SPACES OF RATIONAL CURVES IN HYPERSURFACES

ROYA BEHESHTI AND N. MOHAN KUMAR

To Professor Seshadri on his 80th birthday, who inspired the second author and enriched his mathematical life

ABSTRACT. This is a brief survey of some old and new results on families of rational curves on hypersurfaces.

1. Introduction

In this survey, we will describe some results on families of rational curves on hypersurfaces in $\mathbb{P}^N$. The case of families of rational curves on $\mathbb{P}^N$ itself is well understood. If we denote by $R_e(X)$, the family of all smooth rational curves of degree e with respect to a given polarization on a variety X, then $R_e(\mathbb{P}^N)$ (with the usual polarization) is smooth, irreducible and of dimension $e(N+1)+N-3$ [for example see [6]]. Though similar results are false for hypersurfaces in general, they are expected to be true for small degree hypersurfaces. We are far from a complete solution to this problem.

If $X \subset \mathbb{P}^N$ is a hypersurface of degree d, then $R_1(X)$ is empty for general X with $d \geq 2N-2$ [Corollary 2.5, [6]]. So, the question is interesting only for small d. On the other hand, if $d < N$, there is a line through every point of X [Lemma 2.9, [6]]. In this range, for a general X, conjecturally $R_e(X)$ is expected to be irreducible and of the expected dimension $E(d) = e(N+1-d)+N-4$. Recently, this was settled for $d < \frac{N+1}{2}$, by Harris, Roth and Starr [4]. Improving on their techniques, Roya Beheshti and the author improved this bound to $d < \frac{2N}{3}$ [1]. See the text for more precise statements.

As you would imagine, proving that $R_e(X)$ has the expected dimension for all e can be difficult. But a powerful theorem, which crucially depends on Mori's bend-and-break techniques, by Harris et. al. reduces the result to proving similar results for small e. For example, when $d < \frac{N+1}{2}$, one only need to prove the result for $R_1(X)$, that is, the family of lines. This is an oversimplification, but see the text again for more precise statements. Similarly, for handling the case $d < \frac{2N}{3}$, one is reduced to studying $R_1(X)$ and $R_2(X)$. So, in some sense, our results are really about families of conics in X.

One remark to be made is about the need for such results in enumerative geometry. If one wishes to count say, rational curves on X which intersects a general linear space (or a collection of such) of appropriate codimension, these numbers can be calculated using the theory of Gromov-Witten invariants. But, these are virtual numbers and acquire enumerative significance only if $R_e(X)$ are known to be of the expected dimension.

We would have liked this survey to be as self contained as possible. But, due to various reasons, we have not included details on Kontsevich moduli stack or Gromov-Witten invariants. These are explained in detail in [7, 3, 5] if the reader is interested. Another result we have used but not proved in these notes is the one that we alluded to earlier by Harris et. al. I refer the reader to their paper [4].

The second author thank the organizers of Seshadrifest, Professors V. Balaji and Pramath Sastry, for kindly including me among the speakers and making his stay in CMI, Chennai a memorable experience.

2. Lines on Hypersurfaces

In general, we will always work with hypersurfaces of degree $d \geq 2$ in $\mathbb{P}^N$ with $N \geq 3$ and our base field will always be $\mathbb{C}$, the field of complex numbers. We recall some well known results about lines on hypersurfaces [for e. g.. see Kollar's book, [6]] and some of the recent results of Harris, Roth and Starr [4].

We start with a basic result from deformation theory [10, Section 4.5].

If $X \subset Y \subset \mathbb{P}^N$ are closed subschemes, we have canonical maps of normal sheaves, $N_{X/\mathbb{P}^N} \to N_{Y/\mathbb{P}^N|X}$ and $N_{Y/\mathbb{P}^N} \to N_{Y/\mathbb{P}^N|X}$ and then as usual we denote by,

$$H^0(N_{X/\mathbb{P}^N}) \times_{H^0(N_{Y/\mathbb{P}^N|X})} H^0(N_{Y/\mathbb{P}^N}),$$

the fiber product, which is the set of all elements (α, β) with $\alpha \in H^0(N_{X/\mathbb{P}^N}), \beta \in H^0(N_{Y/\mathbb{P}^N})$ where the images of α, β in $H^0(N_{Y/\mathbb{P}^N|X})$ are equal.

Theorem 2.1 (Flag Hilbert Schemes). *Let H_i be certain Hilbert schemes of closed subschemes in $\mathbb{P}^N$ for $1 \leq i \leq r$ and let $\mathbb{Y} \subset H_1 \times H_2 \times \cdots \times H_r$ be the incidence variety, $(X_1, X_2, \ldots, X_r)$ with $X_i \in H_i$ and $X_i \subset X_{i+1}$ for $1 \leq i < r$. For ease of notation, let $N_i = N_{X_i/\mathbb{P}^N}$ be the normal sheaf of X_i in $\mathbb{P}^N$ and let $M_i = N_{i+1|X_i}$ be the restriction of the normal* *sheaf of X_{i+1} to X_i. Then the tangent space $T_{(X_1,\dots,X_r),\mathbb{Y}}$ of $\mathbb{Y}$ at the point $(X_1,\dots,X_r)\in\mathbb{Y}$ is*

$$H^0(N_1)\times_{H^0(M_1)} H^0(N_2)\times\cdots\times_{H^0(M_{r-1})} H^0(N_r).$$

Lemma 2.2. *Let $X\subset\mathbb{P}^N$ be a smooth subavriety of codimension larger than $\dim X$. Letting I_X be its ideal sheaf, assume that $I_X(d)$ is globally generated for some $d>0$. Then there exists a smooth hypersurface of degree d containing X.*

Proof. Let $V=H^0(I_X(d))$. Since $I_X(d)$ is globally generated, given any point $p\notin X$, there exists $f\in V$ such that $f(p)\neq 0$. So, by Bertini, a general $f\in V$ has the property that $f=0$ is smooth outside X. We will prove the same for points in X and then we would be done.

The conormal bundle I_X/I_X^2 is of rank $=N-\dim X>\dim X$ by assumption and it is globally generated after twisting by d. So, by Serre's theorem [see for e. g. [9, pp. 148]], a general element $f\in V$ is a nowhere vanishing section of the conormal bundle twisted by d. I claim that $f=0$ is smooth at every point of X. Near any $p\in X$, I_X is generated by $u_i, 1\le i\le N-d$, which form a system of regular parameters, since X is smooth. So, locally, I_X/I_X^2 is generated freely by the images of the u_i's. One has $f=\sum a_iu_i$ which does not vanish at p as a section of I_X/I_X^2, and so at least one of the $a_i\neq 0$ at p. Then, it is clear that $f=0$ is smooth at p. □

Corollary 2.3. *Let $L\subset\mathbb{P}^N$ be a line with $N\ge 3$. For any $d\ge 1$, there exists a smooth hypersurface of degree d containing L.*

Proof. L is smooth, codimension of L is $N-1>1=\dim L$ and $I_L(d)$ is globally generated for any $d\ge 1$. So, the lemma applies. □

Let $\mathbb{G}$ be the Grassmannians of lines in $\mathbb{P}^N$ and $\mathbb{P}_d$ be the family of hypersurfaces of degree d (or sometimes an open subset). Let $\mathbb{Y}_d\subset\mathbb{G}\times\mathbb{P}_d$ be the incidence variety, (L,X), where L is a line and X is a hypersurface of degree d with $L\subset X$. Let $p_1:\mathbb{Y}_d\to\mathbb{G}$ and $p_2:\mathbb{Y}_d\to\mathbb{P}_d$ be the two projections.

Lemma 2.4. *With notation as above, for $d>0$ and $N>1$, p_1 makes $\mathbb{Y}_d$ into a projective space bundle over $\mathbb{G}$ and $\dim\mathbb{Y}_d=2N-3-d+\dim\mathbb{P}_d$.*

Proof. Let $L\in\mathbb{G}$, and then we have an exact sequence,

$$0\to I_L(d)\to\mathcal{O}_{\mathbb{P}^N}(d)\to\mathcal{O}_L(d)\to 0,$$

which by taking cohomologies, show that

$$\dim H^0(I_L(d))=\dim H^0(\mathcal{O}_{\mathbb{P}^N}(d))-\dim H^0(\mathcal{O}_L(d))\neq 0,$$

since the natural map $H^0(\mathcal{O}_{\mathbb{P}^N}(d)) \to H^0(\mathcal{O}_L(d))$ is onto. Thus p_1 is surjective and $p_1^{-1}(L)$ is a projective space of dimension $\dim H^0(I_L(d))-1$. Since $\dim H^0(\mathcal{O}_L(d)) = d+1$ we see that $\mathbb{Y}_d$ is a projective space bundle over $\mathbb{G}$ and $\dim \mathbb{Y}_d = \dim \mathbb{G} + \dim \mathbb{P}_d - d - 1 = 2N-3-d+\dim \mathbb{P}_d$. □

Corollary 2.5. *A general hypersurface of degree d does not contain a line if $d \geq 2N-2$.*

Proof. If $d > 2N-3$, $\dim \mathbb{Y}_d < \dim \mathbb{P}_d$ and thus p_2 is not surjective, which proves the corollary.

We give an alternate proof. Let $\mathbb{G}$ be as above and let $\Gamma \subset \mathbb{G} \times \mathbb{P}^N$ be the incidence variety (L, x) with $x \in L$ and let $p_1 : \Gamma \to \mathbb{G}$ and $p_2 : \Gamma \to \mathbb{P}^N$ be the two projections. We have the natural exact sequence,

$$0 \to M \to H^0(\mathbb{P}^N, \mathcal{O}(d)) \otimes \mathcal{O}_{\mathbb{P}^N} \to \mathcal{O}_{\mathbb{P}^N}(d) \to 0,$$

which when pulled back to Γ gives a similar exact sequence. If $L \in \mathbb{G}$, then the fiber in Γ is just the line L and restricting, we get an exact sequence,

$$0 \to M_{|L} \to H^0(\mathbb{P}^N, \mathcal{O}(d)) \otimes \mathcal{O}_L \to \mathcal{O}_L(d) \to 0,$$

which one can easily see is surjective on global sections. Thus $H^1(L, M_{|L}) = 0$. Since this is true for any L, we get $R^1 p_{1*} p_2^* M = 0$. Thus we have an exact sequence,

$$0 \to p_{1*} p_2^* M \to H^0(\mathbb{P}^N, \mathcal{O}(d)) \otimes \mathcal{O}_{\mathbb{G}} \to p_{1*} p_2^* \mathcal{O}_{\mathbb{P}^n}(d) = E \to 0,$$

the last zero, since $R^1 p_{1*} p_2^* M = 0$. Thus E is a globally generated vector bundle on $\mathbb{G}$. If $f \in H^0(\mathbb{P}^N, \mathcal{O}(d))$, the closed subscheme defined by the vanishing of f as a section of E is precisely the scheme of lines contained in $f = 0$. Also, rank of $E = d+1$ and $\dim \mathbb{G} = 2(N-1)$. If $d+1 > 2N-2$, by Serre's theorem [9], we see that a general f has no zeroes on $\mathbb{G}$ and thus there are no lines on $f = 0$. □

So from now on, we assume that $d \leq 2N-3$.

We fix some more notation. Let $S \subset \mathbb{Y}_d$ be the closed subset consisting of (L, X) such that X is singular at some point of L. Let $\mathbb{Y}'_d = \mathbb{Y}_d - S$ and let $R \subset \mathbb{Y}'_d$ be the closed subset (in $\mathbb{Y}'_d$) of (L, X) where $p_2 : \mathbb{Y}'_d \to \mathbb{P}_d$ is not smooth.

Lemma 2.6. *We assume $d \leq 2N-3$ and notation as above.*

(1) $\operatorname{codim}(S, \mathbb{Y}_d) \geq N-2$.
(2) $\operatorname{codim}(R, \mathbb{Y}'_d) \geq 2N-2-d$.

Proof. Denote by $\mathbb{P}_d(L)$, the fiber $p_1^{-1}(L)$ for a line $L \in \mathbb{G}$. It is clear that we need to prove only,

(1) $\operatorname{codim}(S \cap \mathbb{P}_d(L), \mathbb{P}_d(L)) \geq N-2$.
(2) $\operatorname{codim}(R \cap \mathbb{P}_d(L), \mathbb{Y}'_d \cap \mathbb{P}_d(L)) \geq 2N-2-d$.

So, we fix a line $L \in \mathbb{G}$ and choose coordinates so that L is given by $x_0 = x_1 = \cdots = x_{N-2} = 0$. Given $f_i \in H^0(\mathcal{O}_{\mathbb{P}^N}(d-1)), 0 \leq i \leq N-2$, we get a hypersurface given by $f = \sum x_i f_i = 0$, which contains L and conversely, given an $f \in H^0(\mathcal{O}_{\mathbb{P}^N}(d))$ such that $f=0$ contains L, we can find $f_i \in H^0(\mathcal{O}_{\mathbb{P}^N}(d-1))$ (not unique) such that $f = \sum x_i f_i$. Let $T = (f_0, f_1, \ldots, f_{N-2})$ be ordered tuples of degree $d-1$ homogeneous polynomials, at least one of them non-zero. Let $S' \subset T$ be the closed set (f_i) such that $\sum x_i f_i$ is singular at some point of L and let $\mathbb{X} \subset T$ be the complement. Let $R' \subset \mathbb{X}$ be the closed subset (of $\mathbb{X}$) of (f_i) such that p_2 is not smooth at $(L, \sum x_i f_i = 0)$. Then, it suffices to prove,

(1) $\operatorname{codim}(S', T) \geq N-2$.
(2) $\operatorname{codim}(R', \mathbb{X}) \geq 2N-2-d$.

For a point $p \in L$ and $(f_i) \in T$, $\sum x_i f_i$ is singular at p if and only if $f_i(p) = 0$ for all i. Clearly this set has codimension $N-1$ in T. But, S' is the union of these sets as p varies and thus S' is of codimension at least $N-2$ in T. This proves the first part.

The hypersurface $\sum x_i f_i = 0$ is smooth along L if and only if for any point $p \in L$, there exists an f_i such that $f_i(p) \neq 0$. So, if $(f_i) \in \mathbb{X}$, letting X denote $\sum x_i f_i = 0$, we have by Theorem [2.1], the tangent space,

$$T_{(L,X),\mathbb{Y}_d} = H^0(N_{L/\mathbb{P}^N}) \times_{H^0(N_{X/\mathbb{P}^N}|L)} H^0(N_{X/\mathbb{P}^N}).$$

Further, if $(f_i) \in R'$, p_2 is not smooth at (L, X), and so, the map $T_{(L,X),\mathbb{Y}_d} \to T_{X,\mathbb{P}_d} = H^0(N_{X/\mathbb{P}^N})$ is not onto. Since $H^0(N_{X/\mathbb{P}^n}) \to H^0(N_{X/\mathbb{P}^N}|L)$ is onto, this is equivalent to $H^0(N_{L/\mathbb{P}^N}) \to H^0(N_{X/\mathbb{P}^N}|L)$ not being onto. This map can be explicitly described as,

$$H^0(\mathcal{O}_L(1))^{\oplus N-1} \stackrel{(f_i)}{\to} H^0(\mathcal{O}_L(d)).$$

Let $V \subset H^0(\mathcal{O}_L(d))$ be a hyperplane. We will first study the closed subset $R'(V)$ of R' consisting of (f_i) such that the above image is contained in this fixed V. These are tuples (f_i) such that $f_i H^0(\mathcal{O}_L(1)) \subset V$. So, if

$$\operatorname{codim}(\{g \in H^0(\mathcal{O}_L(d-1)) | g H^0(\mathcal{O}_L(1)) \subset V\}, H^0(\mathcal{O}_L(d-1))) = r,$$

then codimension of $R'(V)$ is $r(N-1)$. Of course, since V varies along a d-dimensional variety, namely the set of hyperplanes of $H^0(\mathcal{O}_L(d))$, we will have $\operatorname{codim}(R', \mathbb{X}) \geq r(N-1)-d$. So, to prove the result, suffices to show that $r=2$.

