ such that the lines corresponding to its points cover a divisor in X. If $d \geq n-1$, then $\mathcal{Y}$ is not uniruled.*

The assumption $d \geq n-1$ is necessary: let X be the Fermat hypersurface of degree $d \leq n-2$ in $\mathbf{P}^n$ given by the equation

$$x_0^d + x_1^d + \cdots + x_n^d = 0,$$

and let p be the point with coordinates $(0;\ldots;0;a;b)$ in X. Then the lines on X passing through p form a cone over the hypersurface $x_0^d + \cdots + x_{n-2}^d = 0$ in $\mathbf{P}^{n-2}$, and this hypersurface is uniruled, so we get a uniruled family of lines on X which sweeps out a divisor.

Remark 2.2. Let X be a smooth hypersurface. If $d = n$ and X is general, then $F(X)$ is irreducible and the normal bundle of a general line l on X is isomorphic to $\mathscr{O}_l^{n-3} \oplus \mathscr{O}_l(-1)$ ([9], V.4.4). Hence the lines on X sweep out a divisor and from the theorem above, we can conclude that $F(X)$ is not uniruled. If $d = n-1$, then X is always covered by lines and hence there are many such $\mathcal{Y}$. If $d > n$ and X is general, then the lines on X sweep out a subvariety of codimension at least 2, but for special hypersurfaces, it is possible that they cover a divisor in X; this happens for example in the case of Fermat hypersurfaces.

Proof of Theorem 2.1. Assume on the contrary that $\mathcal{Y}$ is covered by rational curves, and let $m = \dim X = n - l$. Without loss of generality, we can assume that our base field is uncountable. We define Σ_e to be the ruled surface $\mathbf{P}(\mathscr{O}_{\mathbf{P}^1} \oplus \mathscr{O}_{\mathbf{P}^1}(-e))$ over $\mathbf{P}^1$.

Let C be a rational curve on $F(X)$, and denote by $S \subset C \times X$ the family of lines parametrized by C. Let $\nu : \mathbf{P}^1 \to C$ be the normalization. The ruled surface $S \times_C \mathbf{P}^1$ is isomorphic to Σ_e for some nonnegative integer e, and we have the following diagram

$$\begin{array}{ccccc} f : \Sigma_e & \longrightarrow & S \subset C \times X & \longrightarrow & X \\ \downarrow & & \downarrow & & \\ \mathbf{P}^1 & \longrightarrow & C. & & \end{array}$$

Let $\mathcal{M}or(\Sigma_e, X)$ denote the scheme parametrizing morphisms from Σ_e to X. For every integer e, $\mathcal{M}or(\Sigma_e, X)$ has countably many irreducible components. Since our base field is uncountable and the lines corresponding to the points of $\mathcal{Y}$ cover a subvariety of codimension 1 in X, there is a nonnegative integer e_0 and an irreducible subvariety of $\mathcal{M}or(\Sigma_{e_0}, X)$, denoted by $\mathcal{Z}$, with the property that points of $\mathcal{Z}$ correspond to rational curves on $\mathcal{Y}$, and the image of the map

$$\begin{aligned} \phi : \mathcal{Z} \times \Sigma_{e_0} &\longrightarrow X \\ ([g], p) &\longmapsto g(p) \end{aligned}$$

is at least $(m-1)$-dimensional. For the rest of the argument, we put $e = e_0$ and $\Sigma = \Sigma_{e_0}$, and we let $[f]$ denote a general point in $\mathcal{Z}$.

There is an injective morphism from the tangent sheaf of Σ to the pullback of the tangent sheaf of X. Denote the quotient by $\mathscr{G}$

$$0 \to T_\Sigma \to f^*T_X \to \mathscr{G} \to 0. \qquad (1)$$

To get a contradiction, we compute $h^0(\Sigma, (\bigwedge^{m-3}\mathscr{G})/\mathrm{tor})$ in two different ways. First, we use the fact that deformations of f cover a divisor in X to show that $h^0(\Sigma, (\bigwedge^{m-3}\mathscr{G})/\mathrm{tor})$ is positive and then, we prove that it is zero, using some computations with exact sequences of powers of tangent sheaves.

Step 1: $h^0(\Sigma, (\bigwedge^{m-3}\mathscr{G})/\mathrm{tor}) > 0$. Let p be a general point of Σ and consider the evaluation map

$$\alpha : H^0(\Sigma, f^*T_X) \longrightarrow (f^*T_X)_p \cong T_{X,f(p)},$$

and the tangent map

$$\beta : T_{\Sigma,p} \longrightarrow T_{X,f(p)}$$

to f at p.

Lemma 2.3. *For a general point p of Σ the image of the map*

$$H^0(\Sigma, f^*T_X) \oplus T_{\Sigma,p} \xrightarrow{\alpha\oplus\beta} T_{X,f(p)}$$

is at least $(m-1)$-dimensional.

Proof. Since the image of ϕ is at least $(m-1)$-dimensional, the same is true for the image of the differential map $d_{([f],p)}\phi$ at the general point $([f], p)$. The Zariski tangent space to $\mathcal{Z}$ at $[f]$ is a subspace of the Zariski tangent space to $\mathcal{M}or(\Sigma, X)$ at $[f]$ which is isomorphic to $H^0(\Sigma, f^*T_X)$ ([9], I.2.8), and $d_{([f],p)}\phi$ is the restriction of $\alpha \oplus \beta$ to this subspace. This proves the lemma. □

Look at the following commutative diagram.

$$\begin{array}{ccc} H^0(\Sigma, T_\Sigma) & \longrightarrow & H^0(\Sigma, f^*T_X) \\ \downarrow & & \downarrow{\scriptstyle \alpha} \\ T_{\Sigma,p} & \xrightarrow{\beta} & T_{X,f(p)}. \end{array}$$

For a general point p of Σ, the map $H^0(\Sigma, T_\Sigma) \to T_{\Sigma,p}$ is surjective. Therefore, for such a point, $\beta(T_{\Sigma,p}) \subset \alpha(H^0(\Sigma, f^*T_X))$. So by Lemma 2.3, the image of α is at least $(m-1)$-dimensional. Consider now sequence (1). We have shown that the image of α is at least $(m-1)$-dimensional. This implies that the global sections of f^*T_X generate a subspace of dimension at least $m-1$ at a general point of Σ. Therefore global sections of $\mathscr{G}$ generate a subspace of dimension at least $m-3$ at a general point of Σ and hence $h^0(\Sigma, (\bigwedge^{m-3}\mathscr{G})/\mathrm{tor}) > 0$.

Step 2: $h^0(\Sigma, (\bigwedge^{m-3}\mathscr{G})/\mathrm{tor}) = 0$. From sequence (1) we get a morphism

$$\begin{array}{ccc} \bigwedge^2 T_\Sigma \otimes (\bigwedge^{m-3}\mathscr{G})/\mathrm{tor} & \longrightarrow & \bigwedge^{m-1} f^*T_X \\ \xi_1 \wedge \xi_2 \otimes \bar{\eta}_1 \wedge \cdots \wedge \bar{\eta}_{m-3} & \longmapsto & \beta(\xi_1) \wedge \beta(\xi_2) \wedge \eta_1 \wedge \cdots \wedge \eta_{m-3}, \end{array}$$

where η_i is any lifting of $\bar{\eta}_i$ in sequence (1). Furthermore, this map is injective since it is injective at the generic point of Σ and $\bigwedge^2 T_\Sigma \otimes (\bigwedge^{m-3}\mathscr{G})/\mathrm{tor}$ is torsion free.

If we twist the above map with the canonical sheaf ω_Σ of Σ, we get another injective morphism

$$0 \longrightarrow (\bigwedge^{m-3}\mathscr{G})/\mathrm{tor} \longrightarrow \bigwedge^{m-1} f^*T_X \otimes \omega_\Sigma.$$

We compute $h^0(\Sigma, \bigwedge^{m-1} f^*T_X \otimes \omega_\Sigma)$ and show that it is zero. This will conclude the proof of Theorem 2.1. We have

$$\begin{aligned} h^0(\Sigma, \bigwedge^{m-1} f^*T_X \otimes \omega_\Sigma) &= h^2(\Sigma, \bigwedge^{m-1} f^*T_X^\vee) \\ &= h^2(\Sigma, f^*T_X \otimes \det f^*T_X^\vee) \\ &= h^2(\Sigma, f^*T_X \otimes \mathscr{O}_X(d-n-1)). \end{aligned}$$

Pulling back the sequence of tangent sheaves, we get the following exact sequence

$$0 \to f^*T_X \to f^*T_{\mathbf{P}^n}|_X \to \bigoplus_i f^*\mathscr{O}_X(d_i) \to 0. \tag{2}$$

Twisting the above sequence with $\det f^*T_X^\vee$ and applying the long exact sequence of cohomology, we see that to prove the assertion, it suffices to prove two things:

(*a*) $h^1(\Sigma, f^*\mathscr{O}_X(d_i + d - n - 1)) = 0$, for $1 \leq i \leq l$.

(*b*) $h^2(\Sigma, f^*T_{\mathbf{P}^n}|_X \otimes f^*\mathscr{O}_X(d-n-1)) = 0$.

Let F be the class of a fiber of $\pi : \Sigma \to \mathbf{P}^1$ and E the class of the section with $E^2 = -e$. Recall that the Picard group of Σ is the free abelian group generated by F and E, and the intersection products are given by

$$E^2 = -e, \quad F^2 = 0, \quad E \cdot F = 1.$$

Let $f^*\mathscr{O}_X(1) = aE + bF$. The image of a line of ruling of Σ under f is a line, so

$$1 = f^*\mathscr{O}_X(1) \cdot F = (aE + bF) \cdot F = a.$$

Also $f^*\mathscr{O}_X(1) \cdot E \geq 0$, so

$$-e+b=(E+bF)\cdot E\geq 0.$$

Let $d'_i = d_i + d - n - 1$. We have

$$f^*\mathscr{O}_X(d'_i) = d'_iE + bd'_iF.$$

Since the first cohomology group of the sheaf $f^*\mathscr{O}_X(d'_i)$ restricted to a fiber of π vanishes, we have $R^1\pi_*(f^*\mathscr{O}_X(d'_i)) = 0$, so

$$\begin{aligned} h^1(\Sigma, f^*\mathscr{O}_X(d'_i)) &= h^1(\mathbf{P}^1, \pi_*f^*\mathscr{O}_X(d'_i)) \\ &= h^1(\mathbf{P}^1, \mathscr{O}_{\mathbf{P}^1}(bd'_i)\otimes\pi_*(\mathscr{O}_\Sigma(d'_iE))) \\ &= h^1(\mathbf{P}^1, \mathscr{O}_{\mathbf{P}^1}(bd'_i)\otimes(\mathscr{O}_{\mathbf{P}^1}\oplus\mathscr{O}_{\mathbf{P}^1}(-e)\oplus\cdots\oplus\mathscr{O}_{\mathbf{P}^1}(-ed'_i))) \\ &= 0 \quad (\text{since } b\geq e). \end{aligned}$$

This proves (a).

If we pullback the dual of the Euler sequence of $\mathbf{P}^n$ and twist it with $f^*\mathscr{O}_X(d-n-1)$, we get the following exact sequence:

$$0\to f^*\mathscr{O}_X(d-n-1)\to f^*\mathscr{O}_X(d-n)^{n+1}\to f^*T_{\mathbf{P}^n}|_X\otimes f^*\mathscr{O}_X(d-n-1)\to 0. \quad (3)$$

So to prove (b), it is enough to show that:

(b') $h^2(\Sigma, f^*\mathscr{O}_X(d-n)) = h^0(\Sigma, f^*\mathscr{O}_X(n-d)\otimes\omega_\Sigma) = 0.$

To show that (b') is true, we observe that since $d\geq n-1$, the invertible sheaf $f^*\mathscr{O}_X(n-d)\otimes\omega_\Sigma$ has negative intersection with the divisor F and hence has no nonzero global sections. This completes the proof of Theorem 2.1. □

3 Lines having contact to a specific order with a hypersurface

Let $X\subset\mathbf{P}^n$ be a hypersurface of degree d. Take a system of homogeneous coordinates $(x_0;x_1;\ldots;x_n)$ on $\mathbf{P}^n$, and let P be a homogeneous polynomial of degree d vanishing on X. Fix a smooth point p in X. For $1\leq k\leq d$, let Y_p^k be the degree k hypersurface in $\mathbf{P}^n$ given by the homogeneous polynomial

$$\sum_{\substack{(i_1,\ldots i_k)\\ 0\leq i_1,\ldots,i_k\leq n}} \frac{\partial^k P}{\partial x_{i_1}\ldots\partial x_{i_k}}(p)\; x_{i_1}\ldots x_{i_k}. \quad (4)$$

Note that Y_p^1 is just the embedded tangent space to X at p. If $k=2$, then the restriction of (4) to the tangent space $T_{X,p}$ gives a quadric form which is called the *second fundamental form* of X at p.

Denote by Σ_p^k the scheme theoretic intersection of $Y_p^1, Y_p^2, \ldots, Y_p^k$. Since for a point $q=(q_0;\ldots;q_n)$ in $\mathbf{P}^n$ we have

$$P(p+\lambda q) = \sum_{k=0}^{d} \frac{\lambda^k}{k!} \sum_{\substack{(i_1,\ldots,i_k) \\ 0\le i_1,\ldots,i_k\le n}} \frac{\partial^k P}{\partial x_{i_1}\ldots\partial x_{i_k}}(p)\ \ q_{i_1}\ldots q_{i_k},$$

Σ_p^k is a cone with vertex p whose underlying space is the union of all lines in $\mathbf{P}^n$ passing through p and having contact to order k with X at p, and Σ_p^d is exactly the cone of lines on X passing through p.

Since Σ_p^k is the intersection of k hypersurfaces in $\mathbf{P}^n$, its expected dimension is $n-k$. The next theorem says that generally the expected dimension is obtained unless the whole cone lies in X.

Theorem 3.1 (Landsberg [10])**.** *Let $X \subset \mathbf{P}^n$ be a hypersurface, and let p be a general point of X. Any irreducible component of Σ_p^k which has dimension greater than $n-k$ is contained in X.*

We will use this theorem several times in the proof of our main theorem. For later use, we rephrase the above theorem in the following corollary.

Corollary 3.2. *Let $X \subset \mathbf{P}^n$ be a projective variety of dimension m, and let p be a general point of X. Denote by Σ_p the subvariety of X swept out by lines passing through p, and denote by r the dimension of Σ_p. Then Σ_p has degree at most $(m+1-r)!$ and it is contained in the proper intersection of the tangent plane to X at p and $m-r$ hypersurfaces of degrees $2, 3, \ldots, m+1-r$.*

Proof. Assume first that X is a hypersurface in $\mathbf{P}^n$. By Theorem 3.1, all the components of Σ_p^{n-r} are r dimensional, since otherwise Σ_p would be at least $r+1$ dimensional. Hence the hypersurfaces $Y_p^1, \ldots, Y_p^{n-r}$ intersect properly. The cone Σ_p is a component of this intersection and its degree is at most $(n-r)!$.

If X has codimension greater than 1 in $\mathbf{P}^n$, then we consider a general projection of X into $\mathbf{P}^{m+1}$. Since the projection map is generically finite, we get a hypersurface in $\mathbf{P}^{m+1}$ such that the lines passing through its general point sweep out a subvariety of dimension r and we are in the situation of the previous case.

□

In the next subsection, we recall some facts about frame bundles of hypersurfaces. We then prove a proposition on the singularities of the second fundamental form of a hypersurface at a general point. The proposition will be used later in the proof of Theorem 4.2.

3.1 Preliminaries on frame bundles.

For a given integer n, let GL_{n+1} be the space of invertible matrices of size $n+1$ over the base field. Let $\phi : \mathrm{GL}_{n+1} \to \mathbf{P}^n$ be the map given by $\phi([V_0, \ldots, V_n]) = V_0$, where $[V_0, \ldots, V_n]$ is the matrix with columns $V_0, \ldots, V_n$. Let Ω be the matrix of global 1-forms on GL_{n+1} given by

$$\Omega_f = f^{-1}\,\mathrm{d}f \quad \text{for} \quad f \in \mathrm{GL}_{n+1}.$$

Denote the entries of Ω by ω_{ij}, and let x_{ij} be the regular function on GL_{n+1} defined by $x_{ij}(f) = f_{ij}$ for $0 \le i, j \le n$. We have

$$\mathrm{d}x_{ij}(f) = \sum_{0 \le k \le n} f_{ik}\,\omega_{kj}(f) \quad \text{for} \quad f = (f_{ij})_{0 \le i,j \le n} \in \mathrm{GL}_{n+1}. \tag{5}$$

Notation. In what follows, $f = (f_{ij})_{0 \le i,j \le n}$ is always an element of GL_{n+1}. We denote the columns of f by $V_0, \dots, V_n$. Hence the i-th entry of V_j is f_{ij}.

Let X be an integral hypersurface of degree d in $\mathbf{P}^n$ given by the homogeneous polynomial P. For given integers $0 \le i_1, i_2, \dots, i_m \le n$, the regular function $r^m_{i_1,\dots,i_m}$ on GL_{n+1} is defined by

$$r^m_{i_1,\dots,i_m}(f) = \frac{\partial^m P}{\partial \vec{V}_{i_1} \dots \partial \vec{V}_{i_m}}(V_0), \qquad f = [V_0, \dots, V_n].$$

More precisely,

$$r^m_{i_1,\dots,i_m}(f) = \sum_{\substack{(j_1,\dots,j_m) \\ 0 \le j_1,\dots,j_m \le n}} f_{j_1,i_1} \dots f_{j_m,i_m} \frac{\partial^m P}{\partial x_{j_1} \dots \partial x_{j_m}}(V_0). \tag{6}$$

Since the ω_{ij} form a basis for the space of 1–forms at every point of GL_{n+1}, we can express the derivative of the functions $r^m_{i_1,\dots,i_m}$ as linear combinations of the ω_{ij}. The next lemma says what the coefficients of the combination are when $i_1 = \dots = i_m$.

Lemma 3.3. *For $0 \le i \le n$, we have*

$$\mathrm{d}r^m_{i,i,\dots,i} = \sum_{0 \le t \le n} r^{m+1}_{i,\dots,i,t}\,\omega_{t0} + m \sum_{0 \le t \le n} r^m_{i,\dots,i,t}\,\omega_{ti}.$$

Proof. By partial derivation, we have

$$\mathrm{d}r^m_{i,\dots,i} = \sum_{(j_1,\dots,j_m)} \sum_{1 \le l \le m} (f_{j_1,i} \dots \hat{f_{j_l,i}} \dots f_{j_m,i} \frac{\partial^m P}{\partial x_{j_1} \dots \partial x_{j_m}}(V_0))\,\mathrm{d}x_{j_l,i}$$
$$+ \sum_{(j_1,\dots,j_m)} f_{j_1,i} \dots f_{j_m,i}\,\mathrm{d}(\frac{\partial^m P}{\partial x_{j_1} \dots \partial x_{j_m}}(V_0)).$$

Hence the assertion follows from the equality

$$\mathrm{d}(\frac{\partial^m P}{\partial x_{j_1} \dots \partial x_{j_m}}(V_0)) = \sum_{0 \le t \le n} \frac{\partial^{m+1} P}{\partial x_{j_1} \dots \partial x_{j_m} \partial x_t}(V_0)\,\mathrm{d}x_{t0},$$

and equation (5). $\square$

Before going on, we need another easy lemma.

Lemma 3.4. *For $0 \leq i_1, \ldots, i_m \leq n$, we have $r^{m+1}_{i_1,\ldots,i_m,0} = (d-m) r^m_{i_1,\ldots i_m}$, where $d = \deg X$.*

Proof. By definition, we have

$$\begin{aligned} r^{m+1}_{i_1,\ldots,i_m,0} &= \sum_{(j_1,\ldots,j_m)} f_{j_1,i_1} \ldots f_{j_m,i_m} \Big(\sum_t f_{t0} \frac{\partial}{\partial x_t} \frac{\partial^m P}{\partial x_{j_1} \ldots \partial x_{j_m}}(V_0)\Big) \\ &= (d-m) \sum_{(j_1,\ldots,j_m)} f_{j_1,i_1} \ldots f_{j_m,i_m} \frac{\partial^m P}{\partial x_{j_1} \ldots \partial x_{j_m}}(V_0) \\ &= (d-m) r^m_{i_1,\ldots,i_m}. \end{aligned}$$

□

Let $\mathcal{F}_X$ be the set of all invertible matrices with columns $[V_0, \ldots, V_n]$ such that V_0 is a smooth point of the hypersurface X and $V_1, \ldots V_{n-1}$ are in the embedded tangent space to X at V_0, namely the hyperplane defined by the linear polynomial

$$\sum_t \Big(\frac{\partial P}{\partial x_t}(V_0)\Big)\, x_t.$$

The scheme $\mathcal{F}_X$ is a smooth locally closed subvariety of GL_{n+1}; it is a principal bundle over the smooth locus of X and is called the *frame bundle* of X.

Denote by ν_{ij} the image of ω_{ij} under the restriction map $H^0(\mathrm{GL}_{n+1}, \Omega^1_{\mathrm{GL}_{n+1}}) \to H^0(\mathcal{F}_X, \Omega^1_{\mathcal{F}_X})$.

Lemma 3.5. *At every point of $\mathcal{F}_X$, the following hold.*
(a) $\nu_{n0} = 0$.
(b) $\nu_{00}, \nu_{10}, \ldots, \nu_{n-1,0}$ are independent 1-forms.

Proof. (a) We have

$$r_0 = \sum_t f_{t0} \frac{\partial P}{\partial x_t}(V_0) = d\, P(V_0) = 0,$$

so $\mathrm{d}r_0 = 0$. Therefore, it follows from Lemma 3.3 that

$$0 = \mathrm{d}r_0 = \sum_{0 \leq t \leq n} r_{0t}\, \nu_{t0} + \sum_{0 \leq t \leq n} r_t\, \nu_{t0}. \tag{7}$$

Since $V_1, \ldots, V_{n-1}$ are in the tangent space to X at V_0, we have $r_1 = \cdots = r_{n-1} = 0$. Applying Lemma 3.4, we get the equalities

$$r_{0t} = r_t = 0 \quad \text{for} \quad 0 \leq t \leq n-1,$$

hence

$$2r_n\, \nu_{n0} = 0.$$

Because all the matrices in $\mathcal{F}_X$ are invertible, V_n is not in the tangent hyperplane at V_0 and so $r_n \neq 0$ and $\nu_{n0} = 0$.

To prove part (b), observe that the map $\phi : \mathrm{GL}_{n+1} \to \mathbf{P}^n$ factors through the map $\phi' : \mathrm{GL}_{n+1} \to \mathbf{A}^{n+1}$ and $\mathrm{d}x_{t0}$ is the pullback of $\mathrm{d}x_t$ under ϕ'. Let $X' \subset \mathbf{A}^{n+1}$ be the affine cone over X. The map $\phi'|_{\mathcal{F}_X} : \mathcal{F}_X \longrightarrow X'_{smooth}$ is surjective and its fibers are linear, so it is a smooth map. We know that $\mathrm{d}x_0, \ldots, \mathrm{d}x_n$ form a space of dimension n at every smooth point of X', hence the pullbacks of these 1-forms make an n-dimensional space at every point of $\mathcal{F}_X$. On the other hand, f is invertible and $f\Omega = \mathrm{d}f$. Therefore $\nu_{00}, \nu_{10}, \ldots, \nu_{n-1,0}, \nu_{n0}$ form an n-dimensional space at every point and by part (a), $\nu_{n0} = 0$. So the first n 1-forms are linearly independent.

□

For a point p in X, let Y_p^k and Σ_p^k be defined as in the beginning of this section.

Lemma 3.6. *For $f = [V_0, \ldots, V_n] \in \mathcal{F}_X$, the following hold.*

(a) $V_i \in \Sigma_{V_0}^k$ if and only if $r_i^1(f) = r_{i,i}^2(f) = \cdots = r_{i,i,\ldots,i}^k(f) = 0$.

(b) Assume that $V_1 \in \Sigma_{V_0}^k$. Then V_i is in the Zariski tangent space to $\Sigma_{V_0}^k$ at V_1 if and only if

$$r_i^1(f) = r_{1,i}^2(f) = \cdots = r_{1,1,\ldots,1,i}^k(f) = 0.$$

Proof. Part (a) follows from the definition. We prove part (b) in the case of $k = 2$. The proof in the general case is similar.

