Non-uniruledness results for spaces of rational curves in hypersurfaces

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Abstract

We prove that the sweeping components of the space of smooth rational curves in a smooth hypersurface of degree d in $\mathbf{P}^n$ are not uniruled if $(n+1)/2 \leq d \leq n-3$. We also show that for any $e \geq 1$, the space of smooth rational curves of degree e in a general hypersurface of degree d in $\mathbf{P}^n$ is not uniruled roughly when $d \geq e\sqrt{n}$.

1 Introduction

Throughout this paper, we work over an algebraically closed field of characteristic zero $\mathbf{k}$. Let X be a smooth hypersurface of degree d in $\mathbf{P}^n$, and for $e \geq 1$, let $R_e(X)$ denote the closure of the open subscheme of $\mathrm{Hilb}_{et+1}(X)$ parametrizing smooth rational curves of degree e in X. It is known that if $d < \frac{n+1}{2}$ and X is general, then $R_e(X)$ is an irreducible variety of dimension $e(n+1-d)+n-4$, and it is conjectured that the same holds for general Fano hypersurfaces (see [6] and [2]). If X is not general, $R_e(X)$ may be reducible. We call an irreducible component R of $R_e(X)$ a *sweeping component* if the curves parametrized by its points sweep out X or equivalently, if for a general curve C parametrized by R, the normal bundle of C in X is globally generated. If $d \leq n-1$, or if $d = n$ and $e \geq 2$, then $R_e(X)$ has at least one sweeping component.

In this paper, we study the birational geometry of sweeping components of $R_e(X)$. Recall that a projective variety Y of dimension m is called uniruled if there is a variety Z of dimension $m-1$ and a dominant rational map $Z \times \mathbf{P}^1 \dashrightarrow Y$. We are interested in the following question: for which values of n, d, and e, does $R_e(X)$ have non-uniruled sweeping components? Our original motivation for this study comes from the question of whether or not general Fano hypersurfaces of low indices are unirational.

We give a complete answer to the above question when $\frac{n+1}{2} \leq d \leq n-3$:

Theorem 1.1. *Let X be any smooth hypersurface of degree d in $\mathbf{P}^n$, $(n+1)/2 \leq d \leq n-3$. Then for all $e \geq 1$, no sweeping component of $R_e(X)$ is uniruled.*

We also consider the case $d = n-2$ and prove:

Theorem 1.2. *Let X be a smooth hypersurface of degree $n-2$ in $\mathbf{P}^n$, and let C be a smooth rational curve of degree e in X. Every irreducible sweeping component of $R_e(X)$ which contains C is non-uniruled provided that when we split the normal bundle of C in $\mathbf{P}^n$ as a sum of line bundles*

$$N_{C/\mathbf{P}^n} = \mathcal{O}_C(a_1) \oplus \cdots \oplus \mathcal{O}_C(a_{n-1}),$$

we have $a_i + a_j < 3e$ for every $1 \leq i < j \leq n-1$.

When $n = 5$ and $d = 3$, $R_e(X)$ is irreducible for any smooth X (see [2]). In [3], J. de Jong and J. Starr study the birational geometry of $R_e(X)$ with regards to the question of rationality of general cubic fourfolds. Let $\overline{\mathcal{M}}_{0,0}(X, e)$ be the Kontsevich moduli stack of stable maps of degree e from curves of genus zero to X and $\overline{M}_{0,0}(X, e)$ the corresponding coarse moduli scheme. There is an open subscheme of $\overline{M}_{0,0}(X, e)$ parametrizing smooth rational curves of degree e in X. Presenting a general method to produce differential forms on desingularisations of $\overline{M}_{0,0}(X, e)$, de Jong and Starr prove that if X is a general cubic fourfold, then $R_e(X)$ is not uniruled when $e > 5$ is an odd integer, and the general fibers of the MRC fibration of a desingularization of $R_e(X)$ are at most 1-dimensional when $e > 4$ is an even integer.

If X is a general cubic fourfold, then for a general rational curve C of degree e in X, the normal bundle of C in $\mathbf{P}^5$ is isomorphic to $\mathcal{O}_C(\frac{3e-1}{2})^{\oplus 4}$ if $e \geq 5$ is odd and to $\mathcal{O}_C(\frac{3e}{2})^{\oplus 2} \oplus \mathcal{O}_C(\frac{3e}{2} - 1)^{\oplus 2}$ if $e \geq 6$ is an even integer (see [3, Proposition 7.1]). Thus Theorem 1.2 gives a new proof of the result of de Jong and starr when $e \geq 5$ is odd. In section 4 we study the case when e is an even integer and show:

Theorem 1.3. *Let X be a smooth cubic fourfold, and let C be a general smooth rational curve of degree $e \geq 5$ in X.*

- *$R_e(X)$ is not uniruled if e is odd and $N_{C/\mathbf{P}^5} = \mathcal{O}_C(\frac{3e-1}{2})^{\oplus 4}$.*
- *If $\tilde{R}$ is a desingularization of $R_e(X)$, then the general fibers of the MRC fibration of $\tilde{R}$ are at most 1-dimensional if e is even and $N_{C/\mathbf{P}^5} = \mathcal{O}_C(\frac{3e}{2})^{\oplus 2} \oplus \mathcal{O}_C(\frac{3e}{2} - 1)^{\oplus 2}$.*

It is an interesting question whether or not the splitting type of $N_{C/\mathbf{P}^n}$ is always as above for a general rational curve C of degree ≥ 5 in an arbitrary smooth cubic fourfold.

Finally, we consider the case $d < \frac{n+1}{2}$. When $d^2 \leq n$, $R_e(X)$ is uniruled. In fact, in this range a much stronger statement holds: for every $e \geq 2$, the space of based, 2-pointed rational curves of degree e in X is rationally connected in a suitable sense (see [4] and [11]). By [6], when X is general and $d < \frac{n+1}{2}$, $\overline{M}_{0,0}(X, e)$ is irreducible and therefore it is birational to $R_e(X)$. Starr [12] shows that if $d < \min(n - 6, \frac{n+1}{2})$ and $d^2 + d \geq 2n + 2$, then for every $e \geq 1$, the canonical divisor of $\overline{\mathcal{M}}_{0,0}(X, e)$ is big. This suggests that when $d^2 + d \geq 2n + 2$ and X is general, $R_e(X)$ may be non-uniruled. In Section 5, we show:

Theorem 1.4. *Let $X \subset \mathbf{P}^n$ ($n \geq 12$) be a general hypersurface of degree d, and let $m \geq 1$ be an integer. If a general smooth rational curve C in X of degree e is m-normal (that is if the global sections of $\mathcal{O}_{\mathbf{P}^n}(m)$ maps surjectively to those of $\mathcal{O}_{\mathbf{P}^n}(m)|_C$), and if*

$$d^2 + (2m + 1)d \geq (m + 1)(m + 2)n + 2,$$

then $R_e(X)$ is not uniruled.

In particular, since every smooth curve of degree $e \geq 3$ in $\mathbf{P}^n$ is $(e - 2)$-normal, it follows that $R_e(X)$ is not uniruled when X is general and

$$d^2 + (2e - 3)d \geq e(e - 1)n + 2.$$

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2 A Consequence of Uniruledness

In this section, we prove a proposition, analogous to the existence of free rational curves on non-singular uniruled varieties, for varieties whose spaces of smooth rational curves are uniruled. We first fix notation and recall some definitions.

For a morphism $f : Y \to X$ between smooth varieties, by the *normal sheaf of* f we will mean the cokernel of the induced map on the tangent bundles $T_Y \to f^*T_X$.

If Y is an irreducible projective variety, and if $\widetilde{Y}$ is a desingularization of Y, then the maximal rationally connected (MRC) fibration of $\widetilde{Y}$ is a smooth morphism $\pi : Y^0 \to Z$ from an open subset $Y^0 \subset \widetilde{Y}$ such that the fibers of π are all rationally connected, and such that for a very general point $z \in Z$, any rational curve in $\widetilde{Y}$ intersecting $\pi^{-1}(z)$ is contained in $\pi^{-1}(z)$. The MRC fibration of any smooth variety exists and is unique up to birational equivalences [9].

Let Y be an irreducible projective variety, and assume the fiber of the MRC fibration of $\widetilde{Y}$ at a general point is m-dimensional. Then it follows from the definition that there is an irreducible component Z of $\mathrm{Hom}(\mathbf{P}^1, Y)$ such that the map $\mu_1 : Z \times \mathbf{P}^1 \to Y$ defined by $\mu_1([g], b) = g(b)$ is dominant and the image of the map $\mu_2 : Z \times \mathbf{P}^1 \times \mathbf{P}^1 \to Y \times Y$ defined by $\mu_2([g], b_1, b_2) = (g(b_1), g(b_2))$ has dimension $\geq \dim Y + m$.

Proposition 2.1. *Let $X \subset \mathbf{P}^n$ be a nonsingular projective variety. If an irreducible sweeping component R of $R_e(X)$ is uniruled, then there exist a smooth rational surface S with a dominant morphism $\pi : S \to \mathbf{P}^1$ and a generically finite morphism $f : S \to X$ with the following two properties:*

(i) If C is a general fiber of π, then $f|_C$ is a closed immersion onto a smooth curve parametrized by a general point of R.

(ii) If N_f denotes the normal sheaf of f, then $\pi_* N_f$ is globally generated.

Moreover, if the fiber of the MRC fibration of a desingularization of R at a general point is at least m-dimensional, then there are such S and f with the additional property that $\pi_* N_f$ has an ample subsheaf of rank $= m - 1$.

Proof. Let $U \subset R \times X$ be the universal family over R. Since R is uniruled, there exist a quasi-projective variety Z and a dominant morphism $\mu : Z \times \mathbf{P}^1 \to R$. Let $V \subset Z \times \mathbf{P}^1 \times X$ be the pullback of the universal family to $Z \times \mathbf{P}^1$, and denote by $q : V \to Z \times X$ and $p : V \to Z$ the projection maps.

Consider a desingularization $g : \widetilde{V} \to V$, and let $\widetilde{q} = q \circ g$ and $\widetilde{p} = p \circ g$. Let $z \in Z$ be a general point, and denote the fibers of p and $\widetilde{p}$ over z by S and $\widetilde{S}$ respectively. Let $f : S \to X$ be the restriction of q to S, and let $\widetilde{f} = f \circ g : \widetilde{S} \to X$. Since z is general, by generic smoothness, $\widetilde{S}$ is a smooth surface whose general fiber over $\mathbf{P}^1$ is a smooth connected rational curve. We claim that $\widetilde{S}$ and $\widetilde{f}$ satisfy the desired properties. The first property is clearly satisfied.

Since every coherent sheaf on $\mathbf{P}^1$ splits as a torsion sheaf and a direct sum of line bundles, to show that $\pi_* N_f$ is globally generated, it suffices to check that the restriction map $H^0(\mathbf{P}^1, \pi^* N_f) \to N_f|_b$ is surjective for a general point $b \in \mathbf{P}^1$, or equivalently, that the restriction map $H^0(S, N_f) \to H^0(C, N_f|_C)$ is surjective for a general fiber C. To show this, we consider the Kodaira-Spencer map associated to $\widetilde{V}$ at a general point $z \in Z$. Denote by $N_{\widetilde{q}}$ the normal sheaf of the map $\widetilde{q}$. We get a sequence of maps

$$T_{Z,z} \to H^0(\widetilde{S}, \widetilde{p}^* T_Z|_{\widetilde{S}}) \to H^0(\widetilde{S}, \widetilde{q}^* T_{X \times Z}|_{\widetilde{S}}) \to H^0(\widetilde{S}, N_{\widetilde{q}}|_{\widetilde{S}}).$$

Let b be a general point of $\mathbf{P}^1$. Composing the above map with the projection map $T_{Z\times\mathbf{P}^1,(z,b)} \to T_{Z,z}$, we get a map $T_{Z\times\mathbf{P}^1,(z,b)} \to H^0(\widetilde{S}, N_{\widetilde{q}}|_{\widetilde{S}})$. Note that if $N_{\widetilde{f}}$ denotes the normal sheaf of $\widetilde{f}$, then $N_{\widetilde{q}}|_{\widetilde{S}}$ is naturally isomorphic to $N_{\widetilde{f}}$. Also, if C is the fiber of $\pi : \widetilde{S} \to \mathbf{P}^1$ over b, then since b is general, C is smooth, and we have a short exact sequence

$$0 \to N_{C/\widetilde{S}} \to N_{\widetilde{f}(C)/X} \to N_{\widetilde{f}}|_C \to 0.$$

So we get a commutative diagram

$$\begin{array}{ccccc}
T_{Z\times\mathbf{P}^1,(z,b)} & \longrightarrow & T_{Z,z} & \longrightarrow & H^0(\widetilde{S}, N_{\widetilde{f}}) \\
\downarrow{\scriptstyle d\mu_{(z,b)}} & & & & \downarrow \\
T_{R,[\widetilde{f}(C)]} = H^0(\widetilde{f}(C), N_{\widetilde{f}(C)/X}) & & \longrightarrow & & H^0(C, N_{\widetilde{f}}|_C)
\end{array}$$

Since μ is dominant, and since R is sweeping and therefore generically smooth, $d\mu_{(z,b)}$ is surjective. Since the bottom row is also surjective, the map $H^0(\widetilde{S}, N_{\widetilde{f}}) \to H^0(C, N_{\widetilde{f}}|_C)$ is surjective as well. Thus $\widetilde{\pi}_* N_{\widetilde{f}}$ is globally generated.

Suppose now that R is uniruled and that the general fibers of the MRC fibration of R are at least m-dimensional. Let $\dim R = r$. Then there exists a morphism $\mu_1 : Z \times \mathbf{P}^1 \to R$ such that the image of

$$\mu_2 : Z \times \mathbf{P}^1 \times \mathbf{P}^1 \to R \times R$$

$$\mu_2(z, b_1, b_2) = (\mu_1(z, b_1), \mu_1(z, b_2))$$

has dimension $\geq r + m$. Constructing $\widetilde{S}$ and $\widetilde{f}$ as before, and if C_1 and C_2 denote the fibers of π over general points b_1 and b_2 of $\mathbf{P}^1$, then the image of the map

$$d\mu_2 : T_{Z\times\mathbf{P}^1\times\mathbf{P}^1,(z,b_1,b_2)} \to T_{R\times R,([\widetilde{f}(C_1)],[\widetilde{f}(C_2)])} = H^0(C_1, N_{\widetilde{f}(C_1)/X}) \oplus H^0(C_2, N_{\widetilde{f}(C_2)/X})$$

is at least $(r + m)$-dimensional. The desired result now follows from the following commutative diagram

$$\begin{array}{ccccc}
T_{Z\times\mathbf{P}^1\times\mathbf{P}^1,(z,b_1,b_2)} & \longrightarrow & T_{Z,z} & \longrightarrow & H^0(\widetilde{S}, N_{\widetilde{f}}) \\
\downarrow{\scriptstyle (d\mu_2)_{(z,b_1,b_2)}} & & & & \downarrow \\
T_{R\times R,([\widetilde{f}(C_1)],[\widetilde{f}(C_2)])} & & \longrightarrow & & H^0(C_1, N_{\widetilde{f}}|_{C_1}) \oplus H^0(C_2, N_{\widetilde{f}}|_{C_2}),
\end{array}$$

and the observation that the kernel of the bottom row is 2-dimensional. □

The above proposition will be enough for the proof of Theorem 1.1, but to prove Theorem 1.3 in the even case, we will need a slightly stronger variant. Let $f : Y \to X$ be a morphism between smooth varieties, and let N_f be the normal sheaf of f

$$0 \to T_Y \to f^* T_X \to N_f \to 0.$$

Suppose there is a dominant map $\pi : Y \to \mathbf{P}^1$, and let M be the image of the map induced by π on the tangent bundles $T_Y \to \pi^* T_{\mathbf{P}^1}$. Consider the push-out of the above sequence by the map $T_Y \to M$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T_Y & \longrightarrow & f^*T_X & \longrightarrow & N_f & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow = & & \\
0 & \longrightarrow & M & \longrightarrow & N_{f,\pi} & \longrightarrow & N_f & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

The sheaf $N_{f,\pi}$ in the above diagram will be referred to as the *normal sheaf of f relative to π*.

Property (ii) of Proposition 2.1 says that $H^0(S, N_f) \to H^0(C, N_f|_C)$ is surjective. An argument parallel to the proof of Proposition 2.1 shows the following:

Proposition 2.2. *Let X be as in Proposition 2.1. Then property (ii) can be strengthened as follows:*

(ii') If N_f denotes the normal sheaf of f, and if $N_{f,\pi}$ denotes the normal sheaf of f relative to π, then the composition of the maps

$$H^0(S, N_{f,\pi}) \to H^0(C, N_{f,\pi}|_C) \to H^0(C, N_f|_C)$$

is surjective for a general fiber C of π.

Moreover, if the general fibers of the MRC fibration of a desingularization of R are at least m-dimensional, then there are S and f with properties (i) and (ii') such that the image of the map

$$H^0(S, N_{f,\pi} \otimes I_C) \to H^0(C, (N_f \otimes I_C)|_C)$$

is at least $(m-1)$-dimensional.

3 The case when $\frac{n+1}{2} \leq d$

Let X be a smooth hypersurface of degree d in $\mathbf{P}^n$. Assume that a sweeping component R of $R_e(X)$ is uniruled. The following result, along with Proposition 2.1 will prove Theorem 1.1.

Proposition 3.1. *Suppose $d \leq n-3$, and let S and f be as in Proposition 2.1. If C is a general fiber of $\pi : S \to \mathbf{P}^1$ and I_C is the ideal sheaf of C in S, then the restriction map*

$$H^0(S, f^*\mathcal{O}_X(2d-n-1) \otimes I_C^\vee) \to H^0(C, f^*\mathcal{O}_X(2d-n-1) \otimes I_C^\vee|_C)$$

is zero.

Proof of Theorem 1.1. Granting Proposition 3.1, since $H^0(S, f^*\mathcal{O}_X(2d-n-1) \otimes I_C^\vee) \to H^0(C, f^*\mathcal{O}_X(2d-n-1) \otimes I_C^\vee|_C)$ is the zero map, we have

$$H^0(S, f^*\mathcal{O}_X(2d-n-1)) = H^0(S, f^*\mathcal{O}_X(2d-n-1) \otimes I_C^\vee).$$

Thus,

$$\begin{aligned}H^0(\mathbf{P}^1, \pi_* f^*\mathcal{O}_X(2d-n-1)) &= H^0(\mathbf{P}^1, \pi_*(f^*\mathcal{O}_X(2d-n-1)\otimes I_C^\vee)) \\ &= H^0(\mathbf{P}^1, (\pi_* f^*\mathcal{O}_X(2d-n-1))\otimes \mathcal{O}_{\mathbf{P}^1}(1)),\end{aligned}$$

which is only possible if $H^0(\mathbf{P}^1, \pi_* f^*\mathcal{O}_X(2d-n-1)) = 0$. So $H^0(S, f^*\mathcal{O}_X(2d-n-1)) = 0$ and $d < (n+1)/2$. □

Proof of Proposition 3.1. Let ω_S be the canonical sheaf of S. By Serre duality and the long exact sequence of cohomology, it suffices to show that if S and f satisfy the properties of Proposition 2.1, then the restriction map

$$H^1(S, f^*\mathcal{O}_X(n+1-2d)\otimes\omega_S) \to H^1(C, f^*\mathcal{O}_X(n+1-2d)\otimes\omega_S|_C)$$

is surjective. Let N be the normal sheaf of the map $f : S \to X$, and let N' be the normal sheaf of the map $S \to \mathbf{P}^n$. There is a short exact sequence

$$0 \to N \to N' \to f^*\mathcal{O}_X(d) \to 0. \tag{1}$$

Taking the $(n-3)$-rd exterior power of this sequence, we get the following short exact sequence

$$0 \to \bigwedge^{n-3} N \otimes f^*\mathcal{O}_X(-d) \to \bigwedge^{n-3} N' \otimes f^*\mathcal{O}_X(-d) \to \bigwedge^{n-4} N \to 0.$$

For an exact sequence of sheaves of $\mathcal{O}_S$-modules $0 \to E \to F \to M \to 0$ with E and F locally free of ranks e and f, there is a natural map of sheaves $\bigwedge^{f-e-1} M \otimes \bigwedge^e E \otimes (\bigwedge^f F)^\vee \to M^\vee$ which is defined locally at a point $s \in S$ as follows: assume $\gamma_1, \ldots, \gamma_{f-e-1} \in M_s$, $\alpha_1, \ldots, \alpha_e \in E_s$, and $\phi : \bigwedge^f F_s \to O_{S,s}$; then for $\gamma \in M_s$, we set $\gamma_{f-e} = \gamma$, and we define the map to be $\gamma \mapsto \phi(\tilde\gamma_1 \wedge \tilde\gamma_2 \wedge \cdots \wedge \tilde\gamma_{f-e} \wedge \alpha_1 \wedge \cdots \wedge \alpha_e)$ where $\tilde\gamma_i$ is any lifting of γ_i in F_s. Clearly, this map does not depend on the choice of the liftings, and thus it is defined globally. So from the short exact sequence $0 \to T_S \to f^*T_X \to N \to 0$, we get a map

$$\bigwedge^{n-4} N \to N^\vee \otimes f^*\mathcal{O}_X(n+1-d)\otimes\omega_S,$$

and from the short exact sequence $0 \to T_S \to f^*T_{\mathbf{P}^n} \to N' \to 0$, we get a map

$$\bigwedge^{n-3} N' \otimes f^*O_X(-d) \to (N')^\vee \otimes f^*O_X(n+1)\otimes\omega_S.$$

With the choices of the maps we have made, the following diagram, whose bottom row is obtained from dualizing sequence (1) and tensoring with $f^*\mathcal{O}_X(n+1-2d)\otimes\omega_S$, is commutative with exact rows

$$\begin{array}{ccccccccc} 0 \to & \bigwedge^{n-3} N \otimes f^*\mathcal{O}_X(-d) & \longrightarrow & \bigwedge^{n-3} N' \otimes f^*O_X(-d) & \longrightarrow & \bigwedge^{n-4} N & \longrightarrow & 0 \\ & \downarrow & & \downarrow & & \downarrow & & \\ 0 \to & f^*\mathcal{O}_X(n+1-2d)\otimes\omega_S & \to & (N')^\vee \otimes f^*\mathcal{O}_X(n+1-d)\otimes\omega_S & \to & N^\vee \otimes f^*\mathcal{O}_X(n+1-d)\otimes\omega_S & \to & 0 \end{array}$$

Since the cokernel of the first vertical map restricted to C is a torsion sheaf, to show the assertion, it suffices to show that the map

$$H^1(S, \bigwedge^{n-3} N \otimes f^*\mathcal{O}_X(-d)) \to H^1(C, \bigwedge^{n-3} N \otimes f^*\mathcal{O}_X(-d)|_C)$$

is surjective. Applying the long exact sequence of cohomology to the top sequence, the surjectivity assertion follows if we show that

(1) $H^0(S, \bigwedge^{n-4} N) \to H^0(C, \bigwedge^{n-4} N|_C)$ is surjective,

(2) $H^1(C, \bigwedge^{n-3} N' \otimes f^*\mathcal{O}_X(-d)|_C) = 0$.

To prove (1), we consider the commutative diagram

$$\begin{array}{ccc} \bigwedge^{n-4} H^0(S,N) & \longrightarrow & \bigwedge^{n-4} H^0(C,N|_C) \\ \downarrow & & \downarrow \\ H^0(S,\bigwedge^{n-4} N) & \longrightarrow & H^0(C,\bigwedge^{n-4} N|_C). \end{array}$$

The top horizontal map is surjective since $H^0(S,N) \to H^0(C,N|_C)$ is surjective, and the right vertical map is surjective since $N|_C$ is a globally generated line bundles over $\mathbf{P}^1$. By commutativity of the diagram the bottom horizontal map is surjective.

To prove (2), note that there is a surjective map $f^*\mathcal{O}_{\mathbf{P}^n}(1)^{\oplus n+1} \to N'$. Taking the $(n-3)$-rd exterior power, and then tensoring with $f^*\mathcal{O}_X(-d)$, we get a surjective map

$$f^*\mathcal{O}_{\mathbf{P}^n}(n-3-d)^{\oplus \binom{n+1}{n-3}} \to \bigwedge^{n-3} N' \otimes f^*\mathcal{O}_X(-d).$$

Restricting to C, since $n-3-d \geq 0$, we have $H^1(C, \bigwedge^{n-3} N' \otimes f^*\mathcal{O}_X(-d)|_C) = 0$. □

Proof of Theorem 1.2. Suppose that X is a smooth hypersurface of degree $n-2$ in $\mathbf{P}^n$. Let C be a smooth rational curve of degree e in $\mathbf{P}^n$ whose normal bundle $N_{C/\mathbf{P}^n}$ is globally generated. If we write

$$N_{C/\mathbf{P}^n} = \mathcal{O}_C(a_1) \oplus \cdots \oplus \mathcal{O}_C(a_{n-1}),$$

then $\sum_{1\leq i\leq n-1} a_i = e(n+1)-2$. Assume that $a_i + a_j < 3e$ for every $1 \leq i < j \leq n-1$. Then $H^1(C, \bigwedge^{n-3} N_{C/\mathbf{P}^n} \otimes \mathcal{O}_{\mathbf{P}^n}(-d)|_C) = 0$, and so if N' is as in the proof of Theorem 1.1, then

$$H^1(C, \bigwedge^{n-3} N' \otimes f^*\mathcal{O}_X(-d)|_C) = 0.$$

The assertion now follows from the proof of Theorem 1.1. □

We remark that when $d = n-1$ or n, the uniruledness of the sweeping subvarieties of $R_e(X)$ has been studied in [1]. It is proved that if $e \leq n$, then a subvariety of $R_e(X)$ is non-uniruled if the curves parametrized by its points sweep out X or a divisor in X.

4 Cubic Fourfolds

In this section we prove Theorem 1.3. When $e \geq 5$ is odd, the theorem follows from Theorem 1.2 and [3, Proposition 7.1]. So let $e \geq 6$ be an even integer, and assume to the contrary that the general fibers of the MRC fibration of $R_e(X)$ are at least 2-dimensional. Let S and f be as in Proposition 2.2, and let C be a general fiber of π. Set $N = N_f$ and $Q = N_{f,\pi}$. Then by Proposition 2.1 the following properties are satisfied:

- Property (i): The composition of the maps

  $$H^0(S, Q) \to H^0(S, Q|_C) \to H^0(C, N|_C)$$

  is surjective.

- Property (ii): The composition of the maps

  $$H^0(S, Q \otimes I_C) \to H^0(C, Q \otimes I_C|_C) \to H^0(C, N \otimes I_C|_C)$$

  is non-zero.

We show these lead to a contradiction. Note that $I_C|_C$ is isomorphic to the trivial bundle $\mathcal{O}_C$, but we write $I_C|_C$ instead of $\mathcal{O}_C$ to keep track of various maps and exact sequences involved in the proof.

Let Q' be the normal sheaf of the map $S \to \mathbf{P}^5$ relative to π. We have $Q|_C = N_{C/X}$ and $Q'|_C = N_{C/\mathbf{P}^5}$. Since $N_{X/\mathbf{P}^5} = \mathcal{O}_X(3)$, there is a short exact sequence

$$0 \to Q \to Q' \to f^*\mathcal{O}_X(3) \to 0. \tag{2}$$

Taking exterior powers, we obtain the following short exact sequence

$$0 \to \bigwedge^2 Q \otimes f^*\mathcal{O}_X(-3) \to \bigwedge^2 Q' \otimes f^*\mathcal{O}_X(-3) \to Q \to 0. \tag{3}$$

Since this sequence splits locally, its restriction to C is also a short exact sequence

$$0 \to \bigwedge^2 Q \otimes f^*\mathcal{O}_X(-3)|_C \to \bigwedge^2 Q' \otimes f^*\mathcal{O}_X(-3)|_C \to Q|_C \to 0. \tag{4}$$

To get a contradiction, we show that the image of the boundary map

$$\gamma : H^0(C, Q|_C) \to H^1(C, \bigwedge^2 Q \otimes f^*\mathcal{O}_X(-3)|_C)$$

is of codimension at least 2 in $H^1(C, \bigwedge^2 Q \otimes f^*\mathcal{O}_X(-3)|_C)$. This is not possible since by our assumption $N_{C/\mathbf{P}^5} = \mathcal{O}_C(\frac{3e}{2})^{\oplus 2} \oplus \mathcal{O}_C(\frac{3e}{2} - 1)^{\oplus 2}$, and so

$$\begin{aligned} H^1(C, \bigwedge^2 Q' \otimes f^*\mathcal{O}_X(-3)|_C) &= H^1(C, \bigwedge^2 N_{C/\mathbf{P}^5} \otimes f^*\mathcal{O}_X(-3)|_C) \\ &= H^1(C, \mathcal{O}_C(-2) \oplus \mathcal{O}_C(-1)^{\oplus 4} \oplus \mathcal{O}_C) \\ &= \mathbf{k}. \end{aligned}$$

Lemma 4.1. *The kernel of the map $f^*T_X \to Q$ is a line bundle which contains $\bigwedge^2 T_S \otimes \pi^*\omega_{\mathbf{P}^1}$ as a subsheaf.*

Proof. The kernel of $f^*T_X \to Q$ is equal to the kernel of the map induced by π on the tangent bundles $T_S \to \pi^*T_{\mathbf{P}^1}$ which we denote by F

$$0 \to F \to T_S \to \pi^*T_{\mathbf{P}^1}.$$

Since F is reflexive, it is locally free on S, and it is clearly of rank 1. Also the composition of the maps

$$\bigwedge^2 T_S \otimes \pi^*\omega_{\mathbf{P}^1} \to \bigwedge^2 T_S \otimes \Omega_S = T_S \to \pi^*T_{\mathbf{P}^1}$$

is the zero-map. So $\bigwedge^2 T_S \otimes \pi^*\omega_{\mathbf{P}^1}$ is a subsheaf of F. □

Given a section $r \in H^0(C, Q \otimes I_C|_C)$, we can define a map

$$\beta_r : H^1(C, \bigwedge^2 Q \otimes f^*\mathcal{O}_X(-3)|_C) \longrightarrow H^1(C, \omega_S|_C) = \mathbf{k}$$

as follows. Let F be the line bundle from the proof of Lemma 4.1. It follows from the proof of the lemma that there is an injection $\bigwedge^2 T_S \otimes \pi^*\omega_{\mathbf{P}^1} \to F$, and from the short exact sequence

$$0 \to F \to f^*T_X \to Q \to 0$$

we get a generically injective map of sheaves

$$\bigwedge^3 Q \otimes F \to \bigwedge^4 f^*T_X.$$

Combining these, we get a morphism

$$\bigwedge^3 Q \otimes (\omega_S \otimes \pi^*T_{\mathbf{P}^1})^\vee \to \bigwedge^4 f^*T_X.$$

Since $\bigwedge^4 f^*T_X = f^*\mathcal{O}_X(3)$, we get a generically injective map

$$\Psi : \bigwedge^3 Q \otimes f^*\mathcal{O}_X(-3) \otimes I_C \to \omega_S \otimes \pi^*T_{\mathbf{P}^1} \otimes I_C,$$

and by restricting to C, we get a map

$$\Psi|_C : (\bigwedge^3 Q \otimes f^*\mathcal{O}_X(-3) \otimes I_C)|_C \to \omega_S|_C.$$

Finally, r gives a map

$$\Phi_r : \bigwedge^2 Q \otimes f^*\mathcal{O}_X(-3)|_C \xrightarrow{\wedge r} \bigwedge^3 Q \otimes f^*\mathcal{O}_X(-3) \otimes I_C|_C,$$

and we define β_r to be the map induced by the composition $\Psi|_C \circ \Phi_r$. Note that β_r is non-zero if $r \neq 0$.

Lemma 4.2. *For $r, r' \in H^0(C, Q \otimes I_C|_C)$, $\ker(\beta_r) = \ker(\beta_{r'})$ if and only if r and r' are scalar multiples of each other.*

Proof. By Serre duality, it is enough to show that the images of the maps

$$H^0(C, I_C^\vee|_C) = H^0(C, \omega_S^\vee|_C \otimes \omega_C) \overset{\beta_r^\vee}{\underset{\beta_{r'}^\vee}{\rightrightarrows}} H^0(C, (\bigwedge^2 Q^\vee \otimes f^*\mathcal{O}_X(3))|_C \otimes \omega_C)$$

are the same if and only if r and r' are scalar multiples of each other. Since $Q|_C = N_{C/X}$, we have $\bigwedge^3 Q|_C = \bigwedge^3 N_{C/X} = f^*\mathcal{O}_X(3) \otimes \omega_C$, so

$$(\bigwedge^2 Q^\vee \otimes f^*\mathcal{O}_X(3))|_C \otimes \omega_C = Q|_C,$$

and the map

$$\beta_r^\vee : H^0(C, I_C^\vee|_C) \to H^0(C, Q|_C)$$

is simply given by r. Similarly, $\beta_{r'}^\vee$ is given by r', and the lemma follows. □

Recall that by definition, we have a short exact sequence

$$0 \to \pi^* T_{\mathbf{P}^1}|_C \to Q|_C \to N|_C \to 0,$$

and $\pi^* T_{\mathbf{P}^1}|_C = I_C^{-1}|_C$. If we tensor this sequence with $I_C|_C$, we get the following short exact sequence

$$0 \to \mathcal{O}_C \to Q \otimes I_C|_C \to N \otimes I_C|_C \to 0.$$

Let i be a non-zero section in the image of $H^0(C, \mathcal{O}_C) \to H^0(C, Q \otimes I_C|_C)$. Then i induces a map

$$\beta_i : H^1(C, \bigwedge^2 Q \otimes f^*\mathcal{O}_X(-3)|_C) \longrightarrow H^1(C, \omega_S|_C) = \mathbf{k}$$

as described before. Let

$$\gamma : H^0(C, Q|_C) \to H^1(C, \bigwedge^2 Q \otimes f^*\mathcal{O}_X(-3)|_C)$$

be the connecting map in sequence (4).

Lemma 4.3. *We have* $\operatorname{image}(\gamma) \subset \ker \beta_i$.

