RATIONAL SURFACES IN INDEX-ONE FANO HYPERSURFACES

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Abstract. We give the first evidence for a conjecture that a general, index-one, Fano hypersurface is not unirational: (i) a general point of the hypersurface is contained in no rational surface ruled, roughly, by low-degree rational curves, and (ii) a general point is contained in no image of a Del Pezzo surface.

1. Introduction

For complex, projective varieties a classical notion is unirationality: A variety rationally dominated by projective space is *unirational*. A modern notion is rational connectedness: A variety is *rationally connected* if every pair of points is contained in a rational curve. Every unirational variety is rationally connected. The two notions agree for curves and surfaces. Conjecturally they disagree in higher dimensions.

Conjecture 1.1. For every integer $n \geq 4$ there exists a non-unirational, smooth, degree-n hypersurface in $\mathbb{P}^n$.

A smooth hypersurface in $\mathbb{P}^n$ of degree $d \leq n$ is an *index-$(n+1-d)$, Fano manifold*. By [2] and [9], every Fano manifold is rationally connected. Versions of Conjecture 1.1 have been around for decades. The specific case $n = 4$ is attributed to Iskovskikh and Manin, [7].

In [8], Kollár suggested an approach to proving Conjecture 1.1. A general point of an n-dimensional, unirational variety is contained in a k-dimensional, rational subvariety for each $k < n$. Thus, Conjecture 1.1 for $n \geq 5$ follows from the next conjecture.

Conjecture 1.2. For every integer $n \geq 5$, there exists a smooth, degree-n hypersurface in $\mathbb{P}^n$ whose general point is contained in no rational surface.

In fact the conjecture fails for $n = 4$. The following argument was related to us by Rahul Pandharipande and Joe Harris and independently by Miles Reid. For every smooth degree 4 hypersurface $X \subset \mathbb{P}^4$, for general $p \in X$, the set C_p of lines L osculating to X to order 3 at p is a smooth conic in the projective tangent bundle $\mathbb{P}T_pX \cong \mathbb{P}^2$. Of course $L \cap X = 3p + q_L$ for a point q_L. Varying L in C_p, the points q_L sweep out a rational curve B_p (of degree 6). Varying p in a rational curve D, the union of the curves B_p is a rational surface B_D. Finally, varying D among rational curves in X, a general point of X is contained in a rational surface B_D.

We give the first evidence for Conjecture 1.2.

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Theorem 1.3. *For every integer $n \geq 5$, every smooth, degree-n hypersurface X in $\mathbb{P}^n$ contains a countable union of closed, codimension-2 subvarieties containing the image of every generically-finite, rational transformation $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow X$ mapping a general fiber $\{t\} \times \mathbb{P}^1$ isomorphically to an $(n-1)$-normal, smooth, rational curve in X.*

Theorem 1.4. *For every integer $n \geq 5$, every smooth, degree-n hypersurface in $\mathbb{P}^n$ contains a countable union of closed, codimension-2 subvarieties containing the image of every generically finite, regular morphism from a Del Pezzo surface to X.*

A projective variety is *k-normal* if every global section of the restriction of $\mathcal{O}_{\mathbb{P}^n}(k)$ is the restriction of a global section on $\mathbb{P}^n$.

We present two approaches here. First, given a rational surface S and a regular morphism $f : S \to X$, to prove deformations of f are contained in a codimension-2 subvariety of X it suffices to prove $\bigwedge^{n-4}(f^*T_X/T_S)/\text{Torsion}$ has no nonzero section. In Section 2 this is used to prove Theorems 1.3 and 1.4.

Second, a rational surface with a pencil of rational curves gives a morphism from $\mathbb{P}^1$ to a parameter space of rational curves on X. There is a construction of algebraic differential forms on the parameter space. Since $\mathbb{P}^1$ has only the zero form, these forms impose restrictions on rational curves in the parameter space. In Section 3, these restrictions are used to prove the following generalization of Theorem 1.3.

Theorem 1.5. *For every integer $n \geq 5$, every smooth, degree-n hypersurface X in $\mathbb{P}^n$ contains a countable union of closed, codimension-2 subvarieties containing the image of every generically-finite, rational transformation $S \dashrightarrow X$ from a surface with a pencil of curves mapping the general curve isomorphically to an $(n-1)$-normal, smooth curve of genus 0 or 1, also assumed non-degenerate if the genus is 1.*

As the second approach does not apply to Theorem 1.4, the first approach is more productive. However, further progress in proving Conjecture 1.2 will likely use both approaches, as well as new ideas.

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2. The first approach

Let X be a smooth, degree-n hypersurface in $\mathbb{P}^n$, $n \geq 5$. Denote by $\text{Hilb}(X)$ the Hilbert scheme of X. Theorem 1.3 follows easily from the next theorem.

Theorem 2.1. *Let Z be an irreducible subscheme of Hom$(\mathbb{P}^1, \text{Hilb}(X))$ satisfying,*

(i) *the associated morphism $Z \times \mathbb{P}^1 \to \text{Hilb}(X)$ does not factor through the projection $Z \times \mathbb{P}^1 \to Z$, and*
(ii) *the image of a general point of $Z \times \mathbb{P}^1$ parametrizes a smooth, $(n-1)$-normal, rational curve in X.*

Then there exists a codimension ≥ 2 subvariety of X containing all curves parametrized by $Z \times \mathbb{P}^1$.

A morphism $\mathbb{P}^1 \to \mathrm{Hilb}(X)$ is equivalent to a closed subscheme $S' \subset \mathbb{P}^1 \times X$, flat over $\mathbb{P}^1$. If a general point of $\mathbb{P}^1$ parametrizes a smooth rational curve, then S' is an irreducible surface. Any desingularization S of S' is a surface fitting in a diagram,

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "\pi"] & X \\ \mathbb{P}^1 & \end{tikzcd} \tag{1}$$

Associated to this diagram there is a derivative map

$$df : T_S \to f^*T_X$$

and a torsion-free sheaf

$$\bigwedge^{n-4}(f^*T_X/T_S)/\mathrm{Torsion}.$$

Proposition 2.2. *Let Z be an irreducible subvariety of Hom$(\mathbb{P}^1, \mathrm{Hilb}(X))$ satisfying,*

(i) *the associated morphism $Z \times \mathbb{P}^1 \to \mathrm{Hilb}(X)$ does not factor through the projection $Z \times \mathbb{P}^1 \to Z$,*
(ii) *a general point of $Z \times \mathbb{P}^1$ parametrizes a smooth curve in X, and*
(iii) *there is no codimension 2 subvariety of X containing all curves parametrized by $Z \times \mathbb{P}^1$.*

Then, for the morphism $\mathbb{P}^1 \to \mathrm{Hilb}(X)$ parametrized by a general point of Z, the torsion-free sheaf $\bigwedge^{n-4}(f^*T_X/T_S)/$ Torsion associated to the diagram in Equation 1 has a nonzero global section.

Remark 2.3. This proposition holds if $\mathbb{P}^1$ is replaced by any other curve, and even if $\mathrm{Hom}(\mathbb{P}^1, \mathrm{Hilb}(X))$ is replaced by the scheme of all maps from smooth projective connected curves to $\mathrm{Hilb}(X)$.

Proof. Replacing Z by a dense, Zariski open subset if necessary, we may assume Z is smooth. Let $V' \subset Z \times \mathbb{P}^1 \times X$ be the pullback of the universal family to $Z \times \mathbb{P}^1$.

Let $\phi : V \to V'$ be a desingularization of V'. Denote by g' the projection map from V' to $Z \times X$, and denote by p' the projection map from V' to Z. Let $g = g' \circ \phi$, and let $p = p' \circ \phi$.

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[rr, bend left, "g"] \arrow[ddr, "p"'] & V' \arrow[r, "g'"] \arrow[d] \arrow[dd, bend left, "p'"] & Z \times X \\ & Z \times \mathbb{P}^1 \arrow[d] & \\ & Z. & \end{tikzcd}$$

Replacing Z by a dense, Zariski open subset if necessary, we may assume p is smooth, cf. [6, Corollary III.10.7]. Associated to the morphism g is the derivative map,

$$dg : T_V \to g^*T_X.$$

Associated to the morphism p is the derivative map,

$$dp : T_V \to p^*T_Z.$$

By hypothesis, dp is surjective. Denote by T_p the kernel of dp. Because $Z \times \mathbb{P}^1 \to \text{Hilb}(X)$ does not factor through Z, the restriction of g to a general fiber of p maps generically finitely to its image. Therefore the following sheaf homomorphism is generically injective,

$$dg : T_p \to g^*T_X.$$

As T_p is locally free and V is integral, the sheaf homomorphism dg is injective on all of X.

Denote $\text{Coker}(dg)$ by $\mathcal{N}$,

$$\mathcal{N} := \text{Coker}(dg : T_p \to g^*T_X).$$

The following is a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & T_p \arrow[r] \arrow[d, "="'] & T_V \arrow[r, "dp"] \arrow[d, "dg"] & p^*T_Z \arrow[r] & 0 \\ 0 \arrow[r] & T_p \arrow[r] & g^*T_X \arrow[r] & \mathcal{N} \arrow[r] & 0 \end{tikzcd}$$

By the universal property of cokernels, there is a unique sheaf homomorphism,

$$u : p^*T_Z \to \mathcal{N},$$

such that the following diagram commutes,

$$\begin{tikzcd} 0 \arrow[r] & T_p \arrow[r] \arrow[d, "="'] & T_V \arrow[r, "dp"] \arrow[d, "dg"] & p^*T_Z \arrow[r] \arrow[d, "u"] & 0 \\ 0 \arrow[r] & T_p \arrow[r] & g^*T_X \arrow[r] & \mathcal{N} \arrow[r] & 0 \end{tikzcd}$$

By generic smoothness, the rank of dg at a general point of V equals the dimension of the closure of $\text{Image}(g)$. By hypothesis, this is $\geq n-2$. Therefore the rank of u at a general point is $\geq n-4$. Thus, the restriction of u to a general $(n-4)$-plane in the fiber of p^*T_Z has rank $n-4$. A general $(n-4)$-plane is the tangent space of a general $(n-4)$-dimensional subvariety of Z. We may replace Z be the smooth locus inside a general $(n-4)$-dimensional subvariety of Z. Therefore we may assume Z is $(n-4)$-dimensional and u is generically injective.

Associated to u, there is an induced map,

$$\bigwedge^{n-4}(u) : p^* \bigwedge^{n-4} T_Z \to \bigwedge^{n-4} \mathcal{N}/\text{Torsion}.$$

Because u is generically injective and $n \geq 5$, this map is generically injective.

Let z be a general point of Z, and denote by S' and S the fibers of p' and p over z, respectively. Since V is smooth, S is a smooth surface. Let $f : S \to X$ be the restriction of g to S. The restriction of $\mathcal{N}$ to S is precisely f^*T_X/T_S. The restriction of p^*T_Z to S is precisely the trivial vector bundle $T_{Z,z} \otimes_{\mathbb{C}} \mathcal{O}_S$. Since

z is general, the restriction of $\bigwedge^{n-4}(u)$ is generically injective. Therefore, it is a nonzero map,

$$\bigwedge^{n-4}(u)|_S : (\bigwedge^{n-4} T_{Z,z}) \otimes_{\mathbb{C}} \mathcal{O}_S \to (\bigwedge^{n-4} f^*T_X/T_S)/\text{Torsion}.$$

By our assumption $T_{Z,z}$ is $(n-4)$-dimensional. Therefore this nonzero map is equivalent to a nonzero global section of $(\bigwedge^{n-4} f^*T_X/T_S)/\text{Torsion}$ (well-defined up to nonzero scaling). $\square$

Proposition 2.4. *Let $\mathbb{P}^1 \to \text{Hilb}(X)$ be a morphism with associated diagram as in Equation 1. If the curve parametrized by a general point of $\mathbb{P}^1$ is smooth, rational and $(n-1)$-normal then*

$$h^0(S, \omega_S \otimes \bigwedge^{n-2} f^*T_X) = 0.$$

Proof. Pulling back the short exact sequence of tangent bundles

$$0 \to T_X \to T_{\mathbb{P}^n}|_X \to N_{X/\mathbb{P}^n} \cong \mathcal{O}_X(n) \to 0$$

to S and taking its $(n-1)^{\text{st}}$ exterior power gives another short exact sequence

$$0 \to \bigwedge^{n-1} f^*T_X \to \bigwedge^{n-1} f^*T_{\mathbb{P}^n} \to f^*\mathcal{O}_X(n) \otimes \bigwedge^{n-2} f^*T_X \to 0.$$

Tensoring this sequence with $\omega_S \otimes f^*\mathcal{O}_X(-n)$ gives the following short exact sequence

$$0 \to \omega_S \otimes f^*\mathcal{O}_X(-n) \otimes \bigwedge^{n-1} f^*T_X \to \omega_S \otimes f^*\mathcal{O}_X(-n) \otimes \bigwedge^{n-1} f^*T_{\mathbb{P}^n} \to \omega_S \otimes \bigwedge^{n-2} f^*T_X \to 0.$$

By applying the long exact sequence of cohomology, we conclude that

$$h^0(S, \omega_S \otimes \bigwedge^{n-2} f^*T_X) = 0$$

if both

(i) $h^0(S, \omega_S \otimes f^*\mathcal{O}_X(-n) \otimes \bigwedge^{n-1} f^*T_{\mathbb{P}^n}) = 0$, and
(ii) $h^1(S, \omega_S \otimes f^*\mathcal{O}_X(-n) \otimes \bigwedge^{n-1} f^*T_X) = 0$.

