1. Evaluate

$$\int_0^2 \int_0^1 4xy + 3x^2 \, dy \, dx.$$

(a) 2

(b) 1

(c) -8

(d) 8

(e) $\sqrt{3}$

(f) 12

$$\int_0^2 \int_0^1 4xy + 3x^2 \, dy \, dx = \int_0^2 2xy^2 + 3x^2 y \Big|_{y=0}^{y=1} dx = \int_0^2 2x + 3x^2 \, dx$$

$$= x^2 + x^3 \Big|_0^2$$

$$= 12$$

2. Which of the following integrals results from reversing the order of integration of

$$\int_0^2 \int_{x^2}^{\sqrt{8x}} x + y^3 \, dy \, dx?$$

(a) $\int_0^4 \int_{\frac{y^2}{8}}^{\sqrt{y}} x + y^3 \, dx \, dy$

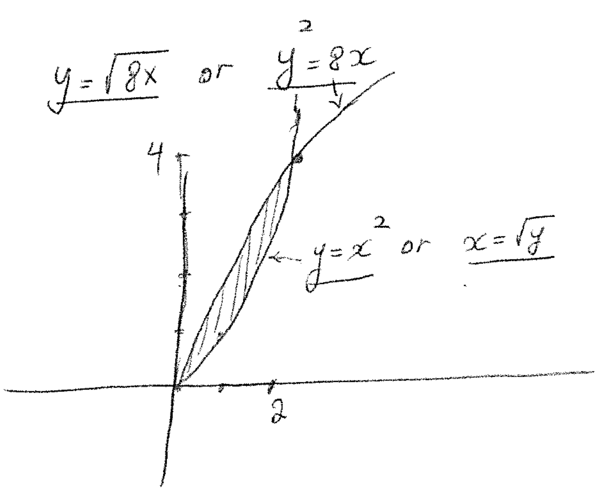
(b) $\int_0^8 \int_{\sqrt{y}}^{y^2} x + y^3 \, dx \, dy$

(c) $\int_0^2 \int_{y^2}^{\sqrt{8y}} x + y^3 \, dx \, dy$

(d) $\int_0^2 \int_{y^2}^{\sqrt{8y}} x^3 + y \, dx \, dy$

(e) $\int_1^2 \int_{\frac{y^2}{64}}^{y} x + y^3 \, dx \, dy$

(f) $\int_{-4}^4 \int_{-\sqrt{y}}^{\frac{\sqrt{2y}}{2}} x^3 + y \, dx \, dy$



$0 \le y \le 4,\quad \frac{y^2}{8} \le x \le \sqrt{y}$

3. Suppose D is the triangular region with vertices $(0,0)$, $(1,1)$, $(2,0)$. Find

$$\iint_D y\,dA.$$

(a) $\frac{3}{5}$

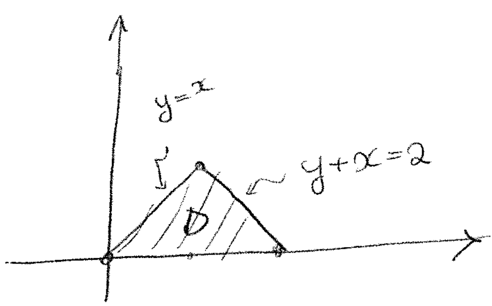
(b) $\frac{1}{3}$

(c) 0

(d) 1

(e) -2

(f) $\frac{5}{2}$



$$\iint_D y\,dA = \int_0^1 \int_y^{2-y} y\,dx\,dy$$

$$= \int_0^1 yx \Big|_{x=y}^{x=2-y} dy$$

$$= \int_0^1 y(2-y) - y^2\,dy$$

$$= y^2 - \frac{2y^3}{3}\Big|_0^1 = 1 - \frac{2}{3} = \frac{1}{3}$$

4. Estimate the volume of the solid that lies below the surface

$$z = 7x + 2y^2$$

and above the rectangle $[0, 2] \times [0, 4]$ using a Riemann sum with $m = n = 2$ and taking the sample points to be lower right corners.

(a) 64

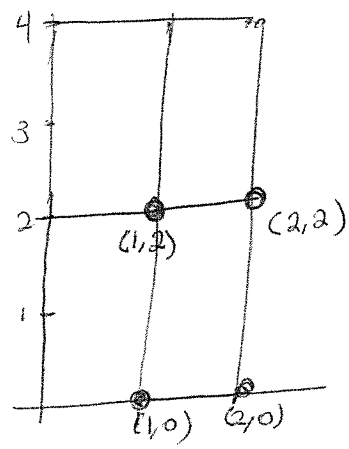
(b) 86

(c) 104

(d) 108

(e) 116

(f) 124



$\Delta A = 2$

$f(x,y) = 7x + 2y^2$

Volume $\approx \left(f(1,0) + f(2,0) + f(1,2) + f(2,2)\right) \Delta A$

$= (7 + 14 + 15 + 22)\ 2$

$= 58 \times 2 = 116$

5. Find the volume of the solid in the first octant enclosed by the surface $z = 4 - y^2$ and the plane $x = 3$.

(a) 4

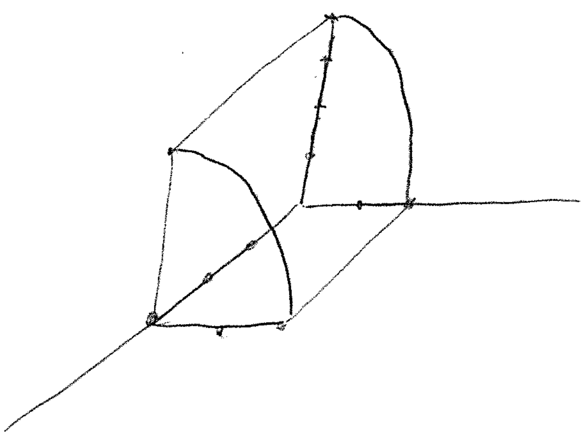
(b) 8

(c) $\frac{15}{4}$

(d) 16

(e) 18

(f) $\frac{25}{4}$



$$\text{volume} = \int_0^2 \int_0^3 4-y^2 \, dx \, dy$$

$$= \int_0^2 4x - y^2 x \Big|_{x=0}^{x=3} dy$$

$$= \int_0^2 12 - 3y^2 \, dy = 12y - y^3 \Big|_0^2$$

$$= 16$$

6. Find the area of the region bounded by the parabolas $x = y^2 - 1$ and $x = 2y^2 - 2$.

(a) $\frac{1}{2}$

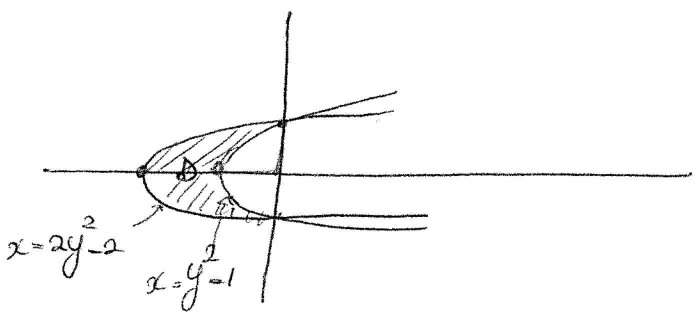
(b) $\frac{2}{3}$

(c) $\frac{4}{3}$

(d) $\sqrt{3}$

(e) 2

(f) 3



$D: \ -1 \le y \le 1 \qquad 2y^2-2 \le x \le y^2-1$

$$\text{area of } D = \iint_D 1 \, dA = \int_{-1}^{1} \int_{2y^2-2}^{y^2-1} 1 \, dx \, dy = \int_{-1}^{1} x \Big|_{x=2y^2-2}^{x=y^2-1} dy$$

$$= \int_{-1}^{1} y^2-1-(2y^2-2) \, dy$$

$$= -\frac{y^3}{3} + y \Big|_{-1}^{1} = -\frac{1}{3} + 1 - \left(\frac{1}{3} - 1\right)$$

$$= 2 - \frac{2}{3} = \frac{4}{3}$$

7. Let E be the solid between the paraboloid $y = x^2 + z^2$ and the plane $y = 1$. If

$$\iiint_E f(x,y,z)\,dV = \int_a^b \int_{g_1(z)}^{g_2(z)} \int_{h_1(x,z)}^{h_2(x,z)} f(x,y,z)\,dy\,dx\,dz,$$

then what is $g_2(z)$?

(a) $1 + z$

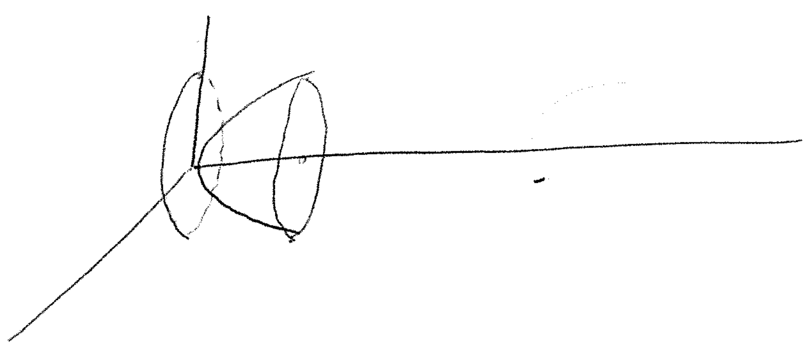
(b) z

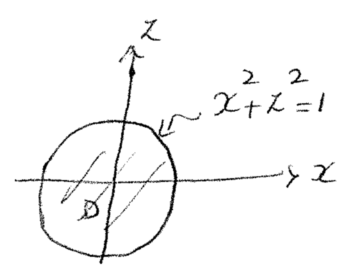
(c) 1

(d) $\sqrt{1 - z^2}$

(e) 0

(f) $x^2 + z^2$





$-1 \leq z \leq 1$

D: $-\sqrt{1-z^2} \leq x \leq \sqrt{1-z^2}$

8. Find the Jacobian of the transformation $x = \frac{1}{u}$, $y = \frac{u}{v}$.

(a) $\frac{1}{v}$

(b) 1

(c) $\frac{2}{v}$

(d) $\frac{v}{u^2}$

(e) $\frac{1}{uv^2}$

(f) $\frac{1}{u^2(u+v)}$

$\frac{\partial x}{\partial u} = \frac{-1}{u^2}$ $\quad \frac{\partial x}{\partial v} = 0$ $\quad \frac{\partial y}{\partial u} = \frac{1}{v}$ $\quad \frac{\partial y}{\partial v} = \frac{-u}{v^2}$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -\frac{1}{u^2} & 0 \\ \frac{1}{v} & -\frac{u}{v^2} \end{vmatrix} = \frac{u}{u^2v^2} = \frac{1}{uv^2}$$

9. Let D be the region in the first quadrant bounded above by $y = \sqrt{2x - x^2}$ and below by $y = x$. Find a desciption of D in polar coordinates.

(a) $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq r \leq 2\cos\theta$

(b) $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq r \leq 2$

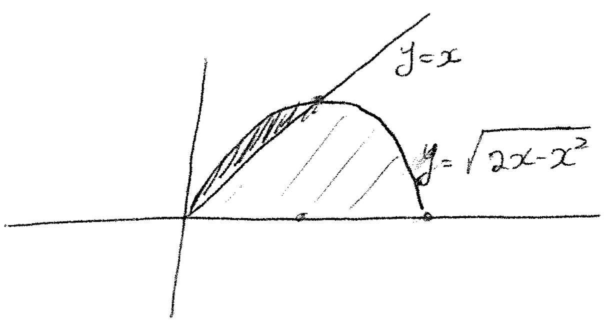
(c) $0 \leq \theta \leq \frac{\pi}{4}, \quad 0 \leq r \leq 2$

(d) $0 \leq \theta \leq \frac{\pi}{2}, \quad 2\cos\theta \leq r \leq 2$

(e) $0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq r \leq 2$

(f) $0 \leq \theta \leq \frac{\pi}{2}, \quad 2\sin\theta \leq r \leq 2\cos\theta$

$y = \sqrt{2x - x^2} \Rightarrow y^2 = 2x - x^2 \Rightarrow y^2 + (x-1)^2 = 1$



$D: \frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$ $\quad 0 \leq r \leq 2\cos\theta$

10. Find

$$\iint_D \frac{1}{\sqrt{x^2+y^2}}\, dA$$

where D is the region bounded by the circle of radius 1 and the curve $r = 3 + \cos\theta$.