Fix a basis u, v of $H^0(\mathcal{O}_L(1))$. Then monomials in u, v form basis for forms of any degree. So, any element in $H^0(\mathcal{O}_L(d))$ can be represented as $\sum_{i=0}^{d} a_i u^i v^{d-i}$ for $a_i \in \mathbb{C}$. Let V be defined by $\sum_{i=0}^{d} c_i a_i = 0$ for some non-zero vector (c_i). If $g = \sum_{i=0}^{d-1} b_i u^i v^{d-1-i}$ is a form such that $gu, gv \in V$, then we get, $\sum_{i=0}^{d-1} c_i b_i = 0$ and $\sum_{i=0}^{d-1} c_{i+1} b_i = 0$. If these equations are linearly independent, we get the required codimension to be two. Otherwise, we see that $sc_i = tc_{i+1}, 0 \leq i \leq d-1$ for some $p = (s,t) \in \mathbb{P}^1$. Then, $(c_0, \ldots, c_d)$ is proportional to $(t^d, t^{d-1}s, \ldots, s^d)$. Then $V = H^0(\mathcal{O}_L(d)(-p))$. Since $(f_i) \in R'(V) \subset \mathbb{X}$, there is some i such that $f_i(p) \neq 0$. Since at least one of u, v does not vanish at p, we see that at least one of $f_i u, f_i v \notin V$, which is not the case.

□

Corollary 2.7. *If $d \leq 2N-3$, the scheme $R_1(X) = p_2^{-1}(X)$ is smooth of dimension $2N-3-d$ for general $X \in \mathbb{P}_d$.*

Proof. I claim that it suffices to prove that p_2 is surjective. Then, since $\dim \mathbb{Y}_d = 2N - d - 3 + \dim \mathbb{P}_d$ from Lemma 2.4, general fiber of p_2 has the claimed dimension. Since $\mathbb{Y}_d$ is smooth, generic smoothness also follows.

From the previous lemma, we know that the set of points $(L, X) \in \mathbb{Y}_d$ where p_2 is smooth is a non-empty dense open set of $\mathbb{Y}_d$. Thus p_2 is dominant and hence surjective.

□

Thus we get a result which can be found in [6].

Corollary 2.8. *If $d \leq 2N-4$ and $N \geq 4$, then all fibers of p_2 are connected.*

Proof. For this, suffices to prove that general fibers are connected. So, let $l \subset \mathbb{P}_d$ be a general line and let $Y = p_2^{-1}(l)$. The set of points where p_2 is not smooth as a map from Y to l has codimension at least $\min(2N-d-2, N-2) \geq 2$. Let $Y \to Z \to l$ be the Stein factorization. If $Z \to l$ is ramified at some point, then every point on the fiber of this point in Y is non-smooth for p_2. This set has codimension one, which is not possible by the above estimate. So, $Z \to l$ is etale and thus an isomorphism. This proves that fibers are connected. □

Next, we study the situation when $d < N$.

Lemma 2.9. *If $X \subset \mathbb{P}^N$ is a smooth hypersurface of degree $d < N$, there is a line through any point of X contained in X.*

Proof. Without loss of generality, we may assume $p = (0, \ldots, 0, 1) \in X$ is the point of interest. Then the equation of X, f can be written in these coordinates as, $f = a_1x_N^{d-1} + a_2X_N^{d-2} + \cdots + a_d$, where a_i is a homogeneous polynomial of degree i in the remaining variables. It is clear that the set of lines through p contained in X is given by the common zeroes of the a_i's in $\mathbb{P}^{N-1}$, the set of lines in $\mathbb{P}^N$ through p. So, if $d < N$, these have a common zero, which proves the lemma. □

We wish to study the a_i's as above a little more carefully. Before that, we state an elementary lemma which will be useful.

Lemma 2.10. (1) *Let $Y \subset \mathbb{P}^N$ be an irreducible variety of dimension r. Then for any $d \geq 0$, the codimension of $H^0(I_Y(d))$ in $H^0(\mathcal{O}_{\mathbb{P}^N}(d))$ is at least $\binom{r+d}{r}$.*
(2) *If $r \geq 1$ and Y spans a linear space of dimension s, then the codimension of $H^0(I_Y(d))$ in $H^0(\mathcal{O}_{\mathbb{P}^N}(d))$ is at least $ds+1$, if $d \geq 2$.*

Proof. For the first part, suffices to prove that the dimension of the image of $H^0(\mathcal{O}_{\mathbb{P}^N}(d))$ in $H^0(\mathcal{O}_Y(d))$ is at least $\binom{r+d}{r}$. Let $\pi : Y \to \mathbb{P}^r$ be a generic linear projection. Then the above image contains $\pi^*H^0(\mathcal{O}_{\mathbb{P}^r}(d))$ and this has the declared dimension.

For the second part, let $C \subset Y$ be an irreducible curve whose linear span has the same dimension s. (In other words, choose a general curve passing through sufficiently many general points.) Clearly we may assume that $Y = C$. Consider an exact sequence,

$$0 \to \mathcal{O}_Y(-1) \to \mathcal{O}_Y^{\oplus 2} = E \to \mathcal{O}_Y(1) \to 0,$$

by choosing two general sections of $\mathcal{O}_Y(1)$ coming from $\mathcal{O}_{\mathbb{P}^N}(1)$. By taking symmetric powers, we get an exact sequence,

$$0 \to \mathcal{O}_Y(-1) \otimes S^{d-2}E \to S^{d-1}E \to \mathcal{O}_Y(d-1) \to 0,$$

which twisted by $\mathcal{O}_Y(1)$ gives,

$$0 \to S^{d-2}E \to S^{d-1}E \otimes \mathcal{O}_Y(1) \to \mathcal{O}_Y(d) \to 0.$$

Taking global sections and noting the the image of $H^0(\mathcal{O}_{\mathbb{P}^N}(1))$ in $H^0(\mathcal{O}_Y(1))$ has dimension at least $s+1$, we see that the dimension of the image of $H^0(\mathcal{O}_{\mathbb{P}^N}(d))$ in $H^0(\mathcal{O}_Y(d))$ is at least, $(s+1)d-(d-1) = sd+1$. This proves the second part. □

Letting $\mathbb{P}_k$ to be the parameter space of all hypersurfaces of degree $k > 0$ as before, for any point

$$\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{P}_1 \times \cdots \times \mathbb{P}_k,$$

denote by $X_\alpha \subset \mathbb{P}^N$, the subscheme defined as the intersection of $\alpha_i, 1 \leq i \leq k$.

Lemma 2.11. *With notation as above, if $k \leq N$, the closed subset $\Delta_k \subset \mathbb{P}_1 \times \cdots \times \mathbb{P}_k$ of points α with codimension of $X_\alpha < k$ has codimension at least $N+1$.*

Proof. Proof is by induction on k. If $k = 1$, then $\Delta_1 = \emptyset$ and since $\dim \mathbb{P}_1 = N$, we are done. Assume result proved for $k-1$. Then Δ_{k-1} has codimension at least $N+1$ in $\mathbb{P}_1 \times \cdots \times \mathbb{P}_{k-1}$ and $\Delta_{k-1} \times \mathbb{P}_k \subset \Delta_k$. So, we only need to prove that $\Delta_k - \Delta_{k-1} \times \mathbb{P}_k$ has codimension at least $N+1$ in $\mathbb{X} \times \mathbb{P}_k$, where $\mathbb{X} = \mathbb{P}_1 \times \cdots \times \mathbb{P}_{k-1} - \Delta_{k-1}$. Thus, in turn, we only need to prove that the codimension of $\Delta_k - \Delta_{k-1} \cap x \times \mathbb{P}_k \subset x \times \mathbb{P}_k$ for any $x \in \mathbb{X}$ has codimension at least $N+1$.

Since $x \in \mathbb{X}$, for the corresponding scheme the dimension of all its irreducible components is precisely $N - k + 1 \geq 1$, by hypothesis. So, it suffices to prove that the set of hypersurfaces of degree k containing an irreducible variety Y of dimension $N - k + 1 > 0$ has codimension at least $N+1$. That is, codimension of $H^0(I_Y(k)) \subset H^0(\mathcal{O}_{\mathbb{P}^N}(k))$ is at least $N+1$. But this follows from the above lemma 2.10.

□

For a point $p \in \mathbb{P}^N$, we denote by $\mathbb{P}_d(p)$, the projective space of hypersurfaces of degree d passing through p. For an $X \in \mathbb{P}_d$, we write $R_e(X)$ for the set of smooth rational curves of degree e contained in X and for a point $p \in X$, we write $R_e(X,p)$ for the set of curves in $R_e(X)$ which pass through p.

Consider a hypersurface X of degree $d < N$, the family of lines $R_1(X)$ on it and let $\mathcal{L} \subset R_1(X) \times X$ be the incidence variety, (L,p) such that $p \in L$. We have seen that $R_1(X)$ is a smooth irreducible variety of dimension $2N-d-3$ for general $X \in \mathbb{P}_d$ (Corollary 2.7) and thus $\mathcal{L}$ is a smooth irreducible variety of dimension $2N-2-d$. So $R_1(X,p)$, the set of lines in X passing through $p \in X$, has dimension at least $N-d-1$ with equality for a general p.

Corollary 2.12. *Let $\mathbb{P}_d(p)$ the projective space of hypersurfaces of degree $d < N$ passing through a point $p \in \mathbb{P}^N$ and let $Z \subset \mathbb{P}_d(p)$ be the closed subset of hypersurfaces X such that* $\dim R_1(X,p) > N-d-1$. *Then* $\operatorname{codim}(Z, \mathbb{P}_d(p)) \geq N$.

Proof. We may and shall assume that $d \geq 2$.

Let $V_i = H^0(\mathbb{P}^{N-1}, \mathcal{O}_{\mathbb{P}^{N-1}}(i))$. Then, we have seen that $\mathbb{P}_d(p)$ is just $\mathbb{P}(\oplus_{i=1}^d V_i)$. Let Z' be the points of $\mathbb{P}_d(p)$, under this identification, $(v_1, \ldots, v_d)$ with $v_i \in V_i$ and at least one $v_i = 0$. Since $\dim V_i \geq N$ for all $i > 0$ and $d \geq 2$, $\operatorname{codim}(Z', \mathbb{P}_d(p)) \geq N$. Thus it suffices to prove that $\operatorname{codim}(Z - Z', \mathbb{P}_d(p) - Z') \geq N$.

We have a natural map $\mathbb{P}_d(p) - Z' \to \mathbb{P}'_1 \times \cdots \times \mathbb{P}'_d = W$, where $\mathbb{P}'_k$ is just $\mathbb{P}(V_k)$, degree k hypersurfaces in $\mathbb{P}^{N-1}$. This is a smooth fibration with fibers $\mathbb{G}_m^{d-1}$. Furthermore, there exists a closed subscheme $\Delta_d \subset W$ (as defined in lemma 2.11, except we are now working on $\mathbb{P}^{N-1}$ instead of $\mathbb{P}^N$) whose inverse image under this map is just $Z - Z'$. Lemma 2.11 now finishes the proof. □

Theorem 2.13 (H-R-S). *For a general X (with $\deg X < N$), for any $p \in X$, all irreducible components of $R_1(X,p)$ have dimension $N-d-1$.*

Proof. As usual, we set up our incidence varieties. Let $\mathbb{X} \subset \mathbb{P}^N \times \mathbb{G} \times \mathbb{P}_d$ be the incidence variety, (p, L, X) such that $p \in L \subset X$. The projection $\pi : \mathbb{X} \to \mathbb{P}^N \times \mathbb{P}_d$ is contained in $\mathcal{H} = \{(p,X) | p \in X\}$, the universal hypersurface of degree d. Since $\pi^{-1}((p,X)) = R_1(X,p)$, we know that all fibers of π have (all its irreducible components) dimension at least $N-d-1$. So, consider the closed subset $Z \subset \mathcal{H}$ consisting of (p,X) such that $\dim R_1(X,p) > N-d-1$. We claim that $\operatorname{codim}(Z,\mathcal{H}) \geq N$. This will finish the proof, since this implies the projection from $Z \to \mathbb{P}_d$ is not onto, and thus for a general X and any point $p \in X$, $\dim R_1(X,p) = N-d-1$.

To check the codimension, suffices to check that $\operatorname{codim}(Z \cap p \times \mathbb{P}_d(p), \mathbb{P}_d(p)) \geq N$ and this was precisely the content of the previous corollary. □

3. Kontsevich Moduli

While studying higher degree smooth rational curves on hypersurfaces, we encounter the problem that these spaces are not complete, unlike the case of lines. One way to overcome this problem would be to study the appropriate Hilbert schemes. But, for various reasons, it is better for the issues at hand to study Kontsevich Moduli spaces. We shall briefly discuss them leaving the details which can be found in [7, 2, 4, 3, 5].

Let $X \subset \mathbb{P}^N$ be any projective variety. Let $e > 0$ and $k \geq 0$ be integers. Then, the Kontsevich Moduli stack $\overline{\mathcal{M}}_{0,k}(X,e)$ consists of the following.

(1) A reduced connected curve C of arithmetic genus zero with k (ordered) marked points $p_1, \ldots, p_k$, which are non-singular on C.
(2) A morphism $f : C \to X$ so that $f^*(\mathcal{O}_X(1))$ has total degree e.
(3) If an irreducible component L of C is mapped to a point by f, then L has at least three points which are marked or nodes of C.

One has a natural morphism $\overline{\mathcal{M}}_{0,k}(X,e) \to X^k$ for $k > 0$, called the *evaluation morphism*, given by,

$$(f : C \to X, p_1, \ldots, p_k) \mapsto (f(p_1), \ldots, f(p_k)) \in X^k.$$

The stack $\overline{\mathcal{M}}$ is complete and $\overline{\mathcal{M}}_{0,0}(X,e)$ contains $R_e(X)$, the space of non-singular rational curves of degree e in X, as an open subset.

The stack $\overline{\mathcal{M}}_{0,k}(\mathbb{P}^N,e)$ is irreducible, smooth of dimension $e(N+1)+N-3+k$ and $\dim R_e(\mathbb{P}^N) = e(N+1)+N-3$. Let $\mathcal{C}$ be the universal curve over $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^N,e)$ and let p (resp. q) be the canonical morphism $p : \mathcal{C} \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}^N,e)$ (resp. $q : \mathcal{C} \to \mathbb{P}^n$). Then $p_*q^*\mathcal{O}_{\mathbb{P}^N}(d)$ is a vector bundle of rank $ed+1$ on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^N,e)$ and the zeroes of a non-zero section of $\mathcal{O}_{\mathbb{P}^N}(d)$ gives a hypersurface X of $\mathbb{P}^N$ while, the zeroes of the same section considered as a section of the above vector bundle gives $\overline{\mathcal{M}}_{0,0}(X,e)$. In particular we see that all irreducible components of $\overline{\mathcal{M}}_{0,0}(X,e)$ are of dimension at least $e(N+1-d)+N-4$ and if equal, these are local complete intersections.

We call this number $E(d) = e(N+1-d)+N-4$, the *expected dimension*.

One has $\dim \overline{\mathcal{M}}_{0,1}(X,e) = \dim \overline{\mathcal{M}}_{0,0}(X,e)+1$ and so, if one is interested in proving that $\overline{\mathcal{M}}_{0,0}(X,e)$ has the expected dimension $E(d)$, it suffices to prove that $\dim \overline{\mathcal{M}}_{0,1}(X,e) = E(d)+1$. This comes with the natural evaluation map $\overline{\mathcal{M}}_{0,1}(X,e) \to X$ and if this map is onto, then it suffices to prove that the fiber, which we denote by $\Gamma_e(X,p)$ over a point $p \in X$, has dimension $E(d)+1-N+1 = e(N+1-d)-2$. The main point of all this is the following powerful result of Harris, Roth and Starr [4].

Denote by $T(d)$, called the *threshold degree*,

$$T(d) = \left\lfloor \frac{N+1}{N+1-d} \right\rfloor.$$

Theorem 3.1 (H-R-S)**.** *Let $d < N$ and assume that for general hypersurfaces of degree d, the map $\overline{\mathcal{M}}_{0,1}(X,e) \to X$ is flat and of relative dimension $e(N+1-d)-2$ for $e \leq T(d)$. Then, the same is true for all e.*

If $d < \frac{N+1}{2}$, then we have $T(d) \leq 1$ and thus by the above result, we need to prove the required statement for $e = 1$. But, this was precisely the content of Theorem 2.13. So, they deduce the following corollary.

Corollary 3.2 (H-R-S)**.** *Assume $d < \frac{N+1}{2}$. Then for a general hypersurface X of degree d in $\mathbb{P}^N$, the natural evaluation map $\overline{\mathcal{M}}_{0,1}(X,e) \to X$ is flat of relative dimension $e(N+1-d)-2$ for all e. In particular,*

$\overline{\mathcal{M}}_{0,0}(X, e)$ *is a local complete intersection of the expected dimension* $E(d) = e(N + 1 - d) + N - 4$.

4. Conics in Hypersurfaces

We prove similar results for conics, thereby improving the above bound from $\frac{N+1}{2}$ to $\frac{2N+2}{3}$ at least for sufficiently large N.

As before, our aim is to study the evaluation map $\overline{\mathcal{M}}_{0,1}(X, 2) \to X$, where X is a hypersurface of degree d in $\mathbb{P}^N$.

For this, as usual, we consider $\Gamma \subset \overline{\mathcal{M}}_{0,1}(\mathbb{P}^N, 2) \times \mathbb{P}_d$, consisting of $((f : C \to \mathbb{P}^n, p), X)$ such that $f(C) \subset X$. Since $d < N$, through any point of X there is a line and thus through every point there is a (reducible) conic. So, the projection $\Gamma \to \mathbb{P}^N \times \mathbb{P}_d$ which clearly factors through the universal hypersurface $H \subset \mathbb{P}^N \times \mathbb{P}_d$ is onto H. The required relative dimension for $e = 2$ is $2N - 2d$ and thus we consider $Z \subset H$ of points whose inverse image has dimension greater than $2N - 2d$. We will also replace $\mathbb{P}_d$ by the open set of smooth hypersurfaces and continue to call this space $\mathbb{P}_d$. If we show that the map from Z to $\mathbb{P}_d$ is not dominant, then it will follow that for a general hypersurface of X degree d, for any point $p \in X$, $(p, X) \notin Z$ and thus by definition of Z, for such an X, every fiber of the evaluation map $\overline{\mathcal{M}}_{0,1}(X, 2) \to X$ has dimension at most $2N - 2d$. We will show from this that the evaluation map is flat of relative dimension $2N - 2d$ and we will be able to appeal to Theorem 3.1 to deduce that for a general hypersurface of degree d, all the necessary dimension results.

So, let us assume that $Z \to \mathbb{P}_d$ is dominant.

We have $\dim \overline{\mathcal{M}}_{0,1}(\mathbb{P}^N, 2) = 3N$ and this is a smooth stack. So, fibers of $\overline{\mathcal{M}}_{0,1}(\mathbb{P}^N, 2) \to \mathbb{P}^N$ are all of dimension $2N$ and smooth. The space of reduced conics through a point in $\mathbb{P}^N$ is of dimension $2N - 1$ and so, this forms a Cartier divisor. So, for a point $z \in Z$, if we denote by $\Gamma(z)$ its inverse image under $\Gamma \to H$, and $\widetilde{\Gamma}(z)$ any one of the irreducible components of $\Gamma(z)$ of dimension greater than $2N - 2d$ (which is non-empty by assumption), the ones with reducible domains which we denote by $\Gamma'(z) \subset \widetilde{\Gamma}(z)$ is a divisor intersected with $\widetilde{\Gamma}(z)$. So, it has three possibilities.

(1) $\Gamma'(z) = \emptyset$.
(2) $\Gamma'(z)$ is a divisor.
(3) $\Gamma'(z) = \widetilde{\Gamma}(z)$.

If $z \in Z$ is general, then by dominance of $Z \to \mathbb{P}_d$, its image is a general hypersurface X. But, we have seen that for a general hypersurface and any point on it, the lines through it has dimension $N - d - 1$, by

Theorem 2.13. So, the reduced conics through any point has dimension $2N-2d-1$. In case two above, this implies that $\dim \widetilde{\Gamma}(z) = 2N-2d$ and in case three, it is $2N-2d-1$. But, our assumption was that its dimension is greater than $2N-2d$ for $z \in Z$. Thus only case one can occur. So, we may assume that for any point $z \in Z$, all irreducible components $\Gamma(z)$ with dimension greater than $2N-2d$ consists entirely of maps from $\mathbb{P}^1$ to X.

If $d < \frac{N+1}{2}$, the required results follow from the Corollary 3.2 of Harris et. al. above. So, we may further assume that $d \geq \frac{N+1}{2}$. Now, let us look at $(f : \mathbb{P}^1 \to X, p)$ in $\widetilde{\Gamma}(z)$ for a general z where the map is not an embedding. Then it must be a double cover of a line in X passing through p. But, X is general by dominance of $Z \to \mathbb{P}_d$ and so the lines through p has dimension $N-d-1$ as before. So, the double covers have dimension $N-d+1$. Since $\widetilde{\Gamma}(z)$ is assumed to have dimension greater than $2N-2d$, it has a closed subset R of dimension at least $N-d-1$ consisting of $f : \mathbb{P}^1 \to X$, which are embeddings as smooth conics.

So, starting with our hypothesis that $Z \to \mathbb{P}_d$ is dominant, we have arrived at the following situation. For a general hypersurface X of degree $d \geq \frac{N+1}{2}$, there exists a point $p \in X$, and an irreducible component of $R_2(X,p)$ with dimension greater than $2N-2d$, and this component contains a complete family T of smooth conics contained in X, passing through p containing a general point of an irreducible component of $R_2(X,p)$ of dimension greater than $2N-2d$ and of dimension at least $N-d-1$.

At this point, our procedure is as follows. From now on, we will write I_p for the ideal sheaf of a point in the appropriates sheaf of rings. We will show under the above hypothesis and for a general point $C \in T$, the natural map

$$H^0(C, N_{C/\mathbb{P}^N} \otimes I_p) \to H^0(C, N_{X/\mathbb{P}^N}|_C \otimes I_p)$$

is onto. Since $N_{C/\mathbb{P}^N} \cong \mathcal{O}_{\mathbb{P}^1}(4) \oplus \mathcal{O}_{\mathbb{P}^1}(2)^{\oplus N-2}$, $h^0(N_{C/\mathbb{P}^N} \otimes I_p) = 2N$. Similarly, $N_{X/\mathbb{P}^N}|_C \cong \mathcal{O}_{\mathbb{P}^1}(2d)$ and thus $h^0(N_{X/\mathbb{P}^N}|_C \otimes I_p) = 2d$. So from the exact sequence,

$$0 \to N_{C/X} \to N_{C/\mathbb{P}^N} \to N_{X/\mathbb{P}^N}|_C \to 0,$$

we see that $h^0(N_{C/X} \otimes I_p) = 2N-2d$. We know that the tangent space $T_{(p,C,X),R_2(X,p)} = H^0(N_{C/X} \otimes I_p)$ and since C is general, it will follow that $\dim R_2(X,p) = 2N-2d$, which is a contradiction.

So, for clarity, let me restart our incidence variety set-up. We have in $\mathbb{P}^N \times R_2(\mathbb{P}^n) \times \mathbb{P}_d$, the incidence variety Γ consisting of (p, C, X)

with $p \in C \subset X$. The projection $\Gamma \to \mathbb{P}^N \times \mathbb{P}^d$ clearly factors through $H = \{(p, X) | p \in X\}$. As before letting $Z \subset H$ to be the closed subset consisting of points (p, X) such that the fiber over this point in Γ has dimension greater than $2N - 2d$, we have shown that Z dominates $\mathbb{P}_d$ under our hypothesis. Replacing Z by an irreducible component of Z which dominates $\mathbb{P}_d$, we further assume that Z is irreducible. Let E be the inverse image of Z in Γ. Again we may replace E by a suitable irreducible component of E which dominates Z and general relative dimension of the map $E \to Z$ is greater than $2N - 2d$. If $(p, C, X) \in E$ is general, we have surjective maps,

$$T_{(p,C,X),E} \to T_{(p,X),Z} \to T_{X,\mathbb{P}_d}.$$

We also have the natural commutative diagram,

$$\begin{array}{ccc} T_{(p,C,X),E} & \hookrightarrow & T_{(p,C,X),\Gamma} \\ \downarrow & & \downarrow \\ T_{(p,X),Z} & \hookrightarrow & T_{(p,X),H} \\ \downarrow & & \downarrow \\ T_{X,\mathbb{P}_d} & = & T_{X,\mathbb{P}_d} \end{array}$$

Since $\dim T_{(p,X),H} = \dim T_{X,\mathbb{P}_d} + N - 1$, we see that the codimension of $T_{(p,X),Z}$ in $T_{(p,X),H}$ is at most $N - 1$.

By Theorem 2.1, we have,

$$T_{(p,C,X),\Gamma} = H^0(N_{p/\mathbb{P}^N}) \times_{H^0(N_{C/\mathbb{P}^N}|_p)} H^0(N_{C/\mathbb{P}^N}) \times_{H^0(N_{X/\mathbb{P}^N}|_C)} H^0(N_{X/\mathbb{P}^N})$$
$$T_{(p,X),H} = H^0(N_{p/\mathbb{P}^N}) \times_{H^0(N_{X/\mathbb{P}^N}|_p)} H^0(N_{X/\mathbb{P}^N})$$

$T_{(p,X),H}$ contains the subspace $W' = \{0\} \times H^0(N_{X/\mathbb{P}^N} \otimes I_p)$ (which can be thought of as just $H^0(N_{X/\mathbb{P}^N} \otimes I_p))$ in the above identification and thus $W = T_{(p,X),Z} \cap W'$ has codimension at most $N-1$ in W'. The inverse image of W' in $T_{(p.C,X),\Gamma}$ is just $\{0\} \times H^0(N_{C/\mathbb{P}^N} \otimes I_p) \times_{H^0(N_{X/\mathbb{P}^N}|_C)} H^0(N_{X/\mathbb{P}^N} \otimes I_p)$. The surjectivity of $T_{(p,C,X),E} \to T_{(p,X),Z}$ implies that given any $\alpha \in W$, there exists an element $(\beta, \alpha) \in H^0(N_{C/\mathbb{P}^N} \otimes I_p) \times_{H^0(N_{X/\mathbb{P}^N}|_C)} H^0(N_{X/\mathbb{P}^N} \otimes I_p)$. That is to say, the image of α can be lifted under the natural map,

$$H^0(N_{C/\mathbb{P}^N} \otimes I_p) \to H^0(N_{X/\mathbb{P}^N} \otimes I_{p|C}).$$

This is true for any $\alpha \in W$ and true for any general C with $(p, C, X) \in E$. The point to note is that once we fix a general X, then there is a point $p \in X$ and a fixed subspace $W \subset H^0(N_{X/\mathbb{P}^N} \otimes I_p)$ of codimension at most $N - 1$ so that for a general curve C with $(p, C, X) \in E$, the image of W in $H^0(N_{X/\mathbb{P}^N} \otimes I_{p|C})$ can be lifted to $H^0(N_{C/\mathbb{P}^N} \otimes I_p)$.

With the above analysis, we are ready to state our technical result, which as observed earlier, will finish what we started off to prove.

Theorem 4.1. *Let T be a complete family of smooth conics in $\mathbb{P}^N$ passing through a point p and contained in a smooth hypersurface of degree $d \geq \frac{N+1}{2}$ with $\dim T \geq N-d-1$. Assume that there exists a subspace $W \subset H^0(N_{X/\mathbb{P}^N} \otimes I_p)$ of codimension at most $N-1$ such that for a general $C \in T$ the image of W in $H^0(N_{X/\mathbb{P}^N} \otimes I_{p|C})$ can be lifted to $H^0(N_{C\mathbb{P}^N} \otimes I_p)$ under the natural map*

$$H^0(N_{C/\mathbb{P}^N} \otimes I_p) \stackrel{\phi_C}{\to} H^0(N_{X/\mathbb{P}^N} \otimes I_{p|C}).$$

Further assume that $\binom{N-d}{2} > N-1$. Then for a general $C \in T$, ϕ_C is surjective.

We will prove this result in the next section.

5. Proof of Theorem 4.1

Under the hypothesis of the theorem, we will show that for any $k, 1 \leq k \leq 2d$, and for general $C \in T$, there exists a section of $H^0(N_{X/\mathbb{P}^N} \otimes I_p|C) = H^0(\mathcal{O}_C(d) \otimes I_p)$ which vanishes at p precisely k times and can be lifted via ϕ_C. This will finish the proof. This is achieved in several steps. Let $F = 0$ define the hypersurface X.

We start with an elementary lemma.

Lemma 5.1. *Let B be a complete family of smooth conics in $\mathbb{P}^N$ passing through two distinct points $p \neq q$. Then $\dim B = 0$.*

Proof. If the lemma is false we can find a smooth complete curve parametrizing smooth conics passing through two distinct points $p \neq q$. Let $\Lambda \subset B \times \mathbb{P}^N$ be the incidence variety with α, β the projections to $B, \mathbb{P}^N$ respectively. Then $\alpha : \Lambda \to B$ is a $\mathbb{P}^1$-bundle and $B \times \{p\}, B \times \{q\}$ are two disjoint sections which are blown down by β to distinct points. This is impossible. ☐

$\mathbf{k = 1}$:

In this case we have for any $C \in T$, the natural commutative diagram,

$$\begin{array}{ccc} \mathcal{O}_C(1)^{N+1} & \stackrel{(\frac{\partial F}{\partial x_i})}{\longrightarrow} & \mathcal{O}_C(d) \\ \downarrow & & \uparrow \phi_C \\ T_{\mathbb{P}^N}|C & \longrightarrow & N_{C/\mathbb{P}^N} \end{array}$$

Since F is smooth, for some i, $\frac{\partial F}{\partial x_i}(p) \neq 0$. We choose a linear equation l such that $l = 0$ meets C transversally at p. Then the image of the section of $\mathcal{O}_C(1)^{N+1} \otimes I_p$ which has zeroes at all coordinates except for the ith coordinate and l in that place goes to the section $\frac{\partial F}{\partial x_i} l \in H^0(\mathcal{O}_C(d) \otimes I_p)$, which clearly vanishes exactly once at p. Tracing this section using the commutative diagram, we get this is in the image of ϕ_C.

$\mathbf{1 < k \leq d}$:

From now on, we choose coordinates so that $p = (1, 0, \ldots, 0)$ and A the hyperplane defined by $x_0 = 0$. For any point $C \in T$ denote by q_C, the point of intersection of the line tangent to C at p with A and $l_C \subset A$ the line obtained as the image of C under projection from p.

Lemma 5.2. *The morphism $T \to A$ given by $C \mapsto q_C$ is finite to the image.*

Proof. If the map is not finite, we can find a smooth complete curve B parametrizing smooth conics through p all having the same tangent line l at p. Let $\Lambda \subset B \times \mathbb{P}^n$ be the incidence variety with $\alpha : \Lambda \to B, \beta : \lambda \to \mathbb{P}^N$, the two projections. α makes Λ a $\mathbb{P}^1$-bundle over B and it has a section E which blows down to p under β. If $q \neq p$ in l, then $\Lambda \cap B \times \{q\} = \emptyset$. Thus projection from q defines a morphism $g : \Lambda \to B \times \mathbb{P}^{N-1}$ over B. For any point $b \in B$, this is just the map from a conic to a line, projection from a point not on the conic. So, g is a finite map of degree two to its image. Let $R \subset \Lambda$ be the ramification locus. Then the map $R \to B$ is a double cover. But, the section $B \times \{p\} \subset R$ and thus the residual part is a section E_q of α. If $q_1 \neq q_2$ different from p, $E_{q_1} \cap E_{q_2} = \emptyset$ and this implies $\Lambda \cong B \times \mathbb{P}^1$. But, E is a section which can be blown down to a point and then by rigidity, the map β must take Λ to a single conic. This is contrary to our hypothesis. $\square$

Let Y denote the image of $T \to A$. By the above lemma, $\dim Y = \dim T \geq N - d - 1$. Since X will only play a peripheral role from now on, we will take the inverse image of W under the natural map $H^0(\mathcal{O}_{\mathbb{P}^N}(d)) \to H^0(\mathcal{O}_X(d)) = H^0(N_{X/\mathbb{P}^N})$ and thus $W \subset H^0(\mathcal{O}_{\mathbb{P}^N}(d) \otimes I_p)$ is of codimension at most $N-1$. On the other hand, we can identify, $H^0(\mathcal{O}_{\mathbb{P}^N}(d) \otimes I_p)$ with $\oplus_{i=1}^d H^0(\mathcal{O}_A(i)) x_0^{d-i}$. Thus, for every $i, 1 \leq i \leq d$, we get subspaces $W_i = W \cap H^0(\mathcal{O}_A(i)) x_0^{d-i}$ of codimension at most $N - 1$. We will abuse notation and sometimes alternately think of $W_i \subset H^0(\mathcal{O}_A(i))$.

Lemma 5.3. *If $f \in H^0(A, \mathcal{O}_A(i))$ is such that $f|_{l_C} \neq 0$ and has a zero of order j at q_C, then the restriction of $x_0^{d-i} f$, considered as a section of $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d))$, to C has a zero of order $i + j$ at p.*

Proof. Let P be the 2-plane spanned by C. Then the divisor of x_0 in this plane is just l_C. The divisor of $f|_{l_C}$ is $jq_C + E$ where E is an effective divisor of degree $i-j$ whose support does not contain q_C. Then the divisor in P of $x_0^{d-i}f$ is $(d-i)l_C + jM + E'$ where M is the tangent line of C at p, E' is a union of $(i-j)$ lines passing through p, none of them equal to M. Thus, the order of its restriction to C at p is just $2j + (i-j) = i+j$.

□

If $\binom{N-d+1}{2} > N-1$, from Lemma 2.10, we see that the image of $W_k \to H^0(\mathcal{O}_Y(k))$ is non-zero for $2 \le k \le d$. Thus, for general points of Y, there exists $f_k \in W_k$ not vanishing there. So, the element $f_k x_0^{d-k} \in W$, which can be lifted via ϕ_C for general C, has the property from the previous lemma that this vanishes to precisely order k at p. Our hypothesis on N, d ensures that the above inequality is satisfied.

$\mathbf{d < k \le 2d-2}$:

Lemma 5.4. *For a general hyperplane $B \subset A$ and for a general point $q_C \in B$, l_C is not contained in B.*

Proof. Let $\mathbb{G}$ be the dual of A, the set of hyperplanes in A. Consider the incidence variety, $\Lambda \subset T \times \mathbb{G}$ consisting of pairs (C, B) such that $l_C \subset B$ and let π_1, π_2 be the two projections from Λ to $T, \mathbb{G}$ respectively. Then the inverse image of any point C by π_1 is the set of hyperplanes containing $l_C \subset A$ and thus of codimension 2 in $\mathbb{G}$. So, $\dim \Lambda = \dim T + \dim \mathbb{G} - 2$. Thus for a general $B \in \mathbb{G}$, the inverse image under π_2 has dimension $\dim T - 2$. Since $\dim B \cap Y = \dim T - 1$, easy to see that the lemma follows. □

Now consider the case of $d < k \le 2d-2$ and let $m = k-d$. Let B be a general hyperplane in A and let $Y' = Y \cap B$. Also choose co-ordinates so that $B : x_1 = 0$.

We can write $H^0(\mathcal{O}_A(d)) = \oplus_{m=0}^{d} H^0(\mathcal{O}_B(d-m))x_1^m$ and let $L_m = W_d \cap H^0(\mathcal{O}_B(d-m))x_1^m$, as usual identified without x_1. Then codimension of L_m in $H^0(\mathcal{O}_B(d-m))$ is at most $N-1$. By Lemma 5.3, suffices to show that for a general C with $q_C \in Y'$ there exists $g_m \in W_d$ such that $g_m|_{l_C}$ vanishes to order m at q_C. So, suffices to show that there exists an $f_m \in L_m$ such that $f_m(q_C) \neq 0$, since then we can take $g_m = f_m x_1^m$ and $g_m \neq 0$ on l_C by the previous lemma.

By Lemma 2.10, the codimension of $H^0(I_{Y'}(d-m))$ in $H^0(\mathcal{O}_B(d-m))$ is at least $\binom{N-d}{2}$, since $d-m \ge 2$. So, if $N-1 < \binom{N-d}{2}$, since L_m has codimension at most $N-1$, we would be done. This is assured by our hypothesis.

$\mathbf{k = 2d - 1, 2d}$:

Lemma 5.5. *Let* $s = \lfloor \frac{\dim T+1}{2} \rfloor$. *If* $C_1, \dots, C_s$ *are general points of* T*, then* $l_{C_1}, \dots, l_{C_s}$ *are linearly independent, i. e. they span a linear subspace of dimension* $2s-1$.

Proof. Assume that $t \leq s$ be the largest number such that for general $C_1, \dots, C_t$, the corresponding lines l_{C_i} are linearly independent and let Λ be their linear span. Then, for any $C \in T$, l_C intersects Λ. Then $\dim \Lambda = 2t - 1$.

If $q \in \Lambda$, let the conics $C \in T$ with l_C passing through q be denoted by $S_q \subset T$. This is a closed subset. I claim that $\dim S_q \leq 1$.

As usual, let $Y \subset S_q \times \mathbb{P}^N$ be the incidence variety. Let l be the line joining p, q and let $Y' = Y \cap S_q \times l$. Then the projection $Y' \to S_q$ is a degree two finite map and it has a component $S_q \times \{p\}$. Let M be the other component. Note that M is isomorphic to S_q and thus in particular has the same dimension as S_q. The projection $M \to l$ has three possibilities.

(1) Image of M is p.
(2) Image of M is $a \neq p$.
(3) Image of M is l.