Proof. Since the short exact sequence $0 \to N \to N' \to f^*\mathcal{O}_X(3) \to 0$ splits locally, there is an exact sequence

$$0 \to \bigwedge^2 N \otimes f^*\mathcal{O}_X(-3) \to \bigwedge^2 N' \otimes f^*\mathcal{O}_X(-3) \to N \to 0.$$

Applying the long exact sequence of cohomology to the restriction of this sequence to C, we get a map

$$H^0(C, N|_C) \to H^1(C, \bigwedge^2 N \otimes f^*\mathcal{O}_X(-3)|_C).$$

Also from the the exact sequence $0 \to T_S \to f^*T_X \to N \to 0$, we get a map $\bigwedge^2 T_S \otimes \bigwedge^2 N \to \bigwedge^4 f^*T_X = f^*\mathcal{O}_X(3)$ and hence a map

$$\bigwedge^2 N \otimes f^*\mathcal{O}_X(-3) \to \omega_S.$$

It follows from the definition of β_i that the map $\beta_i \circ \gamma$ factors through

$$H^0(C, Q|_C) \to H^0(C, N|_C) \to H^1(C, \bigwedge^2 N \otimes f^*\mathcal{O}_X(-3)|_C) \to H^1(C, \omega_S|_C),$$

so we have a commutative diagram

$$\begin{array}{ccccccc} & & H^0(S,N) & \longrightarrow & H^1(S, \bigwedge^2 N \otimes f^*\mathcal{O}_X(-3)) & \longrightarrow & H^1(S,\omega_S) = 0 \\ & & \downarrow & & & & \downarrow \\ H^0(C,Q|_C) & \longrightarrow & H^0(C,N|_C) & & \longrightarrow & & H^1(C,\omega_S|_C). \end{array}$$

Thus we can conclude the assertion from the fact that the restriction map $H^0(S,N) \to H^0(C,N|_C)$ is surjective, and so the image of the composition of the above maps is contained in the image of the restriction map $H^1(S,\omega_S) \to H^1(C,\omega_S|_C)$ which is zero. $\square$

In the following lemma we prove a similar result for the sections of $Q \otimes I_C|_C$ which are obtained by restricting the global sections of $Q \otimes I_C$ to C.

Lemma 4.4. *If $\tilde{r} \in H^0(S, Q \otimes I_C)$, and if $r = \tilde{r}|_C$, then* $\mathrm{image}(\gamma) \subset \ker(\beta_r)$.

Proof. We have a commutative diagram

$$\begin{array}{ccccc} H^0(S,Q) & \longrightarrow & H^1(S, \bigwedge^2 Q \otimes f^*\mathcal{O}_X(-3)) & \longrightarrow & H^1(S,\omega_S) = 0 \\ \downarrow & & \downarrow & & \downarrow \\ H^0(C,Q|_C) & \xrightarrow{\gamma} & H^1(C, \bigwedge^2 Q \otimes f^*\mathcal{O}_X(-3)) & \xrightarrow{\beta_r} & H^1(C,\omega_S|_C) \end{array}$$

and therefore for any $u \in H^0(C,Q|_C)$ in the image of the restriction map $H^0(S,Q) \to H^0(C,Q|_C)$, we have $\beta_r(\gamma(u)) = 0$. Consider the exact sequence

$$0 \to I_C^{-1}|_C \to Q|_C \to N|_C \to 0.$$

From the hypothesis that the composition map $H^0(S,Q) \to H^0(C,Q|_C) \to H^0(C,N|_C)$ is surjective, we see that to prove the statement, it is enough to show that for any non-zero u in the image of $H^0(C, I_C^{-1}|_C) \to H^0(C,Q|_C)$, we have $\gamma(u) \in \ker \beta_r$.

Consider the diagram

$$\begin{array}{ccccc} H^0(C,Q|_C) & \xrightarrow{\gamma} & H^1(C, \bigwedge^2 Q \otimes f^*\mathcal{O}_X(-3)|_C) & \xrightarrow{\beta_r,\ \beta_i} & \\ \wedge i \downarrow \; \downarrow \wedge r & & \wedge i \downarrow \; \downarrow \wedge r & & \\ H^0(C, \bigwedge^2 Q \otimes I_C|_C) & \xrightarrow[\lambda]{} & H^1(C, \bigwedge^3 Q \otimes f^*\mathcal{O}_X(-3) \otimes I_C|_C) & \xrightarrow[\psi]{} & H^1(C,\omega_S|_C) \end{array}$$

where λ is obtained from applying the long exact sequence of cohomology to the third wedge power of sequence (2), and ψ is induced by the map $\Psi|_C$. Then we have

$$\begin{aligned}\beta_r \circ \gamma(u) &= \psi \circ \lambda(u \wedge r) \\ &= \psi \circ \lambda(r \wedge i) \quad \text{(up to a scalar factor)} \\ &= \beta_i \circ \gamma(r) \\ &= 0,\end{aligned}$$

where the last equality comes from the fact that $\gamma(H^0(C, Q|_C)) \subset \ker \beta_i$ by Lemma 4.3. □

Let now $\tilde{r}_0 \in H^0(S, Q \otimes I_C)$ be so that its image in $H^0(C, N \otimes I_C|_C)$ is non-zero. Such $\tilde{r}_0$ exists by Property (ii). Then $r_0 := \tilde{r}_0|_C$ defines a map β_{r_0}. Since the image of r_0 in $H^0(C, N \otimes I_C|_C)$ is non-zero, r_0 and i are not scalar multiples, so according to Lemma 4.2, $\ker \beta_{r_0} \neq \ker \beta_i$. Thus the codimension of $\ker \beta_i \cap \ker \beta_{r_0}$ is at least 2. On the other hand, by the previous lemmas, $\text{image}(\gamma) \subset \ker \beta_i \cap \ker \beta_{r_0}$. This is a contradiction since $\dim H^1(C, \bigwedge^2 Q' \otimes f^*\mathcal{O}_X(-3)|_C) = 1$.

5 The case when $d < \frac{n+1}{2}$

Throughout this section, $X \subset \mathbf{P}^n$ will be a general hypersurface of degree $d < (n+1)/2$. By the main theorem of [6], $R_e(X)$ is irreducible for every $e \geq 1$. If $d^2 \leq n$ and $e \geq 2$, then by [4] and [11], the space of rational curves of degree e in X passing through two general points of X is rationally connected. In particular, $R_e(X)$ is rationally connected for $e \geq 2$. If $e = 1$, then $R_1(X)$ is the Fano variety of lines in X which is rationally connected if and only if $d^2 + d \leq 2n$ [8, V.4.7]. In this section, we will consider the case when $d^2 + d > 2n$.

Assume that $R_e(X)$ is uniruled. Then there are S and f with the two properties given in Proposition 2.1. We can take the pair (S, f) to be minimal in the sense that a component of a fiber of π which is contracted by f cannot be blown down. Let N be the normal sheaf of f, and let C be a general fiber of π with ideal sheaf I_C in S. Denote by H the pullback of a hyperplane in $\mathbf{P}^n$ to S, and denote by K a canonical divisor on S. From the exact sequences $0 \to T_S \to f^*T_X \to N \to 0$ and $0 \to f^*T_X \to f^*T_{\mathbf{P}^n} \to f^*\mathcal{O}_{\mathbf{P}^n}(d) \to 0$ we get

$$\begin{aligned}\chi(N \otimes I_C) &= (n+1)\chi(f^*\mathcal{O}_{\mathbf{P}^n}(1) \otimes I_C) - \chi(f^*\mathcal{O}_{\mathbf{P}^n}(d) \otimes I_C) - \chi(I_C) - \chi(T_S \otimes I_C) \\ &= (n+1)(\frac{(H-C)\cdot(H-C-K)}{2} + 1) - \frac{(dH-C)\cdot(dH-C-K)}{2} - 1 \\ &\quad - \frac{-C\cdot(-C-K)}{2} - 1 - (2K^2 - 14) \\ &= \frac{(n+1-d^2)}{2}H^2 - \frac{(n+1-d)}{2}H\cdot K - 2K^2 - (n+1-d)e + 14.\end{aligned}$$

We claim that $2H + 2C + K$ is base-point free and hence has a non-negative self-intersection number. By the main theorem of [10], if $2H + 2C + K$ is not base point free, then there exists an effective divisor E such that either

$$(2H + 2C)\cdot E = 1, E^2 = 0 \quad \text{or} \quad (2H + 2C)\cdot E = 0, E^2 = -1.$$

The first case is clearly not possible. In the second case, $H \cdot E = 0$, and $C \cdot E = 0$. So E is a component of one of the fibers of π which is contracted by f and which is a (-1)-curve. This contradicts the assumption that (S, f) is minimal. Thus $(2H + 2C + K)^2 \geq 0$. Also, since $H^1(S, f^*\mathcal{O}_X(-1)) = 0$, $H \cdot (H + K) = 2\chi(f^*\mathcal{O}_X(-1)) - 2 \geq -2$, so we can write

$$\begin{aligned} \chi(N \otimes I_C) &= \frac{2n + 2 - d^2 - d}{2} H^2 - (n - d - 15)(e - 1) - 2 \\ &\quad -2(2H + 2C + K)^2 - \frac{n - d - 15}{2}(H \cdot (H + K) + 2) \\ &\leq \frac{2n + 2 - d^2 - d}{2} H^2 - (n - d - 15)(e - 1) - 2, \end{aligned}$$

and therefore $\chi(N \otimes I_C)$ is negative when $d^2 + d \geq 2n + 2$ and $n \geq 30$.

The Leray spectral sequence gives a short exact sequence

$$0 \to H^1(\mathbf{P}^1, \pi_*(N \otimes I_C)) \to H^1(S, N \otimes I_C) \to H^0(\mathbf{P}^1, R^1\pi_*(N \otimes I_C)) \to 0,$$

and by our assumption on S and f, $H^1(\mathbf{P}^1, \pi_*(N \otimes I_C)) = 0$. If we could choose S such that $H^0(\mathbf{P}^1, R^1\pi_*(N \otimes I_C)) = 0$, then we could conclude that $\chi(N \otimes I_C) \geq 0$ and hence $R_e(X)$ could not be uniruled for $d^2 + d \geq 2n + 2$ and $n \geq 30$.

We cannot show that for a general X, a minimal pair (S, f) as in Proposition 2.1 can be chosen so that $H^0(\mathbf{P}^1, R^1\pi_*(N \otimes I_C)) = 0$. However, we prove that if X is general and (S, f) is minimal, then for every $t \geq 1$,

$$H^0(\mathbf{P}^1, R^1\pi_*(N \otimes I_C \otimes f^*\mathcal{O}_X(t))) = 0.$$

We also show that if $t \geq 0$ and $f(C)$ is t-normal, then

$$H^1(\mathbf{P}^1, \pi_*(N \otimes I_C \otimes f^*\mathcal{O}_X(t))) = 0.$$

These imply that $\chi(N \otimes I_C \otimes f^*\mathcal{O}_X(t))$ is non-negative when X is general and $f(C)$ is t-normal. To finish the proof of Theorem 1.4, we compute $\chi(N \otimes I_C \otimes f^*\mathcal{O}_X(t))$ directly and show that it is negative when the inequality in the statement of the theorem holds.

Proof of Theorem 1.4. Let X be a general hypersurface of degree d in $\mathbf{P}^n$. If $R_e(X)$ is uniruled, then there are S and f as in Proposition 2.1. Assume the pair (S, f) is minimal. Let N be the normal sheaf of f, and let C be a general fiber of π. Then $H^0(S, N) \to H^0(C, N|_C)$ is surjective. The restriction map $H^0(S, f^*\mathcal{O}_X(m)) \to H^0(C, f^*\mathcal{O}_X(m)|_C)$ is also surjective since $f(C)$ is m-normal, so the restriction map $H^0(S, N \otimes f^*\mathcal{O}_X(m)) \to H^0(C, N \otimes f^*\mathcal{O}_X(m)|_C)$ is surjective as well. Therefore,

$$H^1(\mathbf{P}^1, \pi_*(N \otimes f^*\mathcal{O}_X(m) \otimes I_C)) = 0.$$

Now let C be an arbitrary fiber of π, and let C^0 be an irreducible component of C. Then by Proposition 5.2, $f^*(T_X(t))|_{C^0}$ is globally generated for every $t \geq 1$, and hence $N \otimes f^*\mathcal{O}_X(t)|_{C^0}$ is globally generated too. So Lemma 5.1 shows that for every $t \geq 1$

$$H^0(\mathbf{P}^1, R^1\pi_*(N \otimes f^*\mathcal{O}_X(t) \otimes I_C)) = 0.$$

By the Leray spectral sequence,

$$\begin{aligned} H^1(S, N \otimes f^*\mathcal{O}_X(m) \otimes I_C) &= H^1(\mathbf{P}^1, \pi_*(N \otimes f^*\mathcal{O}_X(m) \otimes I_C)) \\ &\quad \oplus H^0(\mathbf{P}^1, R^1\pi_*(N \otimes f^*\mathcal{O}_X(m) \otimes I_C)) \\ &= 0, \end{aligned}$$

and therefore, $\chi(N \otimes f^*\mathcal{O}_X(m) \otimes I_C) \geq 0$. We next compute $\chi(N \otimes f^*\mathcal{O}_X(m) \otimes I_C)$. For an integer $t \geq 0$, set

$$a_t = \chi(N \otimes I_C \otimes f^*\mathcal{O}_X(t)).$$

We have

$$a_t = \chi(N \otimes I_C) + \frac{2t(n+1-d) + t^2(n-3)}{2} H^2 - \frac{t(n-5)}{2} H \cdot K - t(n-3)e.$$

So

$$a_t = \frac{b_t}{2} H^2 + \frac{c_t}{2} H \cdot K - 2K^2 + d_t,$$

where

$$b_t = (n+1-d^2) + 2t(n+1-d) + t^2(n-3),$$
$$c_t = -(n+1-d) - t(n-5),$$

and

$$d_t = -t(n-3)e - (n+1-d)e + 14.$$

A computation similar to the computation in the beginning of this section shows that

$$\begin{aligned} a_t &= \frac{b_t - c_t}{2} H^2 - 2(2H + 2C + K)^2 + \frac{c_t + 16}{2}(H \cdot (H+K) + 2) + (d_t - c_t - 32 + 16e) \\ &\leq \frac{b_t - c_t}{2} H^2 + (d_t - c_t - 32 + 16e). \end{aligned}$$

Since

$$d_t - c_t - 32 + 16e = -(e-1)(n - 15 - d + t(n-3)) - 2t - 2,$$

and since $n - 15 - d + t(n-3) \geq 2n - d - 18 \geq 0$ for $t \geq 1$ and $n \geq 12$, we get

$$a_t < \frac{b_t - c_t}{2} H^2.$$

When $d^2 + (2t+1)d \geq (t+1)(t+2)n + 2$, $b_t < c_t$, and so $a_t < 0$. If we let $t = m$, we get the desired result.

□

Lemma 5.1. *If E is a locally free sheaf on S such that for every irreducible component C^0 of a fiber of π, $E|_{C^0}$ is globally generated, then $R^1\pi_*E = 0$.*

Proof. By cohomology and base change [7, Theorem III.12.11], it suffices to prove that for every fiber C of π, $H^1(C, E|_C) = 0$. We first show that if l is the number of irreducible components of C counted with multiplicity, then we can write $C = C_1 + \cdots + C_l$ such that each C_i is an irreducible component of C and for every $1 \leq i \leq l-1$, $(C_1 + \cdots + C_i) \cdot C_{i+1} \leq 1$. This is proven by induction on l. If $l = 1$, there is nothing to prove. Otherwise, there is at least one component C^0 of C which can be contracted. Let r be the multiplicity of C^0 in C. Blowing down C^0, we get a rational surface S' over $\mathbf{P}^1$. Denote by C' the blow-down of C. Then by the induction hypothesis, we can write

$$C' = C'_1 + \cdots + C'_{l-r}$$

such that $(C'_1 + \cdots + C'_i) \cdot C'_{i+1} \leq 1$ for every $1 \leq i \leq l-r-1$. Let C_i be the proper transform of C'_i. Then if in the above sum we replace C'_i by C_i when C_i does not intersect C^0 and by $C_i + C^0$ when C_i intersects C^0, we get the desired result for C.

Since $E|_{C_{i+1}}$ is globally generated, it follows that

$$H^1(C_{i+1}, E(-C_1 - \cdots - C_i)|_{C_{i+1}}) = 0 \quad \text{for every } 0 \leq i \leq l-1.$$

On the other hand, for every $0 \leq i \leq l-2$, we have a short exact sequence of $\mathcal{O}_S$-modules

$$0 \to E(-C_1 - \cdots - C_{i+1})|_{C_{i+2}+\cdots+C_l} \to E(-C_1 - \cdots - C_i)|_{C_{i+1}+\cdots+C_l} \to E(-C_1 - \cdots - C_i)|_{C_{i+1}} \to 0.$$

So a decreasing induction on i shows that for every $0 \leq i \leq l-2$, $H^1(S, E(-C_1 - \cdots - C_i)|_{C_{i+1}+\cdots+C_l}) = 0$. Letting $i = 0$, the statement follows. □

Proposition 5.2. *Let $X \subset \mathbf{P}^n$ be a general hypersurface of degree d.*

(i) For any morphism $h : \mathbf{P}^1 \to X$, $h^*(T_X(1))$ is globally generated.

(ii) If C is a smooth, rational, d-normal curve on X, then $H^1(C, T_X|_C) = 0$.

Proof. (i) This follows from [13, Proposition 1.1]. We give a proof here for the sake of completeness. Consider the short exact sequence

$$0 \to h^*T_X \to h^*T_{\mathbf{P}^n} \to h^*\mathcal{O}_X(d) \to 0.$$

Since X is general, the image of the pull-back map $H^0(X, \mathcal{O}_X(d)) \to H^0(\mathbf{P}^1, h^*\mathcal{O}_X(d))$ is contained in the image of the map $H^0(\mathbf{P}^1, h^*T_{\mathbf{P}^n}) \to H^0(\mathbf{P}^1, h^*\mathcal{O}_X(d))$. Choose a homogeneous coordinate system for $\mathbf{P}^n$. Let p be a point in $\mathbf{P}^1$, and without loss of generality assume that $h(p) = (1 : 0 : \cdots : 0)$. We show that for any $r \in h^*(T_X(1))|_p$, there is $\tilde{r} \in H^0(\mathbf{P}^1, h^*(T_X(1)))$ such that $\tilde{r}|_p = r$.

Consider the exact sequence

$$0 \longrightarrow H^0(\mathbf{P}^1, h^*T_X(1)) \longrightarrow H^0(\mathbf{P}^1, h^*T_{\mathbf{P}^n}(1)) \xrightarrow{\phi} H^0(\mathbf{P}^1, h^*\mathcal{O}_X(d+1)).$$

Denote by s the image of r in $h^*(T_{\mathbf{P}^n}(1))|_p$. There exists $S \in H^0(\mathbf{P}^n, T_{\mathbf{P}^n}(1))$ such that the restriction of $\tilde{s} := h^*(S)$ to p is s. Denote by T the image of S in $H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d+1))$, and let $\tilde{t} = h^*(T)$. Then T is a form of degree $d+1$ on $\mathbf{P}^n$, and since $\tilde{t}|_p = 0$, we can write

$$T = x_1 G_1 + \cdots + x_n G_n,$$

where the G_i are forms of degree d. Our assumption implies that for every $1 \leq i \leq n$, there is $\tilde{s}_i \in H^0(\mathbf{P}^1, h^*T_{\mathbf{P}^n})$ such that $\phi(\tilde{s}_i) = h^*G_i$. Then

$$\phi(\tilde{s} - h^*(x_1)\tilde{s}_1 - \cdots - h^*(x_n)\tilde{s}_n) = \tilde{t} - h^*(x_1G_1) - \cdots - h^*(x_nG_n) = 0,$$

and therefore, $\tilde{s} - h^*(x_1)\tilde{s}_1 - \cdots - h^*(x_n)\tilde{s}_n$ is the image of some $\tilde{r} \in H^0(\mathbf{P}^1, h^*(T_X(1)))$. Since $(\tilde{s} - h^*(x_1)\tilde{s}_1 - \cdots - h^*(x_n)\tilde{s}_n)|_p = \tilde{s}|_p = s$, we have $\tilde{r}|_p = r$.

(ii) There is a short exact sequence

$$0 \to T_X|_C \to T_{\mathbf{P}^n}|_C \to \mathcal{O}_C(d) \to 0.$$

The fact that X is general implies that any section of $\mathcal{O}_C(d))$ which is the restriction of a section of $\mathcal{O}_{\mathbf{P}^n}(d)$ can be lifted to a section of $T_{\mathbf{P}^n}|_C$. Since the first cohomology group of $T_{\mathbf{P}^n}|_C$ vanishes, the result follows. □

Although for every e and n with $e \geq n+1 \geq 4$, there are smooth non-degenerate rational curves of degree e in $\mathbf{P}^n$ which are not $(e-n)$-normal [5, Theorem 3.1], a general smooth rational curve of degree e in a general hypersurface of degree d has possibly a much smaller normality: if a maximal-rank type conjecture holds for rational curves contained in general hypersurfaces (at least when $d < \frac{n+1}{2}$), then it follows that if c is the smallest positive number such that $\binom{n+c}{n} - \binom{n+c-d}{n} \geq ce+1$, a general smooth rational curve of degree e in a general hypersurface of degree d in $\mathbf{P}^n$ is c-normal.

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