Proof of (i). Consider the Euler exact sequence on $\mathbb{P}^n$

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1)^{n+1} \to T_{\mathbb{P}^n} \to 0.$$

Pulling this back to S, and taking its n^{th} exterior power gives the following exact sequence

$$0 \to \bigwedge^{n-1} f^*T_{\mathbb{P}^n} \to f^*\mathcal{O}_X(n)^{\oplus(n+1)} \to f^*\mathcal{O}_X(n+1) \to 0.$$

Tensoring with $\omega_S \otimes f^*\mathcal{O}_X(-n)$ gives an injective map

$$\omega_S \otimes f^*\mathcal{O}_X(-n) \otimes \bigwedge^{n-1} f^*T_{\mathbb{P}^n} \to \omega_S^{\oplus(n+1)}.$$

Thus it suffices to prove $h^0(S, \omega_S) = 0$. Because the general fibers of $S \to \mathbb{P}^1$ is a rational curve, S is a rational surface. Therefore $h^0(S, \omega_S) = 0$.

Proof of (ii). There is a canonical isomorphism

$$\omega_S \otimes f^*\mathcal{O}_X(-n) \otimes \bigwedge^{n-1} f^*T_X \cong \omega_S \otimes f^*\mathcal{O}_X(-n+1).$$

So by Serre duality, it suffices to prove $h^1(S, f^*\mathcal{O}_X(n-1)) = 0$. Let C be a general fiber of the map $\pi : S \to \mathbb{P}^1$. There is a short exact sequence

$$0 \to f^*\mathcal{O}_X(n-1) \otimes I_{C/S} \to f^*\mathcal{O}_X(n-1) \to f^*\mathcal{O}_X(n-1)|_C \to 0, \tag{2}$$

where $\mathcal{I}_{C/S}$ is the ideal sheaf of C in S. By hypothesis, the image of C by f is $(n-1)$-normal in $\mathbb{P}^n$, therefore the map

$$H^0(S, f^*\mathcal{O}_X(n-1)) \to H^0(C, f^*\mathcal{O}_X(n-1)|_C)$$

is surjective. The long exact sequence of cohomology associated to the sequence in Equation 2 gives an isomorphism

$$H^1(S, f^*\mathcal{O}_X(n-1) \otimes I_{C/S}) \cong H^1(S, f^*\mathcal{O}_X(n-1)). \tag{3}$$

Recall a coherent sheaf $\mathcal{F}$ on S is called π-*relatively globally generated* if the following sheaf homomorphism is surjective,

$$\pi^*\pi_*\mathcal{F} \to \mathcal{F}.$$

The surface S is smooth and the general fiber of π is a smooth, rational curve. Therefore if $\mathcal{F}$ is π-relatively globally generated, then $R^1\pi_*\mathcal{F}$ is the zero sheaf.

Of course, since $f^*\mathcal{O}_X(n-1)$ is globally generated, it is π-relatively globally generated. Because $I_{C/S} \cong \pi^*\mathcal{O}_{\mathbb{P}^1}(-1)$, the twist $f^*\mathcal{O}_X(n-1) \otimes I_{C/S}$ is π-relatively globally-generated. Thus the sheaves $R^1\pi_*(f^*\mathcal{O}_X(n-1))$ and $R^1\pi_*(f^*\mathcal{O}_X(n-1) \otimes I_{C/S})$ are each zero. So, by the Leray spectral sequence, there are canonical isomorphisms

$$H^1(S, f^*\mathcal{O}_X(n-1)) \cong H^1(\mathbb{P}^1, \pi_*(f^*\mathcal{O}_X(n-1))), \tag{4}$$

$$H^1(S, f^*\mathcal{O}_X(n-1) \otimes \mathcal{I}_{C/S}) \cong H^1(\mathbb{P}^1, \pi_*(f^*\mathcal{O}_X(n-1) \otimes \mathcal{I}_{C/S})). \tag{5}$$

Taken together, Equations 3, 4 and 5 give a canonical isomorphism

$$H^1(\mathbb{P}^1, \pi_*(f^*\mathcal{O}_X(n-1))) \cong H^1(\mathbb{P}^1, \pi_*(f^*\mathcal{O}_X(n-1)) \otimes \mathcal{O}_{\mathbb{P}^1}(-1)).$$

This is possible only if $h^1(\mathbb{P}^1, \pi_*(f^*\mathcal{O}_X(n-1))) = 0$. $\square$

Proof of Theorem 2.1. Let Z satisfy the hypotheses of Proposition 2.2, and let S and f satisfy the conclusion of Proposition 2.2. The injective sheaf homomorphism $df : T_S \to f^*T_X$ induces a multiplication map,

$$\bigwedge^2 T_S \otimes \bigwedge^{n-4} f^*T_X \to \bigwedge^{n-2} f^*T_X.$$

Because $\bigwedge^3 T_S$ is the zero sheaf, the kernel of the multiplication map contains the image of the following sheaf homomorphism

$$\bigwedge^2 T_S \otimes T_S \otimes \bigwedge^{n-5} f^*T_X \to \bigwedge^2 T_S \otimes \bigwedge^{n-4} f^*T_X.$$

This image is precisely the kernel of the following sheaf homomorphism

$$\bigwedge^2 T_S \otimes \bigwedge^{n-4} f^*T_X \to \bigwedge^2 T_S \otimes \bigwedge^{n-4} (f^*T_X/T_S).$$

Therefore the multiplication map factors uniquely through this surjective sheaf homomorphism, i.e., there is an induced sheaf homomorphism,

$$\bigwedge^2 T_S \otimes \bigwedge^{n-4} (f^*T_X/T_S) \to \bigwedge^{n-2} f^*T_X.$$

Because $\bigwedge^{n-2} f^*T_X$ is locally free and X is integral, every coherent subsheaf is torsion-free. Thus the homomorphism factors through the torsion-free quotient,

$$\bigwedge^2 T_S \otimes \bigwedge^{n-4} (f^*T_X/T_S)/\text{Torsion} \to \bigwedge^{n-2} f^*T_X.$$

On the open dense subset of S where f is unramified, this homomorphism is clearly injective. Because the domain of the homomorphism is torsion-free and S is integral, the sheaf homomorphism is injective on all of X. Tensoring with the canonical bundle of ω_S, this gives an injective sheaf homomorphism

$$(\bigwedge^{n-4}(f^*T_X/T_S))/\text{Torsion} \to \omega_S \otimes \bigwedge^{n-2} f^*T_X.$$

By hypothesis, $(\bigwedge^{n-4}(f^*T_X/T_S))/\text{Torsion}$ has a nonzero global section. Therefore $\omega_S \otimes \bigwedge^{n-2} f^*T_X$ also has a nonzero global section.

On the other hand, for Z satisfying the hypothesis of Theorem 2.1, Proposition 2.4 implies,

$$h^0(S, \omega_S \otimes \bigwedge^{n-2} f^*T_X) = 0.$$

Thus Z does not satisfy the hypothesis of Proposition 2.4, i.e., it does not satisfy Hypothesis (iii). Therefore there exists a codimension 2 subvariety of X containing all the curves parametrized by $Z \times \mathbb{P}^1$. $\square$

Proof of Theorem 1.3. For every generically-finite, rational transformation $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow X$ restricting to a closed immersion on a general fiber, there is an associated rational transformation

$$\mathbb{P}^1 \dashrightarrow \text{Hilb}(X), \quad t \mapsto \text{Image}(\{t\} \times \mathbb{P}^1).$$

By properness of the Hilbert scheme and the valuative criterion, this extends to a regular morphism. Therefore, associated to each rational transformation is an element in the Hom-scheme $\text{Hom}(\mathbb{P}^1, \text{Hilb}(X))$. Those rational transformations satisfying the hypothesis of Theorem 1.3 give a locally closed subset of $\text{Hom}(\mathbb{P}^1, \text{Hilb}(X))$. As $\text{Hom}(\mathbb{P}^1, \text{Hilb}(X))$ is a countable union of quasi-projective varieties, this subset is also a countable union of quasi-projective subvarieties. By Theorem 2.1, for each such subvariety Z, there is a codimension-2 subvariety of X containing every curve parametrized by $Z \times \mathbb{P}^1$. This subvariety contains the image of each rational transformation $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow X$ giving a point in Z. Therefore, there exists a countable union of codimension-2 subvarieties of X containing the image of every rational transformation satisfying the hypothesis of Theorem 2.1. $\square$

The proof of the Theorem 1.4 is similar to the proof of Theorem 2.1. There is a preliminary proposition.

Proposition 2.5. *Let X be a smooth hypersurface of degree n in $\mathbb{P}^n$. For every Del Pezzo surface S and every generically finite morphism $f : S \to X$, the only global section of $\bigwedge^{n-4}(f^*T_X/T_S)/$Torsion is the zero section.*

Proof. The proof is similar to the proof of Theorem 2.1. By the same type of argument, it suffices to prove,

(i) $h^0(S, \omega_S \otimes f^*\mathcal{O}_X(-n) \otimes \bigwedge^{n-1} f^*T_{\mathbb{P}^n}) = 0$, and
(ii) $h^1(S, \omega_S \otimes f^*\mathcal{O}_X(-n) \otimes \bigwedge^{n-1} f^*T_X) = 0$.

The proof of (i) is the same as in the proof of Theorem 2.1, since $h^0(S,\omega_S)=0$. As for (ii), there is a canonical isomorphism

$$\omega_S \otimes f^*\mathcal{O}_X(-n) \otimes \bigwedge^{n-1} f^*T_X \cong (\omega_S^{-1} \otimes f^*\mathcal{O}_X(n-1))^{-1}.$$

Denote $\omega_S^{-1} \otimes f^*\mathcal{O}_X(n-1)$ by L. The sheaf $f^*\mathcal{O}_X(n-1)$ is globally generated. By the hypothesis that S is a Del Pezzo surface, ω_S^{-1} is ample. Thus L is ample. By Kodaira vanishing, $h^1(S,L^{-1})=0$. So, using the canonical isomorphism, $h^1(S,\omega_S \otimes f^*\mathcal{O}_X(-n) \otimes \bigwedge^{n-1} f^*T_X)=0$. $\square$

Proof of Theorem 1.4. By the same countability argument as at the beginning of the section, it suffices to prove that for every flat family

$$\begin{tikzcd} D \arrow[r, "\phi"] \arrow[d, "p"] & X \times V \\ V & \end{tikzcd}$$

such that ϕ is generically finite and a general fiber of p is a Del Pezzo surface, the image of ϕ is contained in a subvariety of codimension ≥ 2. Let S be the fiber of p over a general point of V, and let f be the restriction $\phi|_S : S \to X$. As in the proof of the Proposition 2.2, if the image of ϕ is contained in no subvariety of codimension ≥ 2, then $H^0(S, \bigwedge^{n-4}(f^*T_X/T_S)/\text{Torsion})$ has a nonzero global section. Thus Proposition 2.5 proves the image of ϕ is contained in a subvariety of codimension ≥ 2. $\square$

3. The second approach

Let $n \geq 4$ be an integer. Let X be a smooth hypersurface in $\mathbb{P}^n$ of degree n. Let M be a proper, smooth variety parametrizing a family of genus g curves in X.

Theorem 3.1. *Assume $g=0$ or 1. Assume dim$(M)=n-3$ and the curves parametrized by M sweep an $(n-2)$-fold in X. Assume a general curve parametrized by M is embedded, smooth, and $(n-1)$-normal. Further, if $g=1$, assume the curve is nondegenerate. Then $h^{n-3,0}(M) \geq 1$. In particular, M is not uniruled.*

Here is the outline of the proof.

(i) There is a natural correspondence between X and M giving a map
$$\phi : H^1(X,\Omega_X^{n-2}) \to H^0(M,\Omega_M^{n-3}).$$
It suffices to prove $\phi \neq 0$.

(ii) For a point m of M, taking the fiber of a global section of Ω_M^{n-3} at m gives a map
$$\phi_m : H^1(X,\Omega_X^{n-2}) \to \Omega_M^{n-3} \otimes \kappa(m).$$
If m is a general point of M, there is a description of ϕ_m as the composition of a number of explicit "component" maps. To prove $\phi \neq 0$, it suffices to prove ϕ_m is surjective.

(iii) The hypotheses imply the component maps are surjective. For instance, one of the component maps is the "restriction of sections" map
$$H^0(X,\mathcal{O}_X(n-1)) \to H^0(C,\mathcal{O}_X(n-1)|_C)$$

where C is the curve parametrized by m. Surjectivity of this map is precisely the hypothesis that C is $(n-1)$-normal.

There is one preliminary reduction having to do with the *type* of family of curves parametrized by M. Embedded smooth curves in X are parametrized by a scheme that is simultaneously an open subset of three different proper parameter spaces: the Chow variety of X, the Hilbert scheme of X and the Kontsevich space of stable maps to X. Thus there are (at least) three ways to generalize the notion of embedded smooth curves. This last one is most useful to us. The Kontsevich (coarse moduli) space is denoted by $\overline{\mathrm{M}}_{g,0}(X)$.

Because a general curve parametrized by M is embedded and smooth, there is a rational transformation $\zeta : M \dashrightarrow \overline{\mathrm{M}}_{g,0}(X)$. For smooth projective varieties M, $h^{n-3,0}(M)$ is a birational invariant. So we are free to blow-up M without changing $h^{n-3,0}(M)$. Therefore assume ζ is regular on all of M.

There is a universal family of curves over $\overline{\mathrm{M}}_{g,0}(X)$ and a map from this family to X. The pullback family of curves is denoted $\pi : C_M \to M$. The pullback of the map is denoted $f_M : C_M \to X$.

3.1. **Construction of the forms.** First we construct the $(n-3)$-forms on M. Then there will be some work to prove some of these are nonzero. The $(n-3)$-forms are constructed using the obvious correspondence

$$M \xleftarrow{\pi} C_M \xrightarrow{f_M} X$$

between M and X. The first part of this correspondence is straightforward: For every pair of integers (p,q) there is a pullback map

$$f_M^* : H^{q+1}(X, \Omega_X^{p+1}) \to H^{q+1}(C_M, \Omega_{C_M}^{p+1}).$$

The other part of the correspondence is an "integration along fibers" map

$$I : H^{q+1}(C_M, \Omega_{C_M}^{p+1}) \to H^q(M, \Omega_M^p).$$

Naively it is clear what this is, but we include the construction below for completeness.

Integration along fibers. Let $U \subset M$ be the (dense) open subset over which $C_U := \pi^{-1}(U)$ is a family of smooth curves. There is a canonical isomorphism

$$\pi^*\Omega_U^p \otimes \omega_\pi|_U \cong \Omega_{C_u}^{p+1}/\pi^*\Omega_U^{p+1}$$

where $\omega_\pi|_U$ is the relative cotangent bundle $\Omega_{C_U/U}$. Composing with the quotient map gives

$$\phi_p : \Omega_{C_U}^{p+1} \to \pi^*\Omega_U^p \otimes \omega_\pi|_U$$

As proved in [3] this extends to a sheaf homomorphism on all of C_M

$$\phi_p : \Omega_{C_M}^{p+1} \to \pi^*\Omega_M^p \otimes \omega_\pi.$$

This sheaf homomorphism gives a map of cohomology

$$\phi_{p,*} : H^{q+1}(C_M, \Omega_{C_M}^{p+1}) \to H^{q+1}(C_M, \pi^*\Omega_M^p \otimes \omega_\pi).$$

The morphism π determines a Leray spectral sequence for cohomology of sheaves on C_M. The spectral sequence has an abutment map

$$H^{q+1}(C_M, \pi^*\Omega_M^p \otimes \omega_\pi) \to H^q(M, \Omega_M^p \otimes R^1\pi_*\omega_\pi).$$

Of course there is a trace isomorphism

$$R^1\pi_*\omega_\pi \xrightarrow{\cong} \mathcal{O}_M.$$

So the abutment map can be written as

$$a : H^{q+1}(C_M, \pi^*\Omega^p_M \otimes \omega_\pi) \to H^q(M, \Omega^p_M).$$

This gives the *integration along fibers* map

$$I = a \circ \phi_{p,*} : H^{q+1}(C_M, \Omega^{p+1}_{C_M}) \to H^q(M, \Omega^p_M).$$

Lemma 3.2 ([3])**.** *For every pair of integers (p, q) there is a natural map of $\mathbb{C}$-vector spaces*

$$\phi_{p,q} = I \circ f^*_M : H^{q+1}(X, \Omega^{p+1}_X) \to H^q(M, \Omega^p_M).$$

In particular, this gives a map of $\mathbb{C}$-vector spaces

$$\phi = \phi_{n-3,0} : H^1(X, \Omega^{n-2}_X) \to H^0(M, \Omega^{n-3}_M).$$

3.2. **Description of the forms – overview.** The goal is to prove $\phi \neq 0$. Proving this directly from the definition of ϕ seems difficult: the definition is fairly simple, but not very explicit.

What would be a more explicit description of ϕ? Because elements of $H^0(M, \Omega^{n-3}_M)$ are sections of the sheaf Ω^{n-3}_M, one possibility is to try to describe the values of these sections at some point $m \in M$. The fiber $\Omega^{n-3}_M \otimes \kappa(m)$ is the vector space of $(n-3)$-linear alternating forms on the Zariski tangent space T_mM. Thus, given a point $m \in M$, given an element $\alpha \in H^1(X, \Omega^{n-2}_X)$ and given an $(n-3)$-tuple of tangent vectors $\theta_1, \dots, \theta_{n-3} \in T_mM$, the map ϕ is a rule

$$(\alpha, \theta_1, \dots, \theta_{n-3}) \mapsto \langle \phi(\alpha)|_m, \theta_1 \wedge \cdots \wedge \theta_{n-3} \rangle \in \mathbb{C}.$$

Equivalently, it is a map

$$\phi_m : H^1(X, \Omega^{n-2}_X) \to (\bigwedge^{n-3} T_mM)^\vee.$$

An *explicit description* of ϕ is quite simply an explicit description of the rule ϕ_m for general m. In particular, if $\phi_m \neq 0$ for some m then $\phi \neq 0$.

In fact there is such a description, at least when m is in the open subset U parametrizing embedded smooth curves: ϕ_m is the composition of a number of simple, explicit maps. To be honest, there are quite a number of these component maps. For this reason the composition ϕ_m is not very simple. But it is explicit.

Lemma 3.3. *Denote by $\mathcal{L}$ the invertible sheaf on C*

$$\mathcal{L} := \bigwedge^{n-2} N_{C/X} \otimes N^\vee_{X/\mathbb{P}^n}|_C.$$

There is a commutative diagram

$$\begin{tikzcd}
H^0(X,\Omega^n_{\mathbb{P}^n}(2X)|X) \arrow[r, "res"] \arrow[d, "r"'] & H^1(X,\Omega^{n-2}_X) \arrow[ddddd, "\phi_m"] \\
H^0(C,\Omega^n_{\mathbb{P}^n}(2X)|_C) \arrow[d, "H^0(C,\psi)"'] & \\
H^0(C,\omega_C\otimes\mathcal{L}^\vee) \arrow[d, "s"'] & \\
H^1(C,\bigwedge^{n-2}N_{C/X}\otimes N^\vee_{X/\mathbb{P}^n}|_C)^\vee \arrow[d, "\delta^\dagger"'] & \\
H^0(C,\bigwedge^{n-3}N_{C/X})^\vee \arrow[d, "w^\dagger"'] & \\
(\bigwedge^{n-3}H^0(C,N_{C/X}))^\vee \arrow[r, "(\bigwedge d\zeta)^\dagger"] & (\bigwedge^{n-3}T_mM)^\vee
\end{tikzcd} \tag{6}$$

Of course this is meaningless without an explicit description of the component maps, which we give next. But before describing the component maps it is worth putting things in perspective. The proof that $\phi_m\neq 0$ uses only Lemma 3.3, not the definition from Lemma 3.2. So, the reader is probably asking, why did we bother with Lemma 3.2? Think of it this way: We are about to give an explicit map from global $(n-2,1)$-forms on X to $(n-3,0)$-forms on U. Holomorphic $(n-3,0)$-forms on U are *meromorphic* $(n-3,0)$-forms on M (with poles contained in $M-U$). Now a uniruled variety has no nonzero holomorphic forms, but it has *many* nonzero meromorphic forms. Therefore it is crucial that these meromorphic forms are actually holomorphic on all of M. This is precisely what Lemma 3.2 gives, i.e., Lemma 3.2 is a global regularity result.

In order to make the rule ϕ_m explicit, we first must make the *inputs* of the rule explicit: $\alpha\in H^1(X,\Omega^{n-2}_X)$ and the tangent vectors $\theta_1,\dots,\theta_{n-3}$. This is what we do next.

3.3. **Description of the inputs – Griffiths residue calculus.** Griffiths described the primitive middle cohomology of a hypersurface X in $\mathbb{P}^n$ as the residues of meromorphic n-forms on $\mathbb{P}^n$ with poles along X, cf. [5, Section 8]. One can make the description purely algebraic. For the part we need, this is particularly simple. There is a short exact sequence of sheaves on X

$$0\longrightarrow\Omega^{n-2}_X\longrightarrow\Omega^{n-2}_{\mathbb{P}^n}(X)|_X\longrightarrow\Omega^{n-1}_X\otimes\mathcal{O}_{\mathbb{P}^n}(X)|_X\longrightarrow 0. \tag{7}$$

The long exact sequence of cohomology gives a connecting map

$$\text{res}: H^0(X,\Omega^n_{\mathbb{P}^n}(2X)|_X)\to H^1(X,\Omega^{n-2}_X) \tag{8}$$

By a small amount of diagram-chasing

$$h^1(X,\Omega^{n-2}_{\mathbb{P}^n}(X)|_X)=0.$$

Therefore res is surjective, i.e., every element $\alpha \in H^1(X, \Omega_X^{n-2})$ is the image $\mathrm{res}(\beta)$ of an element

$$\beta \in H^0(X, \Omega^n_{\mathbb{P}^n}(2X)|_X).$$

The usual way of saying this is that an $(n-2,1)$-form α on X is the "residue" of a meromorphic n-form β on $\mathbb{P}^n$ with a double pole along X. (Of course Griffiths describes all (p,q)-forms on X in terms of residues, but we only need the $(n-2,1)$-forms.) Thus the linear map ϕ_m determines a linear map

$$\phi_m \circ \mathrm{res} : H^0(X, \Omega^n_{\mathbb{P}^n}(2X)|_X) \to (\bigwedge^{n-3} T_m M)^\vee.$$

Since res is surjective, ϕ_m is uniquely determined by $\phi_m \circ \mathrm{res}$.

While we're at it, the component map r is just restriction of global sections of $\Omega^n_{\mathbb{P}^n}(2X)|_X$ to the curve C parametrized by m,

$$r : H^0(X, \Omega^n_{\mathbb{P}^n}(2X)|_X) \to H^0(C, \Omega^n_{\mathbb{P}^n}(2X)|_C).$$

3.4. **Description of the inputs – the tangent vectors.** The inputs of the rule ϕ_m are an element $\alpha \in H^1(X, \Omega_X^{n-2})$ and an $(n-3)$-tuple of tangent vectors $\theta_1, \ldots, \theta_{n-3} \in T_m M$. The previous section describes α in terms of a more explicit element $\beta \in H^0(X, \Omega^n_{\mathbb{P}^n}(2X)|X)$. Lemma 3.3 asserts that ϕ_m actually depends only on the restriction $r(\beta) \in H^0(C, \Omega^n_{\mathbb{P}^n}(2X)|C)$, where C is the curve in X parametrized by m.

In a similar way, Lemma 3.3 asserts that ϕ_m depends on the tangent vectors θ_i only through certain associated elements, which are more explicit. Recall there is a morphism $\zeta : U \to \overline{\mathcal{M}}_{g,0}(X)$. The derivative of ζ is a map

$$d\zeta : T_m M \to T_{[C]}\overline{\mathcal{M}}_{g,0}(X)$$

where C is the embedded smooth curve parametrized by m. Since

$$T_{[C]}\overline{\mathcal{M}}_{g,0}(X) = H^0(C, N_{C/X})$$

$d\zeta(\theta_i)$ is just a global section of the normal bundle $N_{C/X}$ of C in X.

To be precise, the derivative $d\zeta$ gives a map between exterior powers

$$\bigwedge d\zeta : \bigwedge^{n-3} T_m M \to \bigwedge^{n-3} H^0(C, N_{C/X}).$$

The transpose is a map

$$(\bigwedge d\zeta)^\dagger : (\bigwedge^{n-3} H^0(C, N_{C/X}))^\vee \to (\bigwedge^{n-3} T_m M)^\vee.$$

It will turn out there is a map

$$\phi_{[C]} : H^0(C, \Omega^n_{\mathbb{P}^n}(2X)|_C) \to (\bigwedge^{n-3} H^0(C, N_{C/X}))^\vee$$

such that

$$\phi_m \circ \mathrm{res} = \phi_m^{(1)} = \phi_m^{(2)} \circ r, \quad \phi_m^{(2)} = (\bigwedge d\zeta)^\dagger \circ \phi_{[C]}.$$

There is another manipulation with the inputs $d\zeta(\theta_i)$. Wedging global sections determines a map

$$w : \bigwedge^{n-3} H^0(C, N_{C/X}) \to H^0(C, \bigwedge^{n-3} N_{C/X}).$$

The component map $w^\dagger$ is just the transpose

$$w^\dagger : H^0(C, \bigwedge^{n-3} N_{C/X})^\vee \to (\bigwedge^{n-3} H^0(C, N_{C/X}))^\vee.$$

3.5. **Description of the rule – the tangent bundle sequence.** The next component map relates $H^0(C, \bigwedge^{n-3} N_{C/X})^\vee$ to $H^1(C, \mathcal{L})^\vee$ for a line bundle $\mathcal{L}$. Then Serre duality will relate this to $H^0(C, \omega_C \otimes \mathcal{L}^\vee)$. The final component map will come from a canonical isomorphism of $\omega_C \otimes \mathcal{L}^\vee$ with another natural line bundle on C.

The relation of $H^0(C, \bigwedge^{n-3} N_{C/X})^\vee$ to $H^1(C, \mathcal{L})^\vee$ comes by way of the tangent bundle sequence of X in $\mathbb{P}^n$

$$0 \longrightarrow T_X \longrightarrow T_{\mathbb{P}^n}|_X \longrightarrow N_{X/\mathbb{P}^n} \longrightarrow 0.$$

This gives a sequence of normal bundles on C

$$0 \longrightarrow N_{C/X} \longrightarrow N_{C/\mathbb{P}^n} \longrightarrow N_{X/\mathbb{P}^n}|_C \longrightarrow 0.$$

Taking $(n-2)^{\mathrm{nd}}$ exterior powers turns this into a sequence

$$0 \to \bigwedge^{n-2} N_{C/X} \to \bigwedge^{n-2} N_{C/\mathbb{P}^n} \to (\bigwedge^{n-3} N_{C/X}) \otimes N_{X/\mathbb{P}^n}|_C \to 0.$$

Twisting each term by $N^\vee_{X/\mathbb{P}^n}|_C$ gives an exact sequence,

$$0 \to \bigwedge^{n-2} N_{C/X} \otimes N^\vee_{X/\mathbb{P}^n}|_C \to \bigwedge^{n-2} N_{C/\mathbb{P}^n} \otimes N^\vee_{X/\mathbb{P}^n}|_C \to \bigwedge^{n-3} N_{C/X} \to 0. \qquad (9)$$

The long exact sequence of cohomology turns this into a connecting map

$$\delta : H^0(C, \bigwedge^{n-3} N_{C/X}) \to H^1(C, \bigwedge^{n-2} N_{C/X} \otimes N^\vee_{X/\mathbb{P}^n}|_C).$$

