THE MINIMAL PROJECTIVE BUNDLE DIMENSION AND TORIC 2-FANO MANIFOLDS

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Abstract. Motivated by the problem of classifying toric 2-Fano manifolds, we introduce a new invariant for smooth projective toric varieties, the minimal projective bundle dimension. This invariant $m(X) \in \{1, \dots, \dim(X)\}$ captures the minimal degree of a dominating family of rational curves on X or, equivalently, the minimal length of a centrally symmetric primitive relation for the fan of X. We classify smooth projective toric varieties with $m(X) \geq \dim(X)-2$, and show that projective spaces are the only 2-Fano manifolds among smooth projective toric varieties with $m(X) \in \{1, \dim(X)-2, \dim(X)-1, \dim(X)\}$.

Contents

1. Introduction

Fano varieties are projective varieties with positive first Chern class. Over the complex numbers, this condition is equivalent to the existence of a metric with positive Ricci curvature. Basic examples of Fano varieties include projective spaces and Grassmannians. The positivity condition has further geometric implications, e.g., Fano varieties over the complex numbers are *simply connected*. This has an analogue on the algebro-geometric side: any Fano variety is covered by rational curves [Mor79], and is in fact *rationally connected* [KMM92; Cam92], i.e., there are rational curves connecting any two of its points. In a series of papers, de Jong and Starr introduce and investigate possible candidates for the notion of *higher rational connectedness* [dHS11; dS07; dS06b; dS06c; Sta06], inspired by the natural analogue in topology. In particular, in [dS06b] they define 2-*Fano* manifolds. A smooth projective variety X is 2-Fano if it is Fano and its second Chern character $\mathrm{ch}_2(T_X) = \frac{1}{2}c_1(T_X)^2 - c_2(T_X)$ is positive, i.e., $\mathrm{ch}_2(T_X) \cdot S > 0$ for every surface S in X. In a similar way, one can define k-Fano varieties for any $k \geq 2$, and aim at their classification. For instance, $\mathbb{P}^n$ is n-Fano, and it is conjectured that it is the only n-dimensional n-Fano manifold. The geometry of higher Fano manifolds has been fairly investigated, and in several special cases they are shown to enjoy the expected nice properties. For instance, 2-Fano manifolds satisfying some mild assumptions are covered by rational surfaces [dS07], and similar results hold for higher Fano manifolds [Suz21], [Nag19]. There is a classification of 2-Fano manifolds of high index [AC13] and, more recently, a classification of homogeneous 2-Fano manifolds [Ara+22]. On the other hand, very few examples of higher Fano manifolds are known. Quite strikingly, all known examples of 2-Fano manifolds have Picard rank 1 and relatively large index.

It is natural for algebraic geometers to turn to the pool of toric varieties when looking for intuition or examples. It is well known that projective spaces are the only projective toric manifolds with Picard rank 1. Thus, a classification of toric 2-Fano manifolds could either provide the first examples of 2-Fano manifolds with higher Picard rank, or it could be an evidence that every 2-Fano manifold has Picard rank 1. Geometric properties of a toric variety can often be checked in the combinatorics of the associated fan. This bridge has been exploited in search of new examples of toric 2-Fano manifolds [Nob11], [Nob12], [Sat12], [Sat16], [SS20], [SSS21], [Shr20]. Despite the efforts, a complete (computer aided) classification is only known up to dimension 8 [Nob11], [SSS21], and projective spaces remain the only known examples of toric 2-Fano manifolds. The sparsity of higher Fano manifolds leads to the following conjecture.

Conjecture 1.1. ([SSS21, Conjecture 4.3]) The only toric 2-Fano manifolds are projective spaces.

In this paper, we propose a new strategy to approach Conjecture 1.1. We follow the philosophy introduced in [AC12], namely, to investigate 2-Fano manifolds by studying their *minimal dominating families of rational curves*. By [CFH14], minimal dominating families of rational curves on a smooth projective toric variety X correspond to *primitive relations* of the form

$$x_0 + \cdots + x_m = 0, \tag{1}$$

satisfied by some of the primitive integral generators x_i of the corresponding fan. These primitive relations are called *centrally symmetric of order* $m+1$. By [CFH14], a centrally symmetric primitive relation of order $m+1$ yields a $\mathbb{P}^m$-bundle structure $X^\circ \to T$ on a dense open subset X° of X. If $\dim(T) \geq 1$, and the complement $X \setminus X^\circ$ has codimension at least 2 in X, then one can construct a complete surface $S \subset X^\circ$ such that $\mathrm{ch}_2(T_X) \cdot S \leq 0$, showing that X is not 2-Fano. So our basic strategy consists of trying to describe, in a rather explicit way, a suitable birational map $\varphi : X \dashrightarrow Y$ transforming X into a projective toric variety Y admitting a $\mathbb{P}^m$-bundle structure on a big open subset. We then hope to be able to compare the second Chern characters $\mathrm{ch}_2(T_X)$ and $\mathrm{ch}_2(T_Y)$ to show that X is not 2-Fano, except if $X = \mathbb{P}^m$ and φ is the identity.

To follow this strategy, we introduce a new invariant of a smooth projective toric variety X, the *minimal projective bundle dimension* of X, *minimal* $\mathbb{P}$-*dimension* in short, which is of independent interest (Definition 2.10):

$$m(X) \;=\; \min\left\{\, m \in \mathbb{Z}_{>0} \;\middle|\; \text{there is a relation as in (1)} \,\right\} \;\in\; \{1, \dots, \dim X\}.$$

By going through the database of toric Fano manifolds of low dimension and computing their primitive collections, one obtains Table 1, indicating the number of Fano manifolds for each value of m. Appendix A contains the code used to compute primitive collections.

$\dim(X)$	# Fanos	$\#(m{=}1)$	$\#(m{=}2)$	$\#(m{=}3)$	$\#(m{=}4)$	$\#(m{=}5)$	$\#(m{=}6)$
4	124	107	15	1	1		
5	866	744	112	8	1	1	
6	7622	6333	1174	105	8	1	1

Table 1. The minimal $\mathbb{P}$-dimension of toric Fano manifolds of low dimension.

When $m(X) = 1$, Casagrande constructs in [Cas03b] a sequence of blowdowns and flips from X to a toric variety admitting a global $\mathbb{P}^1$-bundle structure. This allows us to make the basic strategy work, yielding the following result.

Theorem 1.2. Let X be a smooth toric Fano variety with $m(X) = 1$. Then X is not 2-Fano.

Table 1 suggests that this result covers "most" toric varieties, and not just the fringe cases.

Next we turn our attention to toric Fano manifolds X with large values of $m(X)$. Projective spaces are the only smooth projective toric varieties admitting a centrally symmetric primitive relation of order $\dim(X)+1$. In [CFH14, Proposition 3.8], Chen, Fu and Hwang classify toric Fano manifolds admitting a centrally symmetric primitive relation of order $\dim(X)$. There are three such varieties, and two of them also admit a centrally symmetric primitive relation of order 2. As a consequence, the only n-dimensional toric Fano manifold X with $m(X) = n-1$ is the blowup of $\mathbb{P}^n$ along a linear $\mathbb{P}^{n-2}$. In [BW22], Beheshti and Wormleighton investigate smooth projective toric varieties admitting a centrally symmetric primitive relation of order $\dim(X)-1$, showing that they have Picard rank $\rho(X) \leq 5$. Most of these varieties also admit centrally symmetric primitive relations of order 2 or 3, and we prove the following bound for the remaining ones. Theorem 1.4 shows that this bound is sharp.

Theorem 1.3. Let X be a smooth toric Fano variety with $\dim(X) = n \geq 6$ and $m(X) \geq 3$. If X has a centrally symmetric primitive relation of order $n-1$,

$$x_0 + x_1 + \cdots + x_{n-2} = 0,$$

then $\rho(X) \leq 3$. Moreover, $m(X) = n-2$ and the above relation is the only centrally symmetric primitive relation of X.

Using Theorem 1.3 and Batyrev's description of smooth projective toric varieties with Picard rank 3, we are able to classify n-dimensional smooth toric Fano varieties with $m(X) = n-2$. There are eight distinct isomorphism classes when $n \geq 6$, which can be explicitly described. The following statement summarizes the classification of toric Fano manifolds with $m(X) \geq \dim(X) - 2$.

Theorem 1.4. We have the following classification of smooth toric Fano varieties with $m(X) \geq \dim(X) - 2$.

(1) The only n-dimensional smooth toric Fano variety X with $m(X) = n$ is $\mathbb{P}^n$.
(2) For $n \geq 3$, the only n-dimensional smooth toric Fano variety X with $m(X) = n-1$ is the blowup of $\mathbb{P}^n$ along a linear $\mathbb{P}^{n-2}$.
(3) For $n \geq 6$, there are eight distinct isomorphism classes of n-dimensional smooth toric Fano varieties X with $m(X) = n-2$. Namely:
  (a) $X = \mathbb{P}_S(\mathcal{E})$ is a $\mathbb{P}^{n-2}$-bundle over a toric surface S, where $(S, \mathcal{E})$ is one of the following:
    - $S = \mathbb{P}^2$ and $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus n-2}$,
    - $S = \mathbb{P}^2$ and $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus n-3}$,
    - $S = \mathbb{P}^2$ and $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus n-2}$,
    - $S = \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1) \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}^{\oplus n-2}$,
    - $S = \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,0) \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0,1) \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}^{\oplus n-3}$,
    - $S = \mathbb{F}_1$ and $\mathcal{E} = \mathcal{O}_{\mathbb{F}_1}(e+f) \oplus \mathcal{O}_{\mathbb{F}_1}^{\oplus n-2}$, where $e \subset \mathbb{F}_1$ is the -1-curve, and $f \subset \mathbb{F}_1$ is a fiber of $\mathbb{F}_1 \to \mathbb{P}^1$.

    In the first three cases, $\rho(X) = 2$, while in the latter three cases, $\rho(X) = 3$.
  (b) Let $Y \simeq \mathbb{P}_{\mathbb{P}^2}\big(\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus n-2}\big)$ be the blowup of $\mathbb{P}^n$ along a linear subspace $L = \mathbb{P}^{n-3}$, and denote by $E \subset Y$ the exceptional divisor. Then X is the blowup of Y along a codimension 2 center $Z \subset Y$, where:
    - Z is the intersection of E with the strict transform of a hyperplane of $\mathbb{P}^n$ containing the linear subspace L, or
    - Z is the intersection of the strict transforms of two hyperplanes of $\mathbb{P}^n$, one containing the linear subspace L, and the other one not containing it.

    In both cases, $\rho(X) = 3$.

Corollary 1.5. The projective space $\mathbb{P}^n$ is the only smooth n-dimensional toric 2-Fano variety with $m(X) \in \{1, n-2, n-1, n\}$.

This paper is organized as follows. In Section 2, we review some results from toric geometry and fix notation. In particular, we discuss centrally symmetric primitive relations on smooth projective toric varieties, describing explicitly their open subsets admitting a projective space bundle structure (Proposition 2.13). In Section 3, we study smooth toric Fano varieties with $m(X) = 1$, and prove Theorem 1.2. In Section 4, we investigate smooth projective n-dimensional toric varieties admitting a centrally symmetric primitive relation of order $n-1$, and prove Theorem 1.3. In Section 5, we use this result, together with Batyrev's description of smooth projective toric varieties with Picard rank 3, to prove Theorem 1.4.

Acknowledgements. This collaboration started as a working group at the ICERM "Women in Algebraic Geometry Collaborative Research Workshop" in July 2020. A great part of this work was developed during the follow-up "Collaborate@ICERM" meeting in May 2022. We thank ICERM for the financial support and the great working conditions provided to us during our visit. We are very grateful to Cinzia Casagrande and Milena Hering for many enlightening discussions and explanations about toric varieties, and to

Will Reynolds for running his code and providing us with precious data about primitive relations of toric Fano manifolds.

Carolina Araujo was partially supported by CAPES/COFECUB, CNPq and FAPERJ Research Fellowships. Roya Beheshti was supported by NSF grant DMS-2101935. Ana-Maria Castravet was partially supported by the ANR 20-CE40-0023 grant *FanoHK*. Enrica Mazzon was supported by the collaborative research center SFB 1085 *Higher Invariants - Interactions between Arithmetic Geometry and Global Analysis* funded by the Deutsche Forschungsgemeinschaft. Nivedita Viswanathan was supported by the EPSRC New Horizons Grant No.EP/V048619/1.

2. Primitive collections

2.1. **Notation and background.** A toric variety is a normal complex variety X that contains a torus $T = (\mathbb{C}^*)^n$ as a dense open subset, together with an action of T on X that extends the natural action of T on itself. There is a one-to-one correspondence between n-dimensional toric varieties and *fans* in $\mathbb{Q}^n$. Let N be a free abelian group of rank n, and consider the vector space $N_\mathbb{Q} = N \otimes_\mathbb{Z} \mathbb{Q}$. A *fan* in $N_\mathbb{Q}$ is a nonempty finite collection Σ of strongly convex polyhedral cones in $N_\mathbb{Q}$ such that every face of a cone in Σ is also a cone in Σ, and the intersection of two cones in Σ is a face of each. We write $\delta \prec \tau$ to express that δ is a face of τ. One-dimensional cones in Σ are called rays, and each ray is generated by a primitive vector in N. The set of primitive vectors of N generating rays of Σ is denoted by $G(\Sigma)$. We will write a cone $\tau \in \Sigma$ in terms of its primitive generators, $\tau = \langle v_1, \dots, v_l \rangle$, saying that the v_i's generate τ, and setting $G(\tau) \coloneqq \{v_1, \dots, v_l\} \subseteq G(\Sigma)$.

We denote by X_Σ the toric variety corresponding to a fan Σ. Conversely, given a toric variety X, we denote by Σ_X the fan associated to X. There is a one-to-one inclusion-reversing correspondence between cones in Σ and T-orbit closures in X_Σ. Given a cone $\tau \in \Sigma$, we write $V(\tau) \subset X_\Sigma$ for the corresponding T-orbit closure, or $V(v_1, \dots, v_l)$ when $G(\tau) = \{v_1, \dots, v_l\}$. Note that $\dim(\tau) = \operatorname{codim}_{X_\Sigma} V(\tau)$. We refer to [Ful93] and [CLS11] for the background on toric varieties.

In this paper, we are mostly interested in *smooth* and *proper* toric varieties. The smoothness conditions translates into the fan Σ being *regular*, i.e., for each cone $\tau \in \Sigma$, the set of generators $G(\tau)$ is part of a basis of N ([CLS11, Definition 1.2.16]). The properness condition translates into the fan Σ being *complete*, i.e., its support being the whole $N_\mathbb{Q}$. In what follows, smooth and proper toric varieties will be simply called *toric manifolds*. We would like to classify toric 2-Fano manifolds. Given a toric manifold X, there is an exact sequence ([CLS11, Theorem 8.1.1]):

$$0 \to \Omega^1_X \to N^\vee \otimes_\mathbb{Z} \mathcal{O}_X \to \bigoplus_{v \in G(\Sigma_X)} \mathcal{O}_{V(v)} \to 0\,,$$

from which one easily computes:

$$c_1(X) = \sum_{v \in G(\Sigma_X)} V(v) \quad \text{and} \quad \operatorname{ch}_2(X) = \frac{1}{2} \sum_{v \in G(\Sigma_X)} V(v)^2.$$

Definition 2.1. ([Bat91, Definition 2.6]) Let Σ be a regular complete fan in $N_\mathbb{Q}$. A *primitive collection* $P \subseteq G(\Sigma)$ is a nonempty set of primitive vectors of N that does not generate a cone of Σ, but such that any proper subset of P does. Equivalently, $P =$

$\{v_1, \dots, v_r\} \subseteq G(\Sigma)$ is a primitive collection if and only if

$$\langle v_1, \dots, v_r \rangle \notin \Sigma \qquad \text{and} \qquad \langle v_1, \dots, \check{v}_i, \dots, v_r \rangle \in \Sigma$$

for any $i = 1, \dots, r$. We denote by $\mathrm{PC}(\Sigma)$ the set of primitive collections of Σ. For a toric manifold X, we will talk about primitive collections of X and write $\mathrm{PC}(X)$, meaning $\mathrm{PC}(\Sigma_X)$.

Definition 2.2. Let Σ be a regular complete fan in $N_{\mathbb{Q}}$. Given a primitive collection $P = \{v_1, \dots, v_r\} \in \mathrm{PC}(\Sigma)$, let $\sigma(P) = \langle w_1, \dots, w_s \rangle$ be the minimal cone of Σ such that $v_1 + \dots + v_r \in \sigma(P)$. Then there is a relation

$$r(P)\colon\ v_1 + \dots + v_r = \mu_1 w_1 + \dots + \mu_s w_s,$$

where $\mu_j \in \mathbb{Z}_{\geq 0}$ for $j = 1, \dots, s$. We call $r(P)$ the *primitive relation* associated to P. We define the *order* of P as $\mathrm{ord}(P) = |P| = r$, while the *degree* of P as $\deg(P) = r - \sum_{j=1}^{s} \mu_j$.

By [Bat91, Proposition 3.1], for any primitive collection P, we have $P \cap \sigma(P) = \varnothing$. In particular, $\{v_1, \dots, v_r\} \cap \{w_1, \dots, w_s\} = \varnothing$.

Definition 2.3. Let Σ be a regular complete fan in $N_{\mathbb{Q}}$. A primitive collection $P = \{x_0, \dots, x_k\}$ of Σ is called *centrally symmetric* if $\sigma(P) = \{0\}$, i.e.

$$r(P)\colon x_0 + \dots + x_k = 0.$$

Lemma 2.4. Let Σ be a regular complete fan in $N_{\mathbb{Q}}$, and let P, Q be two distinct centrally symmetric primitive collections. Then $P \cap Q = \varnothing$.

Proof. Write $r(P)\colon x_0 + \dots + x_k = 0$ and $r(Q)\colon y_0 + \dots + y_l = 0$. Assume that $P \cap Q \neq \varnothing$, then without loss of generality we may assume that $x_0 = y_0$. But then subtracting this vector from both relations, we get

$$x_1 + \dots + x_k = y_1 + \dots + y_l,$$

which shows that interiors of two distinct cones intersect. This is a contradiction. □

Lemma 2.5. Let Σ be a regular complete fan in $N_{\mathbb{Q}}$, and let P, Q be two distinct centrally symmetric primitive collections. Then $\operatorname{Span} P \cap \operatorname{Span} Q = \{0\}$, in particular $|P| + |Q| - 2 \leq \dim N_{\mathbb{Q}}$.

Proof. Write $r(P)\colon x_0 + \dots + x_k = 0$ and $r(Q)\colon y_0 + \dots + y_l = 0$. Take any vector $v \in \operatorname{Span} P \cap \operatorname{Span} Q$, so we can write it as

$$v = \sum a_i x_i = \sum b_j y_j.$$

By possibly adding $r(P)$ and $r(Q)$ to the sums, we can get that all $a_i, b_j \geq 0$, and up to relabelling the a_i, b_j, we can assume $a_0 = b_0 = 0$. But this shows that v is in the intersection of two cones, $\langle x_1, \dots, x_k \rangle \cap \langle y_1, \dots, y_l \rangle$, and the sets of generators are disjoint by Lemma 2.4, so $\langle x_1, \dots, x_k \rangle \cap \langle y_1, \dots, y_l \rangle = \{0\}$ and $v = 0$.

The last claim follows from considering the dimensions of $\operatorname{Span} P$ and $\operatorname{Span} Q$. □

Let $A_1(X_\Sigma)$ be the group of algebraic 1-cycles on X_Σ modulo numerical equivalence, and set $\mathcal{N}_1(X_\Sigma) = A_1(X_\Sigma) \otimes_{\mathbb{Z}} \mathbb{Q}$. The Mori cone $\mathrm{NE}(X_\Sigma) \subset \mathcal{N}_1(X_\Sigma)$ is the cone generated by the classes of effective curves. A primitive integral class generating an extremal ray of $\mathrm{NE}(X_\Sigma)$ is called an *extremal class*. There is an exact sequence:

$$\begin{array}{ccccccc} 0 \longrightarrow A_1(X_\Sigma) & \longrightarrow & \mathbb{Z}^{G(\Sigma)} & \longrightarrow & N \longrightarrow 0, & & \\ [C] & \longrightarrow & \big(C \cdot V(v)\big)_{v\in G(\Sigma)} & & & & \\ & & \big(\nu_v\big)_{v\in G(\Sigma)} & \longrightarrow & \sum_{v\in G(\Sigma)} \nu_v v. & & \end{array}$$

Thus the elements of $A_1(X_\Sigma)$ are identified with integral relations between the elements of $G(\Sigma)$. If the class $[C]$ corresponds to the relation $\sum_v \nu_v v = 0$, then we have $-K_{X_\Sigma} \cdot C = \sum_v \nu_v$.

Proposition 2.6. ([Cas03a, Lemma 1.4]) Let Σ be a regular complete fan in $N_\mathbb{Q}$. A relation

$$\alpha_1 x_1 + \cdots + \alpha_l x_l - \beta_1 y_1 - \cdots - \beta_m y_m = 0,$$

with $\alpha_i, \beta_j \in \mathbb{Z}_{>0}$, defines an effective class in $\mathcal{N}_1(X_\Sigma)$ provided that $\langle y_1, \ldots, y_m\rangle$ is a cone of Σ.

We will usually write the above relation as

$$\alpha_1 x_1 + \cdots + \alpha_l x_l = \beta_1 y_1 + \cdots + \beta_m y_m.$$

It follows that primitive relations correspond to effective curve classes. By abuse of notation, we will identify a primitive relation $r(P)$ with the corresponding curve class. Note that $\deg(P) = -K_{X_\Sigma} \cdot r(P)$. In the projective case we have the following description of $\mathrm{NE}(X_\Sigma)$.

Proposition 2.7. ([Bat91, Theorem 2.15]) Let Σ be a regular complete fan in $N_\mathbb{Q}$, and assume that X_Σ is projective. Then the Mori cone is generated by primitive relations:

$$\mathrm{NE}(X_\Sigma) = \sum_{P\in \mathrm{PC}(X_\Sigma)} \mathbb{Q}_{\geq 0}\, r(P).$$

Proposition 2.8. ([Rei83, Theorem 2.4]) Let Σ be a regular complete fan in $N_\mathbb{Q}$, and assume that X_Σ is projective. Let γ be an extremal class in $\mathrm{NE}(X_\Sigma)$ whose corresponding primitive relation is

$$r(P)\colon\ v_1 + \cdots + v_r = \mu_1 w_1 + \cdots + \mu_s w_s.$$

Let $\tau = \langle z_1, \ldots, z_l\rangle$ be a cone of Σ such that $G(\tau) \cap P = G(\tau) \cap G(\sigma(P)) = \varnothing$, and such that $\langle \sigma(P), \tau\rangle = \langle w_1, \ldots, w_s, z_1, \ldots, z_l\rangle$ is a cone of Σ. Then, for each $i = 1, \ldots, r$,

$$\langle P \setminus \{v_i\}, \sigma(P), \tau\rangle = \langle v_1, \ldots, \check{v}_i, \ldots, v_r, w_1, \ldots, w_s, z_1, \ldots, z_l\rangle$$

is also a cone of Σ.

2.2. **The minimal $\mathbb{P}$-dimension.** Let X be a toric manifold with regular complete fan Σ_X in $N_\mathbb{Q}$. In this section, we discuss centrally symmetric primitive collections, introduced in Definition 2.3.

Proposition 2.9. ([Bat91, Proposition 3.2]) If X is projective, then Σ_X has a centrally symmetric primitive collection of order $k+1$

$$r(P)\colon\ x_0 + \cdots + x_k = 0 \tag{2}$$

for some $k \in \{1, \ldots, \dim(X)\}$.

Definition 2.10. For a projective toric manifold X, we define the *minimal* $\mathbb{P}$*-dimension* as

$$m(X) \coloneqq \min\left\{ m \in \mathbb{Z}_{>0} \;\middle|\; \begin{array}{c} \Sigma_X \text{ has a centrally symmetric} \\ \text{primitive collection of order } m+1 \end{array} \right\}.$$

The next remark explains the terminology of Definition 2.10 and highlights the significance of studying centrally symmetric primitive collections.

Remark 2.11. In [CFH14], Chen, Fu and Hwang provide a new geometric proof of Proposition 2.9 by relating centrally symmetric primitive collections to *minimal dominating families of rational curves*. We review some aspects of the theory of rational curves on varieties and refer to [Kol96] for details. Given a smooth and proper uniruled variety X, there is a scheme $\mathrm{RatCurves}^n(X)$ parametrizing rational curves on X. A *dominating family of rational curves* on X is an irreducible component of $\mathrm{RatCurves}^n(X)$ parametrizing rational curves that sweep out a dense open subset of X. A dominating family of rational curves H is said to be *minimal* if, for a general point $x \in X$, the subvariety of H parametrizing curves through x is proper. When X is projective, there always exists a minimal dominating family of rational curves on X. For instance, one can take H to be a dominating family of rational curves on X having minimal degree with respect to some fixed ample line bundle on X.

When $X = X_\Sigma$ is a toric variety, there is a one-to-one correspondence between minimal dominating families of rational curves H on X and centrally symmetric primitive collections of Σ ([CFH14, Proposition 3.2]). Moreover, if the centrally symmetric primitive collection has order $k+1$ as in Equation (2) above, then there is a dense T-invariant open subset U of X and a $\mathbb{P}^k$-bundle $\pi\colon U \to W$ such that the general curve parametrized by H is a line on a general fiber π ([CFH14, Corollary 2.6]).

It follows from this discussion that the minimal $\mathbb{P}$-dimension $m(X)$ is the smallest integer k such that X admits a generic $\mathbb{P}^k$-bundle structure. We have

$$m(X) \in \ \{1, \dots, n = \dim(X)\},$$

and $m(X) = n$ if and only if $X \simeq \mathbb{P}^n$. By [CFH14, Proposition 3.8], there are three toric Fano manifolds admitting a centrally symmetric primitive relation of order $n = \dim(X)$, namely: $\mathbb{P}^{n-1} \times \mathbb{P}^1$, the blowup of $\mathbb{P}^{n-1} \times \mathbb{P}^1$ along a linear $\mathbb{P}^{n-2}$, and the blowup of $\mathbb{P}^n$ along a linear $\mathbb{P}^{n-2}$. The first two varieties also admit a generic $\mathbb{P}^1$-bundle structure. As a consequence, the only n-dimensional toric Fano manifold X with $m(X) = n-1$ is the blowup of $\mathbb{P}^n$ along a linear $\mathbb{P}^{n-2}$. In Section 5, we shall classify n-dimensional toric Fano manifolds X with $m(X) = n-2$.

Let $P \in \mathrm{PC}(X)$ be a centrally symmetric primitive collection of order $k+1$. As explained in Remark 2.11, P induces a $\mathbb{P}^k$-bundle structure on a dense T-invariant open subset U of X. In [CFH14, Corollary 2.6], the T-invariant open subset U was taken as small as possible, namely, $U \cong \mathbb{P}^k \times (\mathbb{C}^*)^{n-k}$. For our purposes, we want to take U as big as possible. So our next goal is to describe explicitly the biggest T-invariant open subset of X on which P induces a $\mathbb{P}^k$-bundle structure.

Notation 2.12. Let $P = \{x_0, \dots, x_k\} \in \mathrm{PC}(X)$ be a centrally symmetric primitive collection. Denote by $\mathcal{E}_P$ the set of cones $\sigma = \langle v_1, \dots, v_r \rangle \in \Sigma_X$ such that $P \cap G(\sigma) = \varnothing$, and $\{v_1, \dots, v_r, x_{j_1}, \dots, x_{j_s}\} \in \mathrm{PC}(X)$ for some $s \geq 1$, i.e.,

$$\mathcal{E}_P \coloneqq \{\sigma \in \Sigma_X \mid P \cap G(\sigma) = \varnothing \text{ and } \exists P' \subsetneq P \text{ such that } P' \cup G(\sigma) \in \mathrm{PC}(X)\}\,.$$

We write

$$V(\mathcal{E}_P) \coloneqq \bigcup_{\sigma \in \mathcal{E}_P} V(\sigma) \subset X.$$

Proposition 2.13. Let $P = \{x_0, \dots, x_k\} \in \mathrm{PC}(X)$ be a centrally symmetric primitive collection, and let $V(\mathcal{E}_P)$ be as in Notation 2.12. Then the open subset $U = X \setminus V(\mathcal{E}_P)$ admits a $\mathbb{P}^k$-bundle structure over a smooth toric variety.

In order to prove Proposition 2.13, we first prove two auxiliary lemmas.

Lemma 2.14. Let $P = \{x_0, \dots, x_k\} \in \mathrm{PC}(X)$ be a centrally symmetric primitive collection, let $V(\mathcal{E}_P)$ be as in Notation 2.12, and set $U = X \setminus V(\mathcal{E}_P)$. Then the fan Σ_U of U consists of all cones of Σ_X of the form

$$\tau' = \langle \tau, x_{j_1}, \dots, x_{j_m} \rangle, \tag{3}$$

where $0 \le m \le k$, and $\tau \in \Sigma_X$ is such that $\langle \tau, P \setminus \{x_i\} \rangle \in \Sigma$ for every $i \in \{0, \dots, k\}$. (When $m = 0$, Equation (3) means that $\tau' = \tau$.)

Proof. Recall that a cone $\sigma \in \Sigma_X$ corresponds to a T-orbit, which is dense and open in $V(\sigma)$. Hence, a cone $\sigma \in \Sigma_X$ is in Σ_U if and only if the corresponding orbit does not intersect $V(\mathcal{E}_P)$, which is equivalent to saying that $V(\sigma) \not\subseteq V(\mathcal{E}_P)$. It is immediate that the cones of the form (3) define a fan $\Sigma' \subset \Sigma_X$ in $N_\mathbb{Q}$, and $X_{\Sigma'}$ is a dense open subset of X. We now prove that the toric variety $X_{\Sigma'}$ coincides with U by showing that a cone $\sigma \in \Sigma_X$ is of the form (3) if and only if $V(\sigma) \not\subseteq V(\mathcal{E}_P)$.

Consider $\sigma \in \Sigma_X \setminus \Sigma'$, which means that $\langle G(\sigma) \cup P \setminus \{x_i\} \rangle \notin \Sigma_X$ for some i. Then the set $G(\sigma) \cup P \setminus \{x_i\}$ contains a primitive collection S, so the cone $\tau \coloneqq \langle S \setminus P \rangle$ is in $\mathcal{E}_P$. But notice that $\tau \prec \sigma$, so $V(\sigma) \subseteq V(\tau) \subseteq V(\mathcal{E}_P)$ and hence σ is not in Σ_U. Conversely, if $\sigma \in \Sigma_X \setminus \Sigma_U$, then $V(\sigma) \subseteq V(\mathcal{E}_P)$, hence there exists $\tau \in \mathcal{E}_P$ such that $V(\sigma) \subseteq V(\tau)$ and $G(\tau) \cup P' \in \mathrm{PC}(X)$ for some $P' \subset P$. Since $G(\tau) \subseteq G(\sigma)$, we conclude that $\langle G(\sigma) \cup P' \rangle \notin \Sigma_X$, i.e., $\sigma \notin \Sigma'$. □

Consider the sequences

$$0 \longrightarrow N_P \coloneqq \ker(\phi) \longrightarrow N \xrightarrow{\phi} \overline{N} = N/\mathbb{Z}\langle x_0, \dots, x_k \rangle \longrightarrow 0,$$

$$0 \longrightarrow (N_P)_\mathbb{Q} \longrightarrow N_\mathbb{Q} \xrightarrow{\phi_\mathbb{Q}} \overline{N}_\mathbb{Q} \longrightarrow 0,$$

$$\Sigma_0 \longrightarrow \Sigma_U \longrightarrow \overline{\Sigma}_U,$$

where ϕ is the quotient map, the fan Σ_0 of $(N_P)_\mathbb{Q} \simeq \mathbb{Q}^{k+1}$ is the subfan of Σ_U of cones of the form (3) with $\tau = \{0\}$ (in particular note that $X_{\Sigma_0} \simeq \mathbb{P}^k$), and $\overline{\Sigma}_U = \{\phi_\mathbb{Q}(\sigma) \mid \sigma \in \Sigma_U\}$.

Lemma 2.15. Let the notation be as above. Then $\overline{\Sigma}_U$ is a toric fan, and the linear map $\phi_\mathbb{Q}$ is compatible with the fans Σ_U and $\overline{\Sigma}_U$.

Proof. The cones of $\overline{\Sigma}_U$ are exactly $\phi_\mathbb{Q}(\tau)$ for $\tau \in \Sigma_U$ such that $G(\tau) \cap P = \varnothing$, so for simplicity we only consider these τ.

- It is immediate that the cones of $\overline{\Sigma}_U$ are rational polyhedral, and that the faces of $\phi_\mathbb{Q}(\tau)$ are $\phi_\mathbb{Q}(\delta)$, for all subcones $\delta \prec \tau$.

- We need to show that the cone $\phi_{\mathbb{Q}}(\tau)$ is strongly convex, i.e., if $y \in \phi_{\mathbb{Q}}(\tau)$ and $-y \in \phi_{\mathbb{Q}}(\tau)$, then $y = 0$. This follows automatically from the fact that the images of the generators $\phi_{\mathbb{Q}}(G(\tau)) = \{\overline{v}_1, \dots, \overline{v}_r\}$ are linearly independent, which we prove by contradiction. If they are linearly dependent, then there exist $a_1, \dots, a_r \in \mathbb{Q}$, not all 0, such that $\Sigma_{i=1}^r a_i \overline{v}_i = 0$ in $\overline{N}_{\mathbb{Q}}$. This implies that there exist $b_j \in \mathbb{Q}$ for $j = 1, \dots, k$ such that $\Sigma_{i=1}^r a_i v_i = \sum_{j=1}^k b_j x_j$, which is a contradiction since $\langle G(\tau) \cup P \setminus \{x_0\} \rangle \in \Sigma$ and hence its primitive generators are linearly independent in $N_{\mathbb{Q}}$.
- Σ_U and $\overline{\Sigma}_U$ are compatible with $\phi_{\mathbb{Q}}$ as we have $\phi_{\mathbb{Q}}(\tau') \in \overline{\Sigma}_U$ for any $\tau' \in \Sigma_U$.

□

Proof of Proposition 2.13. Let the notation be as above. Let $\hat{\Sigma}_U$ be the collection of fans of the form (3) above with $m = 0$. It follows from the description of the cones of Σ_U in Lemma 2.14 that

(1) $\phi_{\mathbb{Q}}$ maps each cone $\hat{\tau} \in \hat{\Sigma}_U$ bijectively to a cone $\overline{\tau} \in \overline{\Sigma}_U$ such that $\phi(\hat{\tau} \cap N) = \overline{\tau} \cap \overline{N}$. Furthermore, the map $\hat{\tau} \mapsto \overline{\tau}$ defines a bijection $\hat{\Sigma}_U \to \overline{\Sigma}_U$;
(2) given cones $\hat{\tau} \in \hat{\Sigma}_U$ and $\tau_0 \in \Sigma_0$, the sum $\hat{\tau} + \tau_0$ lies in Σ_U and every cone of Σ_U arises in this way.

In the notation of [CLS11, Definition 3.3.18], we say that Σ_U *is split by* $\overline{\Sigma}_U$ *and* Σ_0. We conclude by [CLS11, Theorem 3.3.19] that $U = X \setminus V(\mathcal{E}_P)$ is a locally trivial fiber bundle over $X_{\overline{\Sigma}_U}$ with fiber $X_{\Sigma_0} \simeq \mathbb{P}^k$. It follows automatically that $X_{\overline{\Sigma}_U}$ is smooth, since it is the base of a locally trivial fibration with a smooth total space. □

2.3. **Some properties of primitive collections.** Before focusing on toric Fano manifolds, we collect here two useful properties of primitive collections of arbitrary toric manifolds. The first one, by Sato, describes the behaviour of primitive collections under a smooth toric blowdown. The second one, by Batyrev, describes primitive collections on toric manifolds of Picard rank 3.

Proposition 2.16. ([Sat00, Corollary 4.9]) Let X be a toric manifold, and let $f: X \to Y$ be the contraction associated to an extremal class in $\mathrm{NE}(X)$, corresponding to a primitive relation of the form

$$r(Q): t_1 + \cdots + t_s = z.$$

Then the fan Σ_Y is obtained from Σ_X by removing the ray generated by z, and X is the blowup of Y along $V(t_1, \dots, t_s)$. Furthermore, the primitive collections of Y are precisely the following $P_Y \in \mathrm{PC}(Y)$:

- $P_Y = P_X$ for some $P_X \in \mathrm{PC}(X)$ such that $z \notin P_X$ and $P_X \neq Q = \{t_1, \dots, t_s\}$;
- $P_Y = (P_X \setminus \{z\}) \cup \{t_1, \dots, t_r\}$ for some $P_X \in \mathrm{PC}(X)$ such that $z \in P_X$ and $(P_X \setminus \{z\}) \cup S \notin \mathrm{PC}(X)$ for any subset $S \subsetneq \{t_1, \dots, t_r\}$.

Proposition 2.17. ([Bat91, Theorem 5.7, Theorem 6.6]) Let X be a projective toric manifold with $\rho(X) = 3$. Then the number of primitive collections of Σ_X is either $l = 3$ or $l = 5$. Moreover, the set of generators $G(\Sigma_X)$ can be written as a disjoint union of l nonempty subsets

$$G(\Sigma_X) = X_0 \sqcup \cdots \sqcup X_{l-1}$$

that define primitive collections and relations as follows:

- Case $l = 3$. Each X_0, X_1, X_2 is a primitive collection, and the corresponding primitive relations are extremal.
- Case $l = 5$. There are five primitive collections of the form $X_i \sqcup X_{i+1}$, $0 \le i \le 4$, where $X_5 \coloneqq X_0$. To describe the primitive relations of X, we use the following notation. We fix a labelling $(v_1, \dots, v_k)$ for the elements of X_i. If $\overline{c} = (c_1, \dots, c_k) \in \mathbb{Z}^k$, then $\overline{c} \cdot X_i$ stands for $c_1 v_1 + \cdots + c_k v_k$. Moreover, we set $\overline{1} = (1, \dots, 1)$. Then there are vectors $\overline{c}$ and $\overline{b}$ of nonnegative integers such that at least one entry in $\overline{c}$ is zero (up to relabelling, we may assume that $c_1 = 0$), and the primitive relations of X are the following:

$$\begin{aligned}
r_0\colon \quad & \overline{1} \cdot X_0 + \overline{1} \cdot X_1 = \overline{c} \cdot X_2 + (\overline{b} + \overline{1}) \cdot X_3 \\
r_1\colon \quad & \overline{1} \cdot X_1 + \overline{1} \cdot X_2 = \overline{1} \cdot X_4, \\
r_2\colon \quad & \overline{1} \cdot X_2 + \overline{1} \cdot X_3 = 0, \\
r_3\colon \quad & \overline{1} \cdot X_3 + \overline{1} \cdot X_4 = \overline{1} \cdot X_1, \\
r_4\colon \quad & \overline{1} \cdot X_4 + \overline{1} \cdot X_0 = \overline{c} \cdot X_2 + \overline{b} \cdot X_3.
\end{aligned}$$

  The relations r_0, r_1 and r_3 are extremal, while $r_2 = r_1 + r_3$ and $r_4 = r_0 + r_3$.

Remark 2.18. In [SS20, Corollary 1.2], Sato and Suyama use Proposition 2.17 to show that projective spaces are the only toric 2-Fano manifolds with Picard rank $\rho \le 3$.

2.4. **Primitive collections on toric Fano manifolds.** Let X be a projective toric manifold with regular complete fan Σ_X in $N_{\mathbb{Q}}$.

Proposition 2.19. ([Bat99, Proposition 2.3.6]) The toric variety X is Fano if and only if all primitive collections of Σ_X have strictly positive degree.

Proposition 2.20. ([Cas03a, Corollary 4.4]) Assume that X is Fano, and let $P \in \mathrm{PC}(X)$. If $\deg(P) = 1$, then the corresponding curve class is extremal.

Proposition 2.21. Assume that X is Fano, and let $x \in G(\Sigma_X)$.

1. There is at most one primitive collection of order 2 and degree 2 containing x. If it exists, then it is of the form $x + (-x) = 0$, and $m(X) = 1$.
2. ([Cas03b, Lemma 3.3]) There are at most two primitive collections of order 2 and degree 1 containing x. If there are exactly two of them, then they are of the form $x + y = (-w)$ and $x + w = (-y)$, and $m(X) = 1$.

Corollary 2.22. Assume that X is Fano and $m(X) > 1$. Then any $x \in G(\Sigma_X)$ is contained in at most one primitive collection of order 2. If there is such a primitive collection, then it is of the form $x + y = z$.

Definition 2.23. Let $x \in G(\Sigma_X)$. We say that $y \in G(\Sigma_X)$ is an *opponent* of x if $\langle x, y \rangle \notin \Sigma_X$.

Notation 2.24. Assume that X is Fano and $m(X) > 1$. By Corollary 2.22, each vector $x \in G(\Sigma_X)$ has at most one opponent. If such an opponent exists, we denote it by x'.

Lemma 2.25. Assume that X is Fano and $m(X) > 1$. Consider a pair of opponents $x, x' \in G(\Sigma_X)$. If there exist $y, z \in G(\Sigma_X)$ such that $x + x' = y + z$, then $z = y'$.

Proof. If $y = x$ or $y = x'$, the claim follows automatically. So we assume otherwise. Note that $\{x, x'\} \in \mathrm{PC}(X)$, and y and z do not form a cone, as otherwise $x + x' = y + z$ would give us a primitive relation of degree 0, which is impossible for a toric Fano manifold. □

Lemma 2.26. Assume that X is Fano and $m(X) > 1$. Assume that there exist x, y, z, u, v in $G(\Sigma_X)$ such that $(*)$ $x + y + z = u + v$ and such that $\langle u, v\rangle \in \Sigma_X$. Then exactly one of the following must happen:

a. The vectors x, y, z are pairwise distinct, and $\{x, y, z\}$ is a primitive collection with primitive relation $(*)$. In particular, the corresponding curve class is extremal.
b. Up to relabeling, $v = z$, $y = x'$ and $x + x' = u$.

Proof. Assume two of $\{x, y, z\}$ do not form a cone. For example, assume x, y do not form a cone. Then $y = x'$, the opponent of x. Let $x + x' = \alpha$, for some $\alpha \in G(\Sigma_X)$. We have that $\alpha + z = u + v$. As $\langle u, v\rangle \in \Sigma_X$, Proposition 2.6 implies that $\alpha + z = u + v$ corresponds to an effective class of degree 0, which therefore implies that $\{\alpha, z\} = \{u, v\}$. Up to relabeling, we may assume $v = z$, and hence, $x + x' = u$ and we are in the situation b.

Assume now that any two vectors in $\{x, y, z\}$ form a cone. Then x, y, z are mutually disjoint and $\langle x, y, z\rangle \notin \Sigma_X$, as otherwise by Proposition 2.6 we obtain an effective curve class of degree -1, contradicting the fact that X is Fano. It follows that $\{x, y, z\}$ is a primitive collection. Since $\langle u, v\rangle \in \Sigma_X$, it follows that $(*)$ is the associated primitive relation. Proposition 2.20 now implies that the corresponding curve class is extremal. □

3. Toric Fano manifolds with $m(X) = 1$

In this section, we study toric Fano manifolds with $m(X) = 1$, and follow the strategy outlined in the introduction to show that they cannot be 2-Fano.

For any $x \in G(\Sigma_X)$, the set of primitive collections containing x is denoted by

$$\mathrm{PC}_x(X) = \{P \in \mathrm{PC}(X) \mid x \in P\}.$$

Proposition 3.1. ([Cas03b, Lemma 3.1]) Assume that X is a toric Fano manifold and that $P = \{x, -x\} \in \mathrm{PC}(X)$.

(1) Any $Q \in \mathrm{PC}_x(X) \setminus \{P\}$ has degree 1 (hence is extremal by Proposition 2.20), and $r(Q)$ is of the form

$$r(Q)\colon\ x + \underbrace{y_1 + \cdots + y_h}_{\in \langle Q \setminus \{x\}\rangle} = \underbrace{z_1 + \cdots + z_h}_{\in \sigma(Q)},$$

where we denote $\langle Q \setminus \{x\}\rangle := \langle y_1, \ldots, y_h\rangle$ and $\sigma(Q) := \langle z_1, \ldots, z_h\rangle$.

(2) For any $R \in \mathrm{PC}_x(X) \setminus \{P, Q\}$, we have

$$V(R \setminus \{x\}) \cap V(Q \setminus \{x\}) = \varnothing \quad \text{and} \quad V(R \setminus \{x\}) \cap V(\sigma(Q)) = \varnothing.$$

(3) For any $Q \in \mathrm{PC}_x(X) \setminus \{P\}$ with $r(Q)\colon x + y_1 + \cdots + y_h = z_1 + \cdots + z_h$ we have $Q' = \{-x, z_1, \ldots, z_h\} \in \mathrm{PC}_{-x}(X)$, Q' has degree 1 (hence is extremal) and

$$r(Q')\colon\ -x + \underbrace{z_1 + \cdots + z_h}_{\in \langle Q' \setminus \{-x\}\rangle = \sigma(Q)} = \underbrace{y_1 + \cdots + y_h}_{\in \sigma(Q') = \langle Q \setminus \{x\}\rangle}.$$

Corollary 3.2. Let X be a toric manifold, and $P = \{x, -x\} \in \mathrm{PC}(X)$. With Notation 2.12,

$$\mathcal{E}_P = \{\langle Q \setminus \{v\}\rangle \mid Q \in \mathrm{PC}_v(X) \setminus \{P\}, \quad v = \pm x\}\,. \tag{4}$$

If moreover X is Fano, then $V(\mathcal{E}_P)$ has 0, 2 or 4 components of codimension 1 in X.

Proof. Let X be a toric manifold, and $P = \{x, -x\} \in \mathrm{PC}(X)$. The description of $V(\mathcal{E}_P)$ in Equation (4) follows from Notation 2.12. In the Fano case, the number of components of codimension 1 of $V(\mathcal{E}_P)$ equals the number of primitive collections of order 2 and degree 1 containing x or $-x$, which is 0, 2 or 4 by Proposition 2.21 and Proposition 3.1. □

Proposition 3.3. Let X be a toric Fano manifold, and $P = \{x, -x\} \in \mathrