1. If $\mathbf{a}$ and $\mathbf{b}$ are two vectors such that $\mathbf{b}$ has length 2 and $\mathbf{a} \cdot \mathbf{b} = 1$, then what is $3\mathbf{b} \cdot (4\mathbf{a} - \mathbf{b})$?

(a) 0

(b) 2

(c) -6

(d) 6

(e) 10

(f) 8

$|\vec{b}| = 2 \Longrightarrow \vec{b} \cdot \vec{b} = 4$

so

$3\vec{b} \cdot (4\vec{a} - \vec{b}) = 12\, \vec{b} \cdot \vec{a} - 3\, \vec{b} \cdot \vec{b} = 12 \times 1 - 3 \times 4 = 0$

2. The vectors $\mathbf{a} = -2\mathbf{i} + (t-1)\mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = \mathbf{j} + t\,\mathbf{k}$ are parallel when

(a) $t = 0$

(b) $t = \frac{1}{3}$

(c) $t = 1$ and $t = 2$

(d) all values of t

(e) no value for t

(f) $t = 2$

$\vec{a} = \langle -2, t-1, 2 \rangle$

$\vec{b} = \langle 0, 1, t \rangle$

3. If $\mathbf{a} \times \mathbf{b} = \langle 1, 1, -1 \rangle$ and $\mathbf{a} \cdot \mathbf{b} = \sqrt{3}$, then what is the angle between $\mathbf{a}$ and $\mathbf{b}$?

(a) $\frac{\pi}{6}$

(b) $\frac{\pi}{4}$

(c) $\cos^{-1}(\frac{2}{3})$

(d) $\frac{\pi}{3}$

(e) $\cos^{-1}(\frac{\sqrt{3}}{3})$

(f) $\cos^{-1}(\frac{3}{2})$

$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta \Rightarrow \sqrt{3} = |\vec{a}||\vec{b}| \sin\theta$

$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta \Longrightarrow \sqrt{3} = |\vec{a}||\vec{b}| \cos\theta$

$\Rightarrow 1 = \tan\theta$

4. What is the center and the radius of the sphere with equation

$$x^2 + y^2 + z^2 - 2x + 4z + \frac{11}{4} = 0?$$

(a) center: $(1, 0, 2)$, radius $= \frac{1}{2}$

(b) center: $(-1, 0, 1)$, radius $= \frac{5}{4}$

(c) center: $(-1, 0, 2)$, radius $= \frac{9}{4}$

(d) center: $(1, 0, -2)$, radius $= \frac{3}{2}$

(e) center: $(1, 0, 1)$, radius $= \frac{5}{4}$

(f) center: $(1, 0, -2)$, radius $= \frac{9}{4}$

$$0 = x^2 + y^2 + z^2 - 2x + 4z + \frac{11}{4} = (x-1)^2 - 1 + y^2 + (z+2)^2 - 4 + \frac{11}{4}$$

$$\Rightarrow (x-1)^2 + y^2 + (z+2)^2 = 5 - \frac{11}{4} = \frac{9}{4}$$

5. What is the area of the triangle with vertices $P(0,1,2)$, $Q(-1,2,2)$ and $R(4,-1,0)$?

(a) $\sqrt{2}$

(b) 2

(c) 4

(d) $\sqrt{8}$

(e) 8

(f) $\sqrt{3}$

$\vec{PQ} = \langle -1, 1, 0 \rangle$ $\quad\quad$ $\vec{PR} = \langle 4, -2, -2 \rangle$

$\vec{PQ} \times \vec{PR} = \langle -2, -2, -2 \rangle$

$$\text{area} = \frac{|\vec{PQ} \times \vec{PR}|}{2} = \frac{\sqrt{12}}{2} = \sqrt{3}$$

6. Let $P = (0, 4, 3)$, and let $Q(x_0, y_0, z_0)$ be the point on the plane $x + y + z = 1$ that is closest to P. What is y_0?

(a) 2

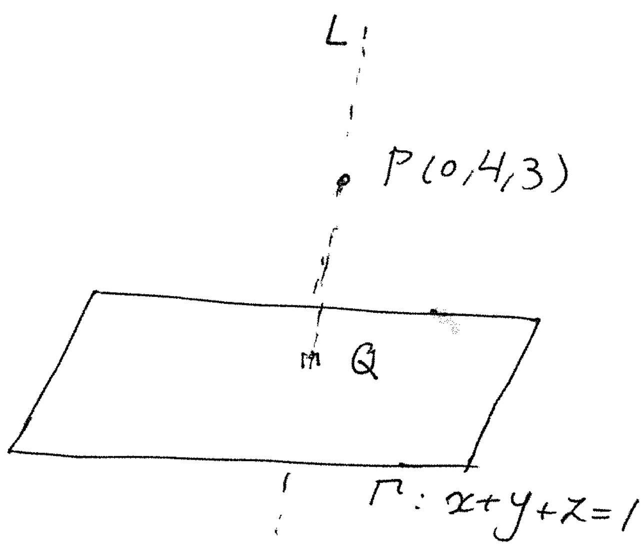
(b) -1

(c) 0

(d) 5

(e) 7

(f) -7



we need to find a line ~~parallel~~ perpendicular to the plane (or in other words parallel to $\vec{n} = \langle 1,1,1 \rangle$) through P

the line has parametric equation $x = t \quad y = 4+t \quad z = 3+t$

we find the intersection of the line with the plane to get Q : $t + (4+t) + (3+t) = 1 \Rightarrow 3t = -6 \Rightarrow t = -2$

$\Rightarrow Q = (-2, 2, 1)$

7. What is the volume of the parallelepiped formed by the three vectors $\langle 2,1,1\rangle$, $\langle -2,0,3\rangle$, and $\langle 0,1,1\rangle$?

(a) 1

(b) 2

(c) 3

(d) 4

(e) 5

(f) 6

$\vec{a} = \langle 2, 1, 1\rangle$

$\vec{b} = \langle -2, 0, 3\rangle$

$\vec{c} = \langle 0, 1, 1\rangle$

$$\text{volume} = \begin{vmatrix} 2 & 1 & 1 \\ -2 & 0 & 3 \\ 0 & 1 & 1 \end{vmatrix} = |2 \times (-3) - (-2) + (-2)| = 6$$

8. Given a point P and a nonzero vector $\mathbf{v}$, the set of points Q such that $\overrightarrow{PQ} \cdot \mathbf{v} = 2$ is

(a) A line through P parallel to $\mathbf{v}$

(b) A plane through P parallel to $\mathbf{v}$

(c) A plane through P perpendicular to $\mathbf{v}$

(d) A line parallel to $\mathbf{v}$ but not passing through P

(e) A plane perpendicular to $\mathbf{v}$ but not passing though P

(f) The line through P and Q

Let $P = (x_0, y_0, z_0)$, $\vec{v} = \langle a, b, c \rangle$, and $Q = (x, y, z)$.

then $\overrightarrow{PQ} \cdot \vec{v} = 2$ becomes

$$\langle x - x_0, y - y_0, z - z_0 \rangle \cdot \langle a, b, c \rangle = 2$$

or $$a(x - x_0) + b(y - y_0) + c(z - z_0) = 2$$

or $$ax + by + cz = \underbrace{2 + ax_0 + by_0 + cz_0}_{d}$$

this is the equation of a plane perpendicular to $\langle a, b, c \rangle$.

It does not pass through P.

9. Find the distance between the planes $x + y - 3z = 5$ and $-2x - 2y + 6z = 12$.

(a) $\sqrt{2}$

(b) $\frac{10}{\sqrt{11}}$

(c) $\sqrt{11}$

(d) $\frac{11}{\sqrt{65}}$

(e) 2

(f) 11

We find a point P on Γ_1 with equation $x+y-3z=5$ and
a point Q on Γ_2 " " $-2x-2y+6z=12$.

set $y=z=0$. then $x=5$, so $P(5,0,0)$ is on Γ_1
set $y=z=0$, then $x=-6$ so $(-6,0,0)$ " " Γ_2

$\vec{n_1} = \langle 1,1,-3\rangle$. $\vec{PQ} = \langle -11,0,0\rangle$

$$\text{distance} = \frac{|\vec{PQ}\cdot\vec{n_1}|}{|\vec{n_1}|} = \frac{11}{\sqrt{11}} = \sqrt{11}$$

10. What is the parametric equation of the line of intersection of the two planes given by equations $x+y+z=0$ and $2x-y+3z=3$?

(a) $x=1+4t, \quad y=-1-t, \quad z=-3t$

(b) $x=1+2t, \quad y=-1-t, \quad z=3t$

(c) $x=1+2t, \quad y=-1, \quad z=-3t$

(d) $x=4t, \quad y=-t, \quad z=-3t$

(e) $x=1+2t, \quad y=-1, \quad z=3t$

(f) $x=1+4t, \quad y=-1, \quad z=3t$

$\vec{n_1} = \langle 1,1,1 \rangle$

$\vec{n_2} = \langle 2,-1,3 \rangle$

$\vec{n_1} \times \vec{n_2} = \langle 4,-1,-3 \rangle$

A point on the plane can be found by setting $z=0$ for example. then $x+y=0$ and $2x-y=3$. Solving these for x and y we get $x=1$ $y=-1$, so $(1,-1,0)$ is on the line.

11. Let L_1 and L_2 be the lines determined by

$$\mathbf{r}_1(t) = \langle 3+2t,\ 1-7t,\ 2+5t\rangle$$
$$\mathbf{r}_2(t) = \langle 1-t,\ 7+3t,\ 4+t\rangle.$$

If Q is the intersection of L_1 and L_2, then what is the distance from Q to the origin.

(a) $\sqrt{11}$

(b) $\sqrt{14}$

(c) $\sqrt{66}$

(d) $\sqrt{290}$

(e) L_1 and L_2 are parallel.

(f) L_1 and L_2 are skew.

we solve:

$3+2t = 1-s \implies 9+6t = 3-3s$

$1-7t = 7+3s \qquad 1-7t = 7+3s$

$\left.\begin{matrix} 9+6t = 3-3s \\ 1-7t = 7+3s \end{matrix}\right\} \Rightarrow 10-t = 10 \Rightarrow t=0 \Rightarrow s=-2$

$2+5t = 4+s$

we see $t=0$, $s=-2$ satisfy all the equation. when $t=0$ we get $\vec{r}_1(0) = \langle 3,1,2\rangle$ so $\underbrace{(3,1,2)}_{Q}$ is the point of intersection

distance to the origin $= \sqrt{9+1+4} = \sqrt{14}$

12. A curve is given by $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$. If $P(x_0, y_0, z_0)$ is a point on the curve at which the tangent line is parallel to $\langle 4, 16, 48 \rangle$, then what is x_0?

(a) -1

(b) 0

(c) 2

(d) 4

(e) 5

(f) 16

$\vec{r}\,'(t) = \langle 1, 2t, 3t^2 \rangle$

If $\vec{r}\,'(t)$ is parallel to $\langle 4, 16, 48 \rangle$, then $2t = 4$, so

$t = 2 \Rightarrow \vec{r}(t) = \langle 2, 4, 8 \rangle$

13. Suppose that the four vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{a}$ are on the same plane. Show that

$$(\mathbf{u} \times \mathbf{v}) \times (\mathbf{w} \times \mathbf{a}) = \mathbf{0}.$$

(6 points)

Since both $\vec{u} \times \vec{v}$ and $\vec{w} \times \vec{a}$ are perpendicular to the same plane, they are parallel, so their cross product is the 0 vector.

14. Let $P = (2, 0, 1)$ and let L be the line defined by the vector equation

$$\mathbf{r}(t) = \langle -1 + t,\ 1 - t,\ 2t \rangle.$$

Answer the following questions. The questions do not build on each other and can be answered independently.

(a) Find parametric equations for the line passing through P that is parallel to L. (6 points)

(b) Find an equation for the plane passing through P that is perpendicular to L. (6 points)

(c) Find an equation for the plane passing through P that has no intersection with L. (5 points)

(d) Find two different planes whose intersection is L. (5 points)

(a) $x = 2 + t \qquad y = -t \qquad z = 1 + 2t$

(b) $\vec{n} = \langle 1, -1, 2 \rangle$ so the equation of the plane is $x - y + 2z = 1 \times 2 + (-1) \times 0 + 2 \times 1 = 4$

(c) It is enough to find a plane through P whose normal vector $\vec{n}$ is perpendicular to L.
Since L is parallel to $\langle 1, -1, 2 \rangle$, for example $\vec{n} = \langle 1, 1, 0 \rangle$ works. This gives.

$x + y = 1 \times 2 + 1 \times 0 + 0 \times 1 = 2 \qquad \boxed{x + y = 2}$

(d) There are many different ways to find 2 planes whose intersection is L. Note that it is enough to find two distinc planes which contain L. This can be done by picking 2 points on L and a 3rd point is space.

It can be also done by looking at the symmetric equation of L:

$$\frac{x+1}{1} = \frac{y-1}{-1} = \frac{z}{2}$$

so points of L satisfy $\frac{x+1}{1} = \frac{y-1}{-1}$, or $-x-1 = y-1$

or $\boxed{x+y=0}$

and $\frac{y-1}{-1} = \frac{z}{2}$ or $2(y-1) = -z$ or $\boxed{2y+z=2}$