The point V_i is in the Zariski tangent space to $\Sigma_{V_0}^2$ at V_1 if and only if it is in the Zariski tangent spaces to the hypersurfaces $Y_{V_0}^1$ and $Y_{V_0}^2$ at V_1. The equations of these tangent spaces are given by

$$\sum_t (\frac{\partial P}{\partial x_t}(V_0))\, x_t = 0,$$

and

$$\begin{aligned} &\sum_t \left(\frac{\partial}{\partial x_t} (\sum_{m,j} \frac{\partial^2 P}{\partial x_m \partial x_j}(V_0) x_m\, x_j)(V_1) \right) x_t \\ &= 2 \sum_{t,m} \frac{\partial^2 P}{\partial x_t \partial x_m}(V_0) f_{m1} x_t. \end{aligned}$$

If we evaluate these two linear polynomials at V_i, we get $r_i(f)$ and $r_{1,i}(f)$. □

3.2 Singularities of the second fundamental form.

Let $X \subset \mathbf{P}^n$ be a hypersurface, and for a smooth point p in X, denote by Z_p^k the intersection of Y_p^k and the embedded tangent space Y_p^1 to X at p. In this subsection, we prove the following proposition on the singularities of Z_p^2 when p is general.

Proposition 3.7. *For a general point p of X, the singular points of Z_p^2 are singular points of Z_p^k for $2 \leq k \leq d = \deg X$.*[1]

To prove the proposition, we need the following lemma.

Lemma 3.8. *For a point $f = [V_0, \dots, V_n] \in \mathcal{F}_X$, V_j is a singular point of $Z_{V_0}^k$ if and only if*

$$r^k_{j,\dots,j,0}(f) = r^k_{j,\dots,j,1}(f) = \cdots = r^k_{j,\dots,j,n-1}(f) = 0.$$

Proof. We can assume Y_p^1 is given by $x_n = 0$. The lemma then follows easily from the definitions.

□

Proof of Proposition 3.7. Since Z_p^2 is a quadric, the singular points of Z_p^2 form a linear subvariety. By ([6], 2.6), this linear subvariety is contained in X and is a fiber of the Gauss map. Let s be the dimension of the singular locus of Z_p^2.

We restrict our functions to those matrices $f = [V_0, \dots, V_n]$ in $\mathcal{F}_X$ such that $V_1, \dots V_s$ are singular points of $Z_{V_0}^2$. So let $\mathcal{H}_X \subset \mathcal{F}_X$ be the set of those matrices such that V_0 is a general point of X and $V_1, V_2, \dots, V_s$ are singular points of $Z_{V_0}^2$, and let ν''_{ij} be the image of ν_{ij} under the restriction map $H^0(\mathcal{F}_X, \Omega^1_{\mathcal{F}_X}) \to H^0(\mathcal{H}_X, \Omega^1_{\mathcal{H}_X})$. Since $V_1, V_2, \dots, V_s$ are in X, it follows from Lemma 3.6 that

$$r^k_{j,\dots,j} = 0 \quad 1 \leq j \leq s,\ 1 \leq k \leq d,$$

therefore

$$0 = \mathrm{d}r_j = \sum_{0 \leq t \leq n-1} r_{jt}\, \nu''_{t0} + \sum_{0 \leq t \leq n} r_t\, \nu''_{tj} \qquad j = 1, \dots, s. \tag{8}$$

Since $V_1, \dots, V_{n-1}$ are in the tangent space to X at V_0, we have $r_1 = r_2 = \cdots = r_{n-1} = 0$. The matrix f is invertible, so V_n is not contained in the tangent space to X at V_0 and $r_n \neq 0$. Also, by the last lemma, we have $r_{j0} = \cdots = r_{j,n-1} = 0$, so equation (8) implies that $\nu''_{nj} = 0$. Applying Lemma 3.3, we get

[1] J. Landsberg mentioned to us that this proposition also follows from [11, Thm 3.1].

$$
\begin{aligned}
0 &= \mathrm{d}r_{jj} \\
&= \sum_{t=0}^{n} r^3_{j,j,t}\ \nu''_{t0} + 2\sum_{t=0}^{n} r^2_{j,t}\ \nu''_{tj} \\
&= \sum_{t=0}^{n-1} r^3_{j,j,t}\ \nu''_{t0}.
\end{aligned}
$$

By Lemma 3.5, the forms $\nu''_{00}, \ldots \nu''_{n-1,0}$ are linearly independent [Notice that in Lemma 3.5, we proved the independence of these forms on $\mathcal{F}_X$. The same proof works here by generic smoothness and the fact that the fibers of $\phi'|_{\mathcal{H}_X}$ are linear.], hence

$$r_{j,j,0} = \cdots = r_{j,j,n-1} = 0.$$

This implies that V_j is a singular point of Z^3_p. By repeating this argument, we see that $V_1, \ldots, V_s$ are singular points of all the Z^k_p. $\square$

4 Dimension of the Fano variety of lines on hypersurfaces

Let $X \subset \mathbf{P}^n$ be a hypersurface of degree d given by a homogeneous polynomial P. Denote by $\mathrm{G}(1,n)$ the Grassmannian which parametrizes lines in $\mathbf{P}^n$. Since X is a hypersurface, it is easy to describe $F(X)$ as a subscheme of $\mathrm{G}(1,n)$. Let S be the universal rank 2 subbundle of $\mathscr{O}^{n+1}_{\mathrm{G}(1,n)}$. The restriction of S to any point $[l]$ in $\mathrm{G}(1,n)$ is identified with the rank 2 linear subspace of $\mathbf{A}^{n+1}$ whose projective space is $l \subset \mathbf{P}^n$. Hence P gives rise to a section of $\mathrm{Sym}^d(S^\vee)$, and the scheme theoretic zero locus of this section is exactly $F(X)$. Therefore the ideal sheaf of $F(X)$ is locally generated by $d+1$ elements and if the corresponding global section of $\mathrm{Sym}^d(S^\vee)$ is regular, then the dimension of $F(X)$ is $\dim \mathrm{G}(1,n) - (d+1) = 2n-d-3$. We refer to the number $2n-d-3$ as the expected dimension of $F(X)$. This description of $F(X)$ shows that the dimension of $F(X)$ is always greater than or equal to $2n-d-3$.

The following well-known lemma asserts that for a general hypersurface X, the scheme $F(X)$ has the expected dimension.

Lemma 4.1. *For every hypersurface X of degree d in $\mathbf{P}^n$, $\dim F(X) \geq 2n-d-3$. For a general X, $F(X)$ has dimension $2n-d-3$ if $d \leq 2n-3$ and is empty otherwise.*

This result can be obtained by calculating the dimension of the tangent space to $F(X)$ at a general point. For a proof see [9, Thm V.4.3] or [5] where a more general statement on the space of linear subvarieties of complete intersections is given.

In this section we prove the following theorem.

Theorem 4.2. *If $X \subset \mathbf{P}^n$ is any smooth Fano hypersurface of degree $d \leq 6$, then $F(X)$ has the expected dimension $2n-d-3$.*

We should remark that for special smooth hypersurfaces, $F(X)$ might be nowhere reduced and so the dimension of the Zariski tangent space to $F(X)$ might be larger than the expected dimension of $F(X)$ at every point. For example, if X is the Fermat hypersurface of degree 4 in $\mathbf{P}^4$, then $F(X)$ has 40 1-dimensional components each with multiplicity 2 (see [4], Exercise 2.5). This example also shows that $F(X)$ can be reducible if X is not general. However, if $d \leq 2n-4$ and X is not a quadric surface, then $F(X)$ is always connected even if X is not smooth (see [9, Thm V.4.3] or [1]).

Reduction to the case of $n = d$. Before giving the proof of the theorem, we show that to prove Question 1.1 holds true for a given degree d, it is enough to consider only the case $n = d$.

Lemma 4.3. *If Question 1.1 is true for $d = n$, then it is true for $d \leq n$.*

Proof. Fix an integer d. We show that if $d \leq m$ and if the statement of the question holds for every smooth hypersurface of degree d in $\mathbf{P}^m$, then it holds for every smooth hypersurface of degree d in $\mathbf{P}^{m+1}$. Let X be a smooth hypersurface of degree d in $\mathbf{P}^{m+1}$, and let X' be a general hyperplane section of X. Since X' is smooth, by our assumption, $\dim F(X') = 2m-d-3$. Let $\mathcal{Y}$ be an irreducible component of $F(X)$. By the following lemma, either a codimension at most 2 subvariety of $\mathcal{Y}$ lies in $F(X')$ or all the lines corresponding to the points of $\mathcal{Y}$ pass through the same point x. In the former case, we get $\dim \mathcal{Y} \leq \dim F(X') + 2 = 2(m+1) - d - 3$. In the latter case, all these lines are contained in the intersection of X and the tangent hyperplane at x. Hence $\dim \mathcal{Y} \leq m-1$ and the equality holds if and only if X is a hyperplane, therefore

$$\dim \mathcal{Y} \leq m - 2 \leq 2(m+1) - d - 3.$$

□

Lemma 4.4. *Let $\mathcal{Y}$ be an irreducible subvariety of $\mathrm{G}(1,n)$. If there is a hyperplane $\Lambda \subset \mathbf{P}^n$ such that the intersection of $\mathcal{Y}$ and the family of lines in Λ has codimension greater than 2 in $\mathcal{Y}$, then all the lines corresponding to the points of $\mathcal{Y}$ pass through the same point.*

Proof. Let $\dim \mathcal{Y} = s$. Notice that by our hypothesis, $s \geq 3$, so the lines corresponding to the points of $\mathcal{Y}$ cannot all lie on the same plane. Therefore if we show that every two lines of $\mathcal{Y}$ intersect, we can conclude that all of them pass through the same point.

Let $I \subset (\mathbf{P}^n)^* \times \mathcal{Y}$ be the incidence correspondence, and let p and q be the projections from I to $(\mathbf{P}^n)^*$ and $\mathcal{Y}$ respectively. Every fiber of q is linear of dimension $n-2$, hence I is irreducible of dimension $s+n-2$. By our assumption, there is a fiber of p whose dimension is at most $s-3$, so p is not dominant and therefore any non-empty fiber of p has dimension at least $s+n-2-(n-1) = s-1$.

For a point $[l] \in \mathcal{Y}$, let $\Lambda_l \subset (\mathbf{P}^n)^*$ be the set of hyperplanes which contain l. Two lines l and l' in $\mathbf{P}^n$ intersect if and only if $\dim(\Lambda_l \cap \Lambda_{l'}) = n-3$. Let $I_l \subset \Lambda_l \times \mathcal{Y}$ be the incidence correspondence, and let p_l and q_l be the projections from I_l to Λ_l and $\mathcal{Y}$. We have shown that any fiber of p_l has dimension at least $s-1$. Thus $\dim I_l \geq s - 1 + \dim \Lambda_l = s+n-3$. So the dimension of any fiber of q_l is at least $s+n-3-\dim \mathcal{Y} = n-3$.

□

4.1 Proof of Theorem 4.2 for $d \leq 5$.

Lemma 4.5. *A smooth hypersurface of degree d in $\mathbf{P}^n$ is not covered by lines if $d \geq n$.*

Proof. Let $X \subset \mathbf{P}^n$ be a smooth hypersurface which is covered by lines. Let $I \subset X \times F(X)$ be the universal family of lines on X, and let $(q, [l])$ be a general point of I. Consider the following commutative diagram:

$$\begin{tikzcd} T_{I,(q,[l])} \arrow[r] \arrow[d, "dp_1"] & T_{F(X),[l]} \cong H^0(l, N_{l/X}) \arrow[d, "\phi"] \\ T_{X,q} \arrow[r, "\psi"] & N_{l/X} \otimes \kappa(q), \end{tikzcd}$$

where p_1 is the projection map from I to X, dp_1 is its differential, and ψ is the map induced by the natural map from the tangent sheaf of X to the normal sheaf of l in X.

Let $N_{l/X} \cong \mathscr{O}_l(a_1) \oplus \cdots \oplus \mathscr{O}_l(a_{n-2})$ be the decomposition of $N_{l/X}$ into line bundles. Since X is covered by lines, the dimension of the image of dp_1 is at least $n-1$, so the dimension of the image of $\psi \circ dp_1$ is at least $n-2$. Therefore $\dim \phi(H^0(l, N_{l/X})) \geq n-2$. This implies that each a_i should be non-negative. On the other hand, by the following exact sequence, $a_1 + \cdots + a_{n-2} = n - 1 - d$,

$$0 \to N_{l/X} \to N_{l/\mathbf{P}^n} \cong \mathscr{O}_l(1)^{n-1} \to N_{X/\mathbf{P}^n}|_l \cong \mathscr{O}_l(d) \to 0$$

thus $n - 1 - d \geq 0$.

□

We are now ready to prove Theorem 4.2 for $d \leq 5$.

$n = d = 3$**.** The lemma above shows that X is not covered by lines, so there are only finitely many lines on X.

$n = d = 4$**.** By the lemma above, the union of all lines on X form a subvariety of dimension at most 2. Any 2-dimensional subvariety of $\mathbf{P}^4$ which contains a 2-parameter family of lines is a linear subvariety. Since X is smooth, it cannot contain a plane: if X contains a plane, then we can assume that the plane is given by $x_0 = x_1 = 0$, so X is defined by an equation of the form $x_0P_0 + x_1P_1$, where P_0 and P_1 are degree 3 polynomials; this is not possible since any point in the intersection $x_0 = x_1 = P_0 = P_1 = 0$ would be a singular point of X. Hence $\dim F(X) = 1$.

$n = d = 5$**.** Assume on the contrary that $\dim F(X) \geq 3$. Let $\mathcal{Y}$ be an irreducible subvariety of $F(X)$ whose dimension is 3, and let X' be the subvariety of X swept out by the lines corresponding to the points of $\mathcal{Y}$. By Lemma 4.5, $\dim X' \leq 3$, and since a surface can contain at most a 2-parameter family of lines, X' is 3-dimensional. Since X' contains a 3-parameter family of lines, there is at least a 1-parameter family of lines passing through its general point. Corollary 3.2 yields that the degree of the cone of such lines is at most 2, hence it is a cone over a rational curve. This contradicts Theorem 2.1.

Remark 4.6. In [2], an alternative proof of the case $n = d = 5$ is given. It is shown, by computing the derivative of the Abel-Jacobi map, that if a smooth quintic threefold contains a 1-parameter family of lines, then its Abel-Jacobi map is nonzero. Hence such a hypersurface cannot be a general hyperplane section of a smooth hypersurface of degree 5 in $\mathbf{P}^5$.

4.2 Proof of Theorem 4.2 for $d = 6$.

Outline of proof. Without loss of generality, we can assume that our base field is uncountable, and by Lemma 4.3, it is enough to consider the case $n = d = 6$. Assume on the contrary that X is a smooth hypersurface of degree 6 in $\mathbf{P}^6$ such that $\dim F(X) > 2n - d - 3 = 3$. Let X' be the subvariety of X swept out by its lines. We show that our assumptions imply that X' has codimension 1 in X and the dimension of the family of lines on X' passing through a general point is 1. Let Σ be the cone of lines passing through a general point of X', and let C be a general hyperplane section of Σ. We use the results of Sections 2 and 3 to show that C is a non-rational curve of degree at most 6 which lies on a non-singular quadric surface in $\mathbf{P}^3$ (Step 1). Then we use the fact that deformations of Σ cover a codimension 1 subvariety of X to show that the space $H^0(\Sigma, \bigwedge^2 N_{\Sigma/X}/\mathrm{tor})$ is non-zero (Step 2), and finally we compute the dimension of this space directly and get a contradiction (Step 3).

Step 1.

Let $\mathcal{Y}$ be an irreducible component of $F(X)$ of dimension $s \geq 4$. Let $I \subset X \times \mathcal{Y}$ be the family of lines parametrized by $\mathcal{Y}$, and denote by π_X and $\pi_{\mathcal{Y}}$ the projections from I to X and $\mathcal{Y}$ respectively. Since $\mathcal{Y}$ is irreducible and the fibers of $\pi_{\mathcal{Y}}$ are lines, I is irreducible of dimension $s + 1$. Let $r = \dim \pi_X(I)$. Note that by Lemma 4.5, X is not covered by lines, hence $r \leq 4$. If $r = 3$, then the fiber of a general point in $\pi_X(I)$ has dimension $s + 1 - 3 \geq 2$, so $\pi_X(I)$ is a linear subvariety of dimension 3 in X. Since X is smooth, it cannot contain a 3-dimensional linear subvariety by a similar argument as in the case of $d = n = 4$. We will argue that $r = 4$ also leads to a contradiction.

Let $X' = \pi_X(I)$ and denote by p a general point of X'. If $r = \dim X' = 4$, then there is at least a 1-parameter family of lines on X' which pass through p.