(a) 2π

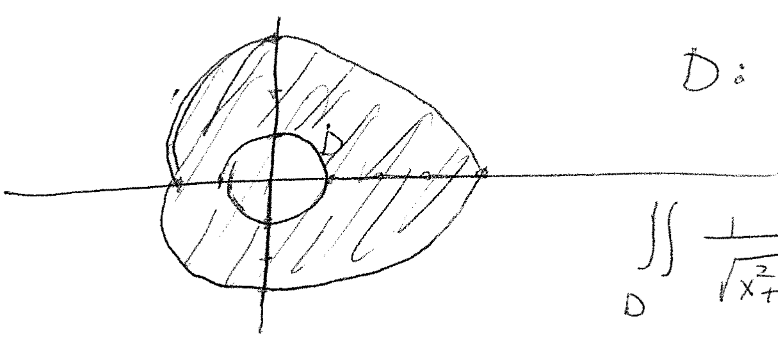
(b) 3π

(c) 4π

(d) 8

(e) 12

(f) 16



D: $0 \le \theta \le 2\pi$
$1 \le r \le 3+\cos\theta$

$$\iint_D \frac{1}{\sqrt{x^2+y^2}}\, dA = \int_0^{2\pi}\int_1^{3+\cos\theta} \frac{1}{r}\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} r \Big|_1^{3+\cos\theta} d\theta$$

$$= 2\theta + \sin\theta \Big|_0^{2\pi} = 4\pi$$

11. Find the volume of the solid bounded by the cone $z = \sqrt{x^2+y^2}$ and the paraboloid $2z = x^2 + y^2$.

(a) 4π

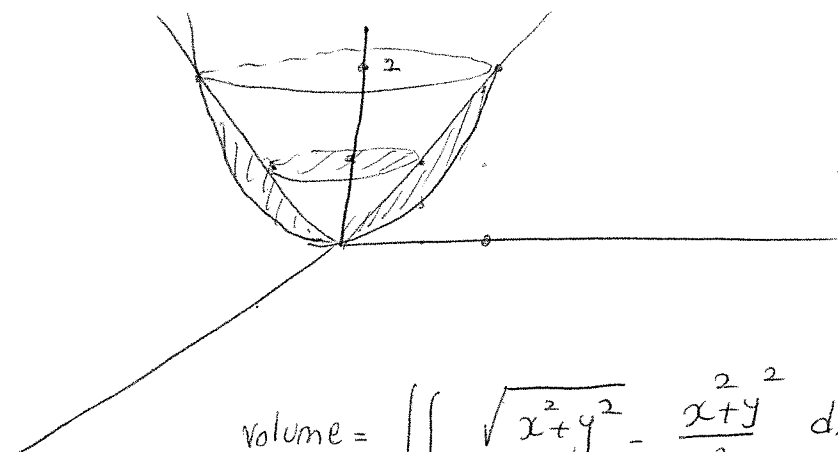
(b) $\frac{4\pi}{3}$

(c) $\frac{\pi}{32}$

(d) $\frac{\pi}{16}$

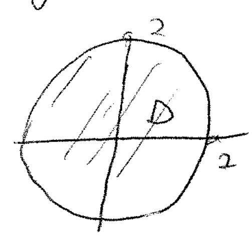
(e) $\frac{\pi}{4}$

(f) $\frac{\pi}{8}$



$$\frac{x^2+y^2}{2} = \sqrt{x^2+y^2}$$

$$\Rightarrow x^2+y^2 = 4$$



$$\text{Volume} = \iint_D \sqrt{x^2+y^2} - \frac{x^2+y^2}{2} \, dA$$

$$= \int_0^{2\pi}\int_0^{2} \left(r - \frac{r^2}{2}\right) r \, dr \, d\theta = \int_0^{2\pi} \frac{r^3}{3} - \frac{r^4}{8} \Big|_0^2 \, d\theta$$

$$= \int_0^{2\pi} \frac{8}{3} - 2 \, d\theta$$

$$= \frac{2}{3} \, 2\pi$$

12. Find the maximum value of the function $f(x, y) = x + y$ subject to the constraint $x^4 + 8y^4 = 6$.

(a) $\frac{3\sqrt{2}}{2}$

(b) $2\sqrt{2}$

(c) $3\sqrt{2}$

(d) $6\sqrt{2}$

(e) 6

(f) $\frac{6\sqrt{2}}{2}$

$g(x,y) = x^4 + 8y^4$

$\nabla f(x) = \lambda \nabla g(x)$

$$\begin{cases} 1 = 4x^3\lambda & \Rightarrow \lambda = \frac{1}{4x^3} \\ 1 = 32y^3\lambda & \Rightarrow \lambda = \frac{1}{32y^3} \\ x^4 + 8y^4 = 6 \end{cases} \Rightarrow \frac{1}{4x^3} = \frac{1}{32y^3}$$

$\Rightarrow x^3 = (2y)^3$

$\Rightarrow x = 2y$

$\Rightarrow (2y)^4 + 8y^4 = 6 \Rightarrow 24y^4 = 6 \Rightarrow y = \frac{1}{\sqrt{2}} \Rightarrow x = \sqrt{2}$

$\Rightarrow x + y = \sqrt{2} + \frac{1}{\sqrt{2}} = \frac{3}{\sqrt{2}}$

13. Use Lagrange multipliers to find the coordinates of the points on the cone $z^2 = x^2 + y^2$ that are closest to the point $(18, 8, 0)$. (12 points)

$d = \sqrt{(x-18)^2 + (y-8)^2 + z^2}$

$d^2 = (x-18)^2 + (y-8)^2 + z^2$ $\qquad f(x,y,z) = (x-18)^2 + (y-8)^2 + z^2$

$\underbrace{z^2 - x^2 - y^2}_{g(x,y,z)} = 0$

$$\begin{cases} 2(x-18) = -2\lambda x \\ 2(y-8) = -2\lambda y \\ 2z = 2\lambda z \\ z^2 - x^2 - y^2 = 0 \end{cases} \implies 2z(1-\lambda) = 0, \text{ so either } \lambda = 1 \text{ or } z = 0$$

If $z=0$, then since $z^2 = x^2 + y^2$, we have $x = y = 0$, but that is not possible since $2(x-18) \neq -2\lambda x$.

If $\lambda = 1$, then $2(x-18) = -2x \Rightarrow x = 9$
and $2(y-8) = -2y \Rightarrow y = 4$

$z^2 = x^2 + y^2 = 81 + 16 = 97$

so the closest points are $(9, 4, \pm\sqrt{97})$

14 (a) Let S be the region inside the ellipse $\frac{x^2}{9}+\frac{y^2}{16}=1$, and let T be the one-to-one transformation sending the rectangle $D : 0 \le u \le 1, 0 \le v \le 2\pi$ to S. Find the Jacobian of the transformation T. (6 points)

$x = 3u \cos v$

$y = 4u \sin v$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 3\cos v & -3u\sin v \\ 4\sin v & 4u\cos v \end{vmatrix} = 12u \cos^2 v + 12u \sin^2 v$$

$$= 12u$$

(b) If R is the region bounded by the curves $x+y=1$, $x+y=2$, $x-y=2$, and $x-y=4$, then use change of variables to find

$$\iint_R y^2 - x^2 \, dA.$$

(10 points)

$u = x+y$ $\quad 1 \le u \le 2$

$v = x-y$ $\quad 2 \le v \le 4$

$$\left.\begin{aligned} x &= \frac{u+v}{2} \\ y &= \frac{u-v}{2} \end{aligned}\right\} \Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

$$\iint_R y^2 - x^2 \, dA = \int_1^2 \int_2^4 -\frac{1}{2} uv \, dv \, du = \int_1^2 -\frac{v^2}{4} u \Big|_{v=2}^{v=4} du$$

$$= \int_1^2 -4u - (-u) \, du$$

$$= -\frac{3}{2} u^2 \Big|_1^2 = -6 - \left(-\frac{3}{2}\right)$$

$$= -6 + \frac{3}{2} = \frac{-9}{2}$$