The component map $\delta^\dagger$ is just the transpose

$$\delta^\dagger : H^1(C, \bigwedge^{n-2} N_{C/X} \otimes N^\vee_{X/\mathbb{P}^n}|_C)^\vee \to H^0(C, \bigwedge^{n-3} N_{C/X})^\vee.$$

Define $\mathcal{L}$ to be the line bundle

$$\mathcal{L} := \bigwedge^{n-2} N_{C/X} \otimes N^\vee_{X/\mathbb{P}^n}|_C.$$

The component map s is the isomorphism given by Serre duality

$$s : H^0(C, \omega_C \otimes \mathcal{L}^\vee) \xrightarrow{\cong} H^1(C, \mathcal{L})^\vee = H^1(C, \bigwedge^{n-2} N_{C/X} \otimes N^\vee_{X/\mathbb{P}^n}|_C)^\vee.$$

3.6. **Description of the rule – the line bundle $\mathcal{L}$.** The last thing we need to understand the rule ϕ_m is an alternative description of the line bundle

$$\mathcal{L} := \bigwedge^{n-2} N_{C/X} \otimes N^\vee_{X/\mathbb{P}^n}|_C.$$

To obtain this, we will identify each of the two factors in this tensor product.

The first factor. The factor $\bigwedge^{n-2} N_{C/X}$ is the determinant of $N_{C/X}$. By adjunction for the embedding $C \hookrightarrow X$ the determinant of $N_{C/X}$ is canonically isomorphic to $\omega_C \otimes (\Omega^{n-1}_X)^\vee|_C$.

On the other hand, adjunction for the embedding $X \hookrightarrow \mathbb{P}^n$ gives an isomorphism $\Omega_X^{n-1} \cong \Omega^n_{\mathbb{P}^n}(X)|_X$. These two adjunction isomorphisms together give

$$\bigwedge^{n-2} N_{C/X} \cong \omega_C \otimes (\Omega^n_{\mathbb{P}^n}(X))^\vee|_C.$$

The second factor. Also, $N_{X/\mathbb{P}^n}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^n}(X)|_X$. Thus the factor, $N^\vee_{X/\mathbb{P}^n}|_C$, is isomorphic to $\mathcal{O}_{\mathbb{P}^n}(X)^\vee|_C$. Together these two identifications give an identification

$$\mathcal{L} := \bigwedge^{n-2} N_{C/X} \otimes N^\vee_{X/\mathbb{P}^n}|_C \cong \omega_C \otimes [\Omega^n_{\mathbb{P}^n}(2X)]^\vee|_C. \tag{10}$$

Taking the transpose and twisting by ω_C gives the isomorphism of $\mathcal{O}_C$-modules

$$\psi : \Omega^n_{\mathbb{P}^n}(2X)|_C \xrightarrow{\cong} \omega_C \otimes [\bigwedge^{n-2} N_{C/X} \otimes N^\vee_{X/\mathbb{P}^n}|_C]^\vee = \omega_C \otimes \mathcal{L}^\vee.$$

The final component map $H^0(C,\psi)$ is the map of global sections associated to ψ

$$H^0(C,\psi) : H^0(C, \Omega^n_{\mathbb{P}^n}(2X)|_C) \to H^0(C, \omega_C \otimes \mathcal{L}^\vee).$$

Proof of Lemma 3.3. In the special case that $n = 5$, this is [3, Theorem 5.1]. The proof in the general case is very similar. It amounts to much more of the same sort of diagram-chasing above. We will leave the details to the reader, referring to the proof of [3, Theorem 5.1] for the key ideas.

□

3.7. **Proof of the theorem.** Finally we are ready to prove Theorem 3.1. It suffices to prove the map ϕ from Lemma 3.2 is nonzero.

Proof of Theorem 3.1. Recall there is a morphism $\zeta : M \to \overline{\mathrm{M}}_{g,0}(X)$. The subvariety of X swept by curves in M equals the subvariety swept by curves in $\zeta(M)$. Thus the dimension of the subvariety swept by curves in M has dimension $\leq 1 + \dim(\zeta(M))$. By hypothesis, the curves parametrized by M sweep an $(n-2)$-dimensional subvariety of X. Thus $\dim(\zeta(M)) = n-3$, i.e., ζ is generically finite. Because our base field is characteristic 0 and because M is smooth, ζ is generically unramified.

The hypothesis that the curves of M sweep an $(n-2)$-dimensional subvariety of X also implies the map $f_U : C_U \to X$ is generically finite, where $U \subset M$ parametrizes only smooth curves and $C_U \to U$ is this family of smooth curves. Because C_U is smooth and the characteristic is zero, f_U is generically unramified, i.e., the locus where f_U is ramified is a proper subvariety of C_U. Therefore, for a general point $m \in U$ the map f_U is unramified at a general point of the fiber $C_m \subset C_U$.

Let V be the maximal open subset of M such that

(i) $\zeta|_V : V \to \overline{\mathrm{M}}_{g,0}(X)$ is unramified,
(ii) every curve C_m parametrized by $m \in V$ is embedded, smooth and $(n-1)$-normal,
(iii) if $g = 1$, C_m is also nondegenerate,
(iv) and f_U is unramified at a general point of $C_m \subset C_U$.

The arguments above prove V is dense in M. The claim is that for every $m \in V$, $\phi_m \neq 0$. Because the target of ϕ_m is 1-dimensional this is equivalent to the claim that ϕ_m is surjective.

Of course it suffices to prove that $\phi_m \circ \mathrm{res}$ is surjective. This is equivalent to the claim that the transpose map $(\phi_m \circ \mathrm{res})^\dagger$ is injective. By Lemma 3.3

$$\phi_m \circ \mathrm{res} = (\bigwedge d\zeta)^\dagger \circ w^\dagger \circ \delta^\dagger \circ s \circ H^0(C, \psi) \circ r.$$

Thus the transpose is

$$(\phi_m \circ \mathrm{res})^\dagger = r^\dagger \circ H^0(C, \psi)^\dagger \circ s^\dagger \circ \delta \circ w \circ \bigwedge d\zeta.$$

We will prove this in stages, by first proving injectivity of the maps

$$(w \circ \bigwedge d\zeta), s^\dagger, H^0(C, \psi)^\dagger, \text{ and } r^\dagger.$$

We reserve injectivity of δ for last, since this is the most difficult to verify.