In the first case, l is the tangent to C for all $C \in S_q$ and thus by Lemma 5.2, $\dim S_q \leq 0$. In the second case, all conics in S_q pass through two distinct points and again by Lemma 5.1, $\dim S_q \leq 0$. In the third case, for any point $a \in l$, the fiber of $M \to l$ has dimension at most zero by the two quoted lemmas and thus $\dim S_q \leq 1$.

Since T is assumed to be the union of S_q as q varies in Λ, we see that $\dim T \leq \dim \Lambda + 1$. Clearly this implies $t = s$.

□

Let s be as above. Consider the space $\mathbb{G}$ of all linear subspaces of A of codimension s. Since $s < \dim T$, by Bertini's theorem, we see that for a general $H \in \mathbb{G}$, the inverse image T' of H under $T \to A$ is irreducible. For general points $C_i \in T, 1 \leq i \leq s$, the points $q_{C_i} \in A$ span a dimension $s-1$ linear subspace of A by the previous lemma and since $\dim H = N - s - 1 \geq s - 1$, we may further assume that H contains the image of s general points. Further, since C_i can be assumed general, we may assume by the previous lemma that l_{C_i} are linearly independent.

Fix points $b_{C_i} \in l_{C_i}$ with $b_{C_i} \neq q_{C_i}$. Let H' be the linear span of the the b_{C_i}. So, $\dim H' = s - 1$ and $H \cap H' = \emptyset$. Thus for a general $C \in T'$, l_C is not contained in H and so the linear span of H, l_C is of codimension $s-1$ and it intersects H' in exactly one point, say b_C.

By this correspondence, the set of such b_C for general $C \in T'$ gives an irreducible quasi-projective variety $Z' \subset H'$. Since $b_{C_i} \in Z'$ which span H', we see that Z' is non-degenerate in H' and has dimension at least one. Let Z be the closure of Z'.

We identify $H^0(\mathcal{O}_{H'}(d))$ as a subspace of $H^0(\mathcal{O}_A(d))$, so that all $f \in H^0(\mathcal{O}_{H'}(d))$ vanish to order at least d times along H. By Lemma 2.10, the codimension of $H^0(I_Z(d))$ in $H^0(\mathcal{O}_{H'}(d))$ is at least $d(s-1)+1 > N-1$. Thus, there exists an $f \in W_d \cap H^0(\mathcal{O}_{H'}(d))$ which does not vanish on Z. So, for general point $C \in T'$, $f(b_C) \neq 0$. Since f vanishes along H, clearly f does not vanish identically on l_C. But f vanishes to order d at $q_C \in H$, so by lemma 5.3, we are done in the case of $2d$.

Repeating the same argument with d replaced by $d-1$ and choosing a form h of degree one on A which does not vanish at such a point q_C, if $(d-1)(s-1)+1 > N-1$ with a $g \in W_{d-1} \cap H^0(\mathcal{O}_{H'}(d-1))$ not vanishing at q_C, we take $f = gh$ to achieve the result for $2d-1$.

Thus we have proved the following theorem.

Theorem 5.6. *Let d be an integer such that either $d < \frac{N+1}{2}$ or $\binom{N-d}{2} > N-1$. Then, for a general hypersurface X of degree d in $\mathbb{P}^N$, the fibers of the evaluation map $\overline{\mathcal{M}}_{0,1}(X,2) \to X$ have constant dimension $2N-2d$.*

This implies that the evaluation map is flat of relative dimension $2N-2d$.

Corollary 5.7. *Let d be as in the above theorem. Then for a general hypersurface X as above of degree d, the evaluation map $ev_X : \overline{\mathcal{M}}_{0,1}(X,2) \to X$ is flat.*

Proof. If $d < \frac{N+1}{2}$, this follows from the result of Harris et. al. (Corollary 3.2). So, we may assume that $d \geq \frac{N+1}{2}$ and $\binom{N-d}{2} > N-1$. Then by the previous theorem, we know that the evaluation map has constant fiber dimension $2N-2d$.

Recall that $\overline{\mathcal{M}}_{0,1}(\mathbb{P}^N,2)$ is a smooth stack of dimension $3(N+1)-3 = 3N$ and that $\overline{\mathcal{M}}_{0,1}(X,2)$ is the zero locus of a section of a locally free sheaf of rank $2d+1$ over $\overline{\mathcal{M}}_{0,1}(\mathbb{P}^N,2)$. Since the fibers of ev are of expected dimension $2(N-d)$, $\overline{\mathcal{M}}_{0,1}(X,2)$ has dimension

$$2(N-d)+N-1 = 3N-(2d+1),$$

so it is a local complete intersection and in particular a Cohen-Macaulay substack of $\overline{\mathcal{M}}_{0,1}(\mathbb{P}^N,2)$. Since a map from a Cohen-Macaulay scheme to a smooth scheme is flat if and only if it has constant fiber dimension ([8, Theorem 23.1]), we have proved the corollary.

□

If $d < \frac{2N+2}{3}$, then the threshold degree,

$$T(d) = \left\lfloor \frac{N+1}{N+1-d} \right\rfloor < 3.$$

Thus coupled with the result of Harris et. al. (Theorem 3.1), we get,

Corollary 5.8. *If $X \subset \mathbb{P}^N$ is a general hypersurface of degree d with $d < \frac{2N+2}{3}$ and either $d < \frac{N+1}{2}$ or $\binom{N-d}{2} > N-1$, then the evaluation map $\overline{\mathcal{M}}_{0,1}(X,e) \to X$ is flat of relative dimension $e(N+1-d)-2$ and $\overline{\mathcal{M}}_{0,0}(X,e)$ is an integral local complete intersection stack of expected dimension $e(N+1-d)+N-4$ for all $e \geq 1$.*

We have proved all statements except the integrality of $\overline{\mathcal{M}}_{0,0}(X,e)$. I refer the reader to [1] for a proof of integrality.

References

1. R. Beheshti and N. Mohan Kumar, *Spaces of rational curves in complete intersections*, To appear in Compositio Math.
2. K. Behrend and Yu. Manin, *Stacks of stable maps and Gromov-Witten invariants*, Duke Math. J. **85** (1996), no. 1, 1–60. MR 1412436 (98i:14014)
3. W. Fulton and R. Pandharipande, *Notes on stable maps and quantum cohomology*, Algebraic geometry—Santa Cruz 1995, Proc. Sympos. Pure Math., vol. 62, Amer. Math. Soc., Providence, RI, 1997, pp. 45–96. MR 1492534 (98m:14025)
4. Joe Harris, Mike Roth, and Jason Starr, *Rational curves on hypersurfaces of low degree*, J. Reine Angew. Math. **571** (2004), 73–106. MR 2070144 (2005e:14067)
5. Joachim Kock and Israel Vainsencher, *An invitation to quantum cohomology*, Progress in Mathematics, vol. 249, Birkhäuser Boston Inc., Boston, MA, 2007, Kontsevich's formula for rational plane curves. MR 2262630 (2007m:14081)
6. János Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 32, Springer-Verlag, Berlin, 1996. MR 1440180 (98c:14001)
7. Maxim Kontsevich, *Enumeration of rational curves via torus actions*, The moduli space of curves (Texel Island, 1994), Progr. Math., vol. 129, Birkhäuser Boston, Boston, MA, 1995, pp. 335–368. MR 1363062 (97d:14077)
8. Hideyuki Matsumura, *Commutative ring theory*, second ed., Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989, Translated from the Japanese by M. Reid. MR 1011461 (90i:13001)
9. David Mumford, *Lectures on curves on an algebraic surface*, With a section by G. M. Bergman. Annals of Mathematics Studies, No. 59, Princeton University Press, Princeton, N.J., 1966. MR 0209285 (35 #187)
10. Edoardo Sernesi, *Deformations of algebraic schemes*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 334, Springer-Verlag, Berlin, 2006. MR 2247603 (2008e:14011)

Department of Mathematics, Washington University in St. Louis, St. Louis, Missouri, 63130
E-mail address: kumar@wustl.edu
URL: http://www.math.wustl.edu/~kumar