Claim 4.7. *No 2-parameter family of lines on X' pass through p.*

Proof. Assume on the contrary that there is a 2-parameter family of such lines. Then by Corollary 3.2, the degree of the cone of lines on X' passing through p is at most 2. Hence it is a cone over a surface of degree at most 2. Any quadric surface is covered by rational curves, so we get a contradiction by Theorem 2.1. □

Let Σ be an irreducible cone of lines on X' passing through p. From the above claim, together with Corollary 3.2, we conclude that Σ is a surface of degree at most 6, and it sits in the proper intersection of two hypersurfaces of degrees 2 and 3 in the embedded tangent plane to X at p. Let C be a hyperplane section of Σ which does not pass through p. The above argument shows that C lies on a $\mathbf{P}^3$ and it is a component of the proper intersection of a quadric Q and a cubic T in $\mathbf{P}^3$.

Lemma 4.8. *With the same notation as above, we have the following:*
(a) C is not rational.
(b) The quadric Q is irreducible and non-singular.

Proof. Part (a) follows from Theorem 2.1. As for (b), if Q is singular at a point q, then by Proposition 3.7, q is contained in T and it is a singular point of T. We can assume that $q = (1; 0; 0; 0)$ and Q is the zero locus of the polynomial $x_1^2 - x_2x_3$ or the polynomial x_1x_2 depending on whether it is irreducible or not. Hence T is given by a polynomial of the form $x_0G_1 + G_2$, where G_1 and G_2 are homogeneous polynomials of degrees 2 and 3 in x_1, x_2, x_3. Therefore, the irreducible components of the intersection of Q and T are rational curves and in particular C is rational. This is not possible by part (a).

□

Step 2.

Recall that X' is a 4-dimensional subvariety of X and that there is a 1-parameter family of lines contained in X' passing through a general point. We denoted by p a general point of X' and by Σ an irreducible cone of lines on X' passing through p. We use these assumptions to show that $\bigwedge^2 N_{\Sigma/X}/\mathrm{tor}$ has a nonzero global section.

Proposition 4.9. *Let $X \subset \mathbf{P}^n$ be a projective variety. Let P be a polynomial, and let $\mathcal{G}$ be a closed subscheme of $\mathrm{Hilb}_P(X)$ whose general point is a reduced subvariety of X of dimension m. If the subschemes of X corresponding to the points of $\mathcal{G}$ cover a k-dimensional subvariety of X, then for a general point $[Y]$ of $\mathcal{G}$, we have $H^0(Y, (\bigwedge^{k-m} N_{Y/X})/\mathrm{tor}) \neq 0$.*

Proof. The proof is similar to the proof of Lemma 4.5. Let $I \subset X \times \mathcal{G}$ be the family of subschemes of X parametrized by $\mathcal{G}$, and denote by $(q, [Y])$ a general point of I. Note that the Zariski tangent space to $\mathrm{Hilb}_P(X)$ at $[Y]$ is isomorphic to $H^0(Y, N_{Y/X})$ ([9], I.2.8), and we have the following commutative diagram:

$$\begin{tikzcd} T_{I,(q,[Y])} \arrow[r] \arrow[d, "dp_1"] & T_{\mathcal{G},[Y]} \subset H^0(Y, N_{Y/X}) \arrow[d, "\phi"] \\ T_{X,q} \arrow[r, "\psi"] & N_{Y/X} \otimes \kappa(q), \end{tikzcd}$$

where $p_1 : I \to X$ is the projection map, dp_1 is its differential, and ψ is the dual of the map

$$\mathscr{I}_{Y/X} \otimes \kappa(q) \longrightarrow \mathfrak{m}_{X,q}/\mathfrak{m}_{X,q}^2.$$

By our assumption, the image of p_1 is k-dimensional and since $(q, [Y])$ is a general point of I, the same is true for the image of the differential map dp_1. Therefore the image of

$\psi \circ dp_1$ is at least $(k-m)$-dimensional. The diagram is commutative, so the image of ϕ is at least $(k-m)$-dimensional. This implies that the map $H^0(Y, \bigwedge^{k-m} N_{Y/X}) \to N_{Y/X} \otimes \kappa(q)$ is nonzero for a general point q in Y, hence $H^0(Y, \bigwedge^{k-m} N_{Y/X}/\,\mathrm{tor}) \neq 0$. □

Corollary 4.10. *In the situation of our problem, $H^0(\Sigma, \bigwedge^2 N_{\Sigma/X}/\mathrm{tor}) \neq 0$, and for a general point $q \in \Sigma$, there are global sections s_1 and s_2 of $N_{\Sigma/X}$ such that $(s_1 \wedge s_2)(q) \neq 0$.*

Proof. The statement is the consequence of the proposition above (and its proof for the second statement) with $k = 4$ and $m = 2$, along with the assumptions that our base field is uncountable and the cones of lines cover a 4-dimensional subvariety of X. □

Step 3.

In this part, we try to compute $h^0(\Sigma, \bigwedge^2 N_{\Sigma/X}/\mathrm{tor})$ directly and get a contradiction using Corollary 4.10.

Lemma 4.11. *Assume $Y \subset X$ are nonsingular varieties. Then for every subvariety Z of Y the sequence of normal sheaves*

$$0 \to N_{Z/Y} \to N_{Z/X} \to N_{Y/X} \otimes \mathscr{O}_Z \to 0$$

is exact.

Proof. It is enough to show that the exact sequence of conormal sheaves

$$\mathscr{I}_{Y/X} \otimes \mathscr{O}_Z \to \mathscr{I}_{Z/X} \otimes \mathscr{O}_Z \to \mathscr{I}_{Z/Y} \otimes \mathscr{O}_Z \to 0 \tag{9}$$

is exact on the left too and splits locally. Since X and Y are nonsingular, we have an exact sequence of $\mathscr{O}_Y$ modules

$$0 \to \mathscr{I}_{Y/X} \otimes \mathscr{O}_Y \xrightarrow{d} \Omega^1_X \otimes \mathscr{O}_Y \to \Omega^1_Y \to 0,$$

which splits locally. Therefore locally there exists a map $s : \Omega^1_X \otimes \mathscr{O}_Y \to \mathscr{I}_{Y/X} \otimes \mathscr{O}_Y$ such that $s \circ d = \mathrm{id}$. Now it is clear that the composition of the maps

$$\mathscr{I}_{Z/X} \otimes \mathscr{O}_Z \xrightarrow{d} \Omega^1_X \otimes \mathscr{O}_Z \xrightarrow{s \otimes \mathrm{id}} \mathscr{I}_{Y/X} \otimes \mathscr{O}_Z$$

splits sequence (9) locally. □

By the above lemma, the sequence of normal sheaves on Σ

$$0 \to N_{\Sigma/X} \to N_{\Sigma/\mathbf{P}^6} \to N_{X/\mathbf{P}^6}|_\Sigma \cong \mathscr{O}_\Sigma(6) \to 0 \tag{10}$$

is exact. Dividing the second exterior power of this sequence by torsions, we get the following short exact sequence

$$0 \to \bigwedge^3 N_{\Sigma/X}(-6)/\mathrm{tor} \to \bigwedge^3 N_{\Sigma/\mathbf{P}^6}(-6)/\mathrm{tor} \to \bigwedge^2 N_{\Sigma/X}/\mathrm{tor} \to 0. \qquad (11)$$

Also, by taking determinants of (10), we get the following isomorphism

$$\bigwedge^4 N_{\Sigma/\mathbf{P}^6}(-12)/\mathrm{tor} \cong \bigwedge^3 N_{\Sigma/X}(-6)/\mathrm{tor}. \qquad (12)$$

By step 1, $\deg \Sigma \leq 6$ and it is a cone over a non-rational curve. Hence $3 \leq \deg \Sigma \leq 6$. The case $\deg \Sigma = 3$ cannot happen because in this case C would be a curve of type $(1,2)$ on the non-singular quartic Q and any such curve is rational. So we have $\deg \Sigma \geq 4$. We now analyze different cases separately. In each case, first we compute $h^0(\Sigma, \bigwedge^2 N_{\Sigma/X}/\mathrm{tor})$ by applying the long exact sequence of cohomology to sequence (11), and then we use Corollary 4.10 to get a contradiction.

The case $\deg \Sigma = 4$ **:**

In this case, C is of type $(1,3)$ or $(2,2)$ as a divisor on Q. By Lemma 4.8, the former cannot happen because a curve of type $(1,3)$ on a quadric surface is rational. In the latter case, C is a complete intersection of two hyperplanes and two quadrics in $\mathbf{P}^6$, and

$$N_{\Sigma/\mathbf{P}^6} \cong \mathscr{O}_\Sigma(2)^2 \oplus \mathscr{O}_\Sigma(1)^2.$$

Hence we have $H^1(\Sigma, \bigwedge^3 N_{\Sigma/X}(-6)) = H^1(\Sigma, \mathscr{O}_\Sigma(-6)) = 0$, and $H^0(\Sigma, \bigwedge^3 N_{\Sigma/\mathbf{P}^6}(-6)) = H^0(\Sigma, \mathscr{O}_\Sigma(-1)^2 \oplus \mathscr{O}_\Sigma(-2)^2) = 0$. This implies, by (11), that $H^0(\Sigma, \bigwedge^2 N_{\Sigma/X}) = 0$, contradicting Corollary 4.10.

The case $\deg \Sigma = 5$ **:**

In this case, C is a possibly singular divisor of type $(2,3)$ on Q. Let U be the complement of the vertex of Σ, and let $\pi : U \to C$ be the projection map. We show that:

(a) $H^0(U, \bigwedge^3 N_{\Sigma/\mathbf{P}^6}(-6)|_U) = 0$.
(b) There is a sheaf of $\mathscr{O}_\Sigma$-modules $\mathscr{G}$ and a map

$$\phi : \bigwedge^3 N_{\Sigma/X}(-6)/\mathrm{tor} \longrightarrow \mathscr{G},$$

such that ϕ is an isomorphism on U and $H^1(\Sigma, \mathscr{G}) = 0$.

Let us first show how we can use these to get a contradiction. The map ϕ induces a map between $\mathrm{Ext}^1(\bigwedge^2 N_{\Sigma/X}/\mathrm{tor}, \bigwedge^3 N_{\Sigma/X}(-6)/\mathrm{tor})$ and $\mathrm{Ext}^1(\bigwedge^2 N_{\Sigma/X}/\mathrm{tor}, \mathscr{G})$. So we can extend sequence (11) to a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \bigwedge^3 N_{\Sigma/X}(-6)/\mathrm{tor} \arrow[r] \arrow[d, "\phi"] & \bigwedge^3 N_{\Sigma/\mathbf{P}^6}(-6)/\mathrm{tor} \arrow[r] \arrow[d, "\psi"] & \bigwedge^2 N_{\Sigma/X}/\mathrm{tor} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathscr{G} \arrow[r] & \mathscr{G}' \arrow[r] & \bigwedge^2 N_{\Sigma/X}/\mathrm{tor} \arrow[r] & 0,
\end{tikzcd}$$

and ψ is an isomorphism on U because ϕ is. Since $H^1(\Sigma, \mathscr{G}) = 0$, every global section of $\bigwedge^2 N_{\Sigma/X}/\mathrm{tor}$ is the image of a global section of $\mathscr{G}'$ and hence zero on U because $H^0(U, \bigwedge^3 N_{\Sigma/\mathbf{P}^6}(-6)|_U) = 0$. This contradicts Corollary 4.10.

Proof of Part (a): Denote by $\mathbf{P}^5 \subset \mathbf{P}^6$ a hyperplane that contains C and does not pass through the vertex of Σ. Recall that π is the projection map from U to C, so the normal sheaf of U in $\mathbf{P}^6$ is the pullback of the normal sheaf of C in $\mathbf{P}^5$ under π. Hence to show the assertion, it is enough to show that $\bigwedge^3 N_{C/\mathbf{P}^5}(-6)$ has no nonzero global sections. Since C lies on the nonsingular quadric Q in $\mathbf{P}^3$, $N_{C/\mathbf{P}^3}$ and hence $N_{C/\mathbf{P}^5}$ are locally free sheaves, and $\bigwedge^3 N_{C/\mathbf{P}^5} \cong N^\vee_{C/\mathbf{P}^5} \otimes \det N_{C/\mathbf{P}^5}$.

There is an exact sequence of conormal sheaves:

$$0 \to \mathscr{O}_C(-2) \oplus \mathscr{O}_C(-1)^2 \cong N^\vee_{Q/\mathbf{P}^5}|_C \to N^\vee_{C/\mathbf{P}^5} \to N^\vee_{C/Q} \to 0.$$

Since $\deg N^\vee_{C/Q} = -C^2 = -12$, from the sequence above we get $\deg(\det N_{C/\mathbf{P}^5}) = 32$. Tensoring the above exact sequence with $\det N_{C/\mathbf{P}^5} \otimes \mathscr{O}_C(-6)$, we get the following exact sequence

$$0 \to \det N_{C/\mathbf{P}^5} \otimes (\mathscr{O}_C(-8) \oplus \mathscr{O}_C(-7)^2) \to \bigwedge^3 N_{C/\mathbf{P}^5}(-6) \to \det N_{C/\mathbf{P}^5}(-6) \otimes N^\vee_{C/Q} \to 0.$$

Since the right and left terms of this sequence are direct sums of negative degree line bundles, they have no nonzero global sections. So the middle term has no nonzero global sections either.

Proof of Part (b): Since Σ lies on a $\mathbf{P}^4$, by Lemma 4.11, the sequence of normal sheaves

$$0 \to N_{\Sigma/\mathbf{P}^4} \to N_{\Sigma/\mathbf{P}^6} \to N_{\mathbf{P}^4/\mathbf{P}^6}|_\Sigma \cong \mathscr{O}^2_\Sigma(1) \to 0$$

is exact. Taking determinants in this sequence, together with equation (12), we get

$$\bigwedge^3 N_{\Sigma/X}(-6)/\mathrm{tor} \cong \bigwedge^2 N_{\Sigma/\mathbf{P}^4}(-10)/\mathrm{tor}.$$

Let $\mathscr{I}_C$ be the ideal sheaf of C in $\mathbf{P}^3$. Since

$$H^0(\mathbf{P}^3, \mathscr{I}_C(3)) \geq H^0(\mathbf{P}^3, \mathscr{O}_{\mathbf{P}^3}(3)) - H^0(C, \mathscr{O}_C(3)) = 6,$$

the homogeneous ideal of C in $\mathbf{P}^3$ is generated by equations of the quadric Q and two cubics, and the intersection of each of the cubics with Q is the union of C and a line, so we get the following exact sequence of $\mathscr{O}_{\mathbf{P}^3}$-modules

$$\mathscr{O}_{\mathbf{P}^3}(-4)^2 \to \mathscr{O}_{\mathbf{P}^3}(-3)^2 \oplus \mathscr{O}_{\mathbf{P}^3}(-2) \to \mathscr{O}_{\mathbf{P}^3} \to \mathscr{O}_C \to 0.$$

(This sequence is exact on the left too, and it is the minimal free resolution of $\mathscr{O}_C$ in $\mathbf{P}^3$, but we won't need that.) If $I \subset \mathbf{k}[x_0, \ldots, x_3]$ is the homogeneous ideal of C in $\mathbf{P}^3$, then the homogeneous ideal of Σ in $\mathbf{P}^4$ is generated by I in $\mathbf{k}[x_0,\ldots,x_4]$. Hence the above sequence gives a similar exact sequence for Σ in $\mathbf{P}^4$

$$\mathscr{O}_{\mathbf{P}^4}(-4)^2 \to \mathscr{O}_{\mathbf{P}^4}(-3)^2 \oplus \mathscr{O}_{\mathbf{P}^4}(-2) \to \mathscr{I}_\Sigma \to 0, \tag{13}$$

where $\mathscr{I}_\Sigma$ is the ideal sheaf of Σ in $\mathbf{P}^4$. Recall that from an exact sequence of modules $T \to M \to N \to 0$ over a ring A, we get an exact sequence of second exterior powers $T \otimes M \to \bigwedge^2 M \to \bigwedge^2 N \to 0$. So restricting sequence (13) to Σ, and then considering its second exterior power, we get the following exact sequence

$$\mathscr{O}_\Sigma(-7)^4 \oplus \mathscr{O}_\Sigma(-6)^2 \to \mathscr{O}_\Sigma(-5)^2 \oplus \mathscr{O}_\Sigma(-6) \to \bigwedge^2 \mathscr{I}_\Sigma/\mathscr{I}_\Sigma^2 \to 0.$$

Finally, if we dualize this sequence and twist it with $\mathscr{O}_\Sigma(-10)$, we get

$$0 \to (\bigwedge^2 \mathscr{I}_\Sigma/\mathscr{I}_\Sigma^2)^\vee \otimes \mathscr{O}_\Sigma(-10) \to \mathscr{O}_\Sigma(-5)^2 \oplus \mathscr{O}_\Sigma(-4) \xrightarrow{\eta} \mathscr{O}_\Sigma(-3)^4 \oplus \mathscr{O}_\Sigma(-4)^2. \tag{14}$$

Denote by $\mathscr{F}$ the image of η. We claim that $H^0(\Sigma,\mathscr{F}) = 0$ and $H^1(\Sigma,\mathscr{O}_\Sigma(-5)^2 \oplus \mathscr{O}_\Sigma(-4)) = 0$. The first part is clear since $\mathscr{F}$ is a subsheaf of $\mathscr{O}_\Sigma(-3)^4 \oplus \mathscr{O}_\Sigma(-4)^2$ which has no nonzero global sections. For the second part of the claim, notice that $\mathscr{I}_{C/Q} \cong \mathscr{O}_Q(-2,-3)$, so $H^1(Q,\mathscr{I}_{C/Q}(m)) = 0$ for every integer m ([8], III, Ex. 5.6). Therefore the map

$$H^0(\mathbf{P}^4,\mathscr{O}_{\mathbf{P}^4}(m)) \longrightarrow H^0(C,\mathscr{O}_C(m))$$

is surjective for every m. The above map factors through the map $H^0(\Sigma,\mathscr{O}_\Sigma(m)) \to H^0(C,\mathscr{O}_C(m))$ and hence this map is also surjective. So $H^1(\Sigma,\mathscr{O}_\Sigma(m)) = 0$ for every integer m.

Let $\mathscr{G} = (\bigwedge^2 \mathscr{I}_\Sigma/\mathscr{I}_\Sigma^2)^\vee \otimes \mathscr{O}_\Sigma(-10)$. It follows from the claim that $H^1(\Sigma,\mathscr{G}) = 0$. Notice that there is a natural map

$$\bigwedge^2 N_{\Sigma/\mathbf{P}^4}(-10) = (\bigwedge^2 (\mathscr{I}_\Sigma/\mathscr{I}_\Sigma^2)^\vee) \otimes \mathscr{O}_\Sigma(-10) \longrightarrow \mathscr{G},$$

and since $\mathscr{G}$ is torsion free, the above map induces a map

$$\phi : \bigwedge^2 N_{\Sigma/\mathbf{P}^4}(-10)/\mathrm{tor} \to \mathscr{G}.$$

Finally, $\mathscr{I}_U/\mathscr{I}_U^2$ is isomorphic to $\pi^*(\mathscr{I}_C/\mathscr{I}_C^2)$, and since the latter is locally free, ϕ is an isomorphism on U.

The case $\deg\Sigma = 6$:

In this case, Σ is the complete intersection of two hyperplanes, a quadric, and a cubic in $\mathbf{P}^6$, and

$$N_{\Sigma/\mathbf{P}^6} \cong \mathscr{O}_\Sigma(1)^2 \oplus \mathscr{O}_\Sigma(2) \oplus \mathscr{O}_\Sigma(3).$$

Therefore $H^1(\Sigma, \bigwedge^4 N_{\Sigma/\mathbf{P}^6}(-12)) = H^1(\Sigma, \mathscr{O}_\Sigma(-5)) = 0$, and $h^0(\Sigma, \bigwedge^3 N_{\Sigma/\mathbf{P}^6}(-6)) = h^0(\Sigma, \mathscr{O}_\Sigma^{\oplus 2} \oplus \mathscr{O}_\Sigma(-1) \oplus \mathscr{O}_\Sigma(-2)) = 2$. So from sequence (11) we get

$$h^0(\Sigma, \bigwedge^2 N_{\Sigma/X}) = 2.$$

Lemma 4.12. *Every nonzero global section of $N_{\Sigma/X}$ has finitely many zeros.*

Proof. Let $r \in H^0(\Sigma, N_{\Sigma/X})$ be a nonzero global section. Recall that $\det N_{\Sigma/X}$ is isomorphic to $\mathscr{O}_\Sigma(1)$ by (12). So by (11) we have an exact sequence:

$$0 \to \mathscr{O}_\Sigma(-5) \xrightarrow{\mu} \mathscr{O}_\Sigma^2 \oplus \mathscr{O}_\Sigma(-1) \oplus \mathscr{O}_\Sigma(-2) \to \bigwedge^2 N_{\Sigma/X} \to 0. \tag{15}$$

The map μ can be described in the following way. Let L_1, L_2, G, and K be homogeneous polynomials defining the two hyperplanes, the quadric, and the cubic in $\mathbf{P}^6$ whose intersection is Σ, and let P be a homogeneous polynomial defining X. There are homogeneous polynomials H_1, H_2, R, S such that

$$P = H_1\, L_1 + H_2\, L_2 + R\, G + S\, K.$$

Then the map μ in sequence (15) is given by

$$\mu : \mathscr{O}_\Sigma(-5) \xrightarrow{(H_1, H_2, R, S)} \mathscr{O}_\Sigma^2 \oplus \mathscr{O}_\Sigma(-1) \oplus \mathscr{O}_\Sigma(-2).$$

By the next lemma, there is $s \in H^0(\Sigma, N_{S/X})$ such that $r \wedge s$ is a nonzero global section of $\bigwedge^2 N_{\Sigma/X}$. Since $H^1(\Sigma, \mathscr{O}_\Sigma(-5)) = 0$, $r \wedge s$ is the image of some t in $H^0(\Sigma, \mathscr{O}_\Sigma^2 \oplus \mathscr{O}_\Sigma(-1) \oplus \mathscr{O}_\Sigma(-2))$. Without loss of generality, we can assume $t = (1, 0, 0, 0)$. So if $r(q) = 0$ for a point $q \in \Sigma$, then $H_2(q) = R(q) = S(q) = 0$. The set $\{q \in \Sigma \mid H_2(q) = R(q) = S(q) = 0\}$ is zero-dimensional, since otherwise there would be a point in the intersection of this set and the hypersurface $H_1 = 0$ and this point would be a singular point of X. Therefore r has finitely many zeros. □

Lemma 4.13. *For every nonzero $r \in H^0(\Sigma, N_{\Sigma/X})$, there exists $s \in H^0(\Sigma, N_{\Sigma/X})$ such that $r \wedge s \neq 0$.*

Proof. Let q be a general point of Σ. By Corollary 4.10, there are global sections s_1 and s_2 of $N_{\Sigma/X}$ such that $(s_1 \wedge s_2)(q) \neq 0$. So the image of the map $H^0(\Sigma, N_{\Sigma/X}) \to N_{\Sigma/X} \otimes \kappa(q)$ is at least 2-dimensional. On the other hand, since q is a general point, $r(q) \neq 0$, so there exists $s \in H^0(\Sigma, N_{\Sigma/X})$ such that $s(q)$ and $r(q)$ are linearly independent in $N_{\Sigma/X} \otimes \kappa(q)$. Such a global section satisfies $s \wedge r \neq 0$.

□

Let $\mathcal{H}$ be the Hilbert scheme of subschemes of X with the same Hilbert polynomial as Σ. Since $\dim X' = 4$ and we chose Σ to be a general cone of lines on X', the dimension of $\mathcal{H}$ at the point $[\Sigma]$ is at least 4, so $h^0(\Sigma, N_{\Sigma/X}) \geq 4$. Also, we showed that $h^0(\Sigma, \bigwedge^2 N_{\Sigma/X}) = 2$, so there are independent global sections r, s of $N_{\Sigma/X}$ such that $r \wedge s = 0$. Let C' be a hyperplane section of Σ which does not contain any zeros of

r. We have $(r \wedge s)|_{C'} = 0$ and $r|_{C'}$ is nowhere zero. Hence there exists a constant c such that $r|_{C'} = c\,s|_{C'}$. Therefore $r - c\,s$ is a global section of $N_{\Sigma/X}$ with a 1-dimensional set of zeros and so by the previous argument it is the zero section. This contradicts our assumption that r and s are linearly independent.

Remark 4.14. The above proof shows that for any smooth hypersurface of degree at least 6 in $\mathbf{P}^6$, the Fano variety of lines is at most 3-dimensional, and as it was mentioned earlier, any Fermat hypersurfaces of degree at least 6 in $\mathbf{P}^6$ contains a 3-dimensional family of lines.

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Max-Planck-Institut für Mathematik, Bonn 53111, Germany.
E-mail address : beheshti@mpim-bonn.mpg.de