Injectivity of $w \circ \bigwedge d\zeta$. It turns out this is precisely Item (iv) in the definition of V. By hypothesis, there exists $p \in C_m$ such that f_U is unramified at p. Consider the sheaf homomorphism

$$df_U : N_{C_m/C_U} \to N_{C_m/X}$$

and take its $(n-3)^{\mathrm{rd}}$ exterior powers

$$\bigwedge df_U : \bigwedge^{n-3} N_{C_m/C_U} \to \bigwedge^{n-3} N_{C_m/X}.$$

Both sheaves are locally free, C_m is integral, and the sheaf homomorphism is injective at p by hypothesis. Therefore the sheaf homomorphism is injective. Thus the map of global sections

$$H^0(C, \bigwedge df_U) : H^0(C, \bigwedge^{n-3} N_{C_m/C_U}) \to H^0(C, \bigwedge^{n-3} N_{C_m/X})$$

is also injective. Because C_m is a fiber of $C_U \to U$,

$$N_{C_m/C_U} = T_m M \otimes_k \mathcal{O}_{C_m}.$$

Therefore we have canonical isomorphisms

$$H^0(C, \bigwedge^{n-3} N_{C_m/C_U}) = \bigwedge^{n-3} T_m M, \quad H^0(C, \bigwedge df_U) = w \circ \bigwedge d\zeta.$$

Because $H^0(C, \bigwedge df_U)$ is injective, $w \circ \bigwedge d\zeta$ is injective.

Injectivity of $H^0(C, \psi)^\dagger$ and $s^\dagger$. Because ψ is an isomorphism, so is $H^0(C, \psi)$, and thus also $H^0(C, \psi)^\dagger$. Similarly, the Serre duality map s is an isomorphism, and thus also $s^\dagger$ is an isomorphism.

Injectivity of $r^\dagger$. This is equivalent to proving surjectivity of r. Recall r is the restriction map

$$r : H^0(X, \Omega^n_{\mathbb{P}^n}(2X)|_X) \to H^0(C, \Omega^n_{\mathbb{P}^n}(2X)|_C).$$

Now $\Omega_{\mathbb{P}^n}(2X)$ is isomorphic to $\mathcal{O}_{\mathbb{P}^n}(-n-1) \otimes \mathcal{O}_{\mathbb{P}^n}(2n) \cong \mathcal{O}_{\mathbb{P}^n}(n-1)$. Therefore r is the same as the restriction map

$$H^0(X, \mathcal{O}_X(n-1)) \to H^0(C, \mathcal{O}_X(n-1)|_C).$$

The hypothesis that C is $(n-1)$-normal precisely says this restriction map is surjective.

Injectivity of δ. Associated to the short exact sequence in Equation 9, there is an exact sequence of cohomology

$$H^0(C, \bigwedge^{n-2} N_{C/\mathbb{P}^n} \otimes N^\vee_{X/\mathbb{P}^n}|_C) \to H^0(C, \bigwedge^{n-3} N_{C/X}) \xrightarrow{\delta} H^1(C, \bigwedge^{n-2} N_{C/X} \otimes N^\vee_{X/\mathbb{P}^n}|_C). \tag{11}$$

Claim 3.4. *$h^0(C, \bigwedge^{n-2} N_{C/\mathbb{P}^n} \otimes N^\vee_{X/\mathbb{P}^n}|_C) = 0$; in particular δ is injective.*

Proof of Claim 3.4. We reduce the original claim to a second claim that is similar. The second claim will follow from the hypotheses on g and on the curve C.

To begin with there is an isomorphism

$$\bigwedge^{n-2} N_{C/\mathbb{P}^n} \cong N^\vee_{C/\mathbb{P}^n} \otimes \bigwedge^{n-1} N_{C/\mathbb{P}^n}.$$

Adjunction for the embedding of C in $\mathbb{P}^n$ gives an isomorphism

$$\bigwedge^{n-1} N_{C/\mathbb{P}^n} \cong \omega_C \otimes (\Omega^n_{\mathbb{P}^n})^\vee|_C \cong \omega_C \otimes \mathcal{O}_{\mathbb{P}^n}(n+1)|_C.$$

Together with the isomorphism $N^\vee_{X/\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^N}(-n)|_X$ this gives

$$N_{C/\mathbb{P}^n} \otimes N^\vee_{X/\mathbb{P}^n}|_C \cong \omega_C \otimes \mathcal{O}_{\mathbb{P}^n}(1)|_C \otimes N^\vee_{C/\mathbb{P}^n}.$$

In other words $N_{C/\mathbb{P}^n} \otimes N^\vee_{X/\mathbb{P}^N}|_C$ is obtained by twisting the vector bundle $N^\vee_{C/\mathbb{P}^n}$ by the line bundle $\omega_C \otimes \mathcal{O}_{\mathbb{P}^n}(1)|_C$.

The vector bundle $N^\vee_{C/\mathbb{P}^n}$ is a subsheaf of $\Omega_{\mathbb{P}^n}|_C$,

$$0 \longrightarrow N^\vee_{C/\mathbb{P}^n} \longrightarrow \Omega_{\mathbb{P}^n}|_C \longrightarrow \Omega_C \longrightarrow 0.$$

Thus the twist $\omega_C \otimes \mathcal{O}_{\mathbb{P}^n}(1)|_C \otimes N^\vee_{C/\mathbb{P}^n}$ is a subsheaf of $\omega_C \otimes \mathcal{O}_{\mathbb{P}^n}(1)|_C \otimes \Omega_{\mathbb{P}^n}|_C$. And so it goes with global sections

$$H^0(C, \omega_C \otimes N^\vee_{C/\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}(1)|_C) \subset H^0(C, \omega_C \otimes \Omega_{\mathbb{P}^n}(1)|_C).$$

Thus Claim 3.4 follows from the following.

Claim 3.5. $h^0(C, \omega_C \otimes \Omega_{\mathbb{P}^n}(1)|_C) = 0$.

Proof of Claim 3.5. The proof will ultimately boil down to injectivity of a certain multiplication map v. This injectivity follows from the hypotheses that $g = 0$ or $g = 1$, that C is embedded and smooth, and that C is nondegenerate if $g = 1$.

There is an exact sequence on $\mathbb{P}^n$.

$$0 \to \Omega_{\mathbb{P}^n}(1) \to H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \otimes_k \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1) \to 0.$$

Pulling back and twisting with ω_C gives an isomorphism of $H^0(C, \omega_C \otimes \Omega_{\mathbb{P}^n}(1)|_C)$ with the kernel of a multiplication map v,

$$H^0(C, \omega_C \otimes \Omega_{\mathbb{P}^n}(1)|_C) \cong \mathrm{Ker}(v),$$

$$v : H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \otimes_k H^0(C, \omega_C) \to H^0(C, \omega_C \otimes \mathcal{O}_{\mathbb{P}^n}(1)|_C).$$

Thus it suffices to prove v is injective.

By hypothesis $g = 0$ or $g = 1$. If $g = 0$ then already $h^0(C, \omega_C) = 0$ and thus $\mathrm{Domain}(v) = 0$. Thus v is injective if $g = 0$.

If $g = 1$ then $\omega_C \cong \mathcal{O}_C$. Thus v is simply the restriction map

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \to H^0(C, \mathcal{O}_{\mathbb{P}^n}(1)|_C).$$

By hypothesis C is nondegenerate, i.e., the restriction map is injective. Thus again v is injective if $g = 1$. This proves Claim 3.5 and thus also Claim 3.4, both when $g = 0$ and when $g = 1$. □

Proof of Theorem 1.5. This follows from Theorem 3.1 by an argument similar to the one in the proof of Theorem 1.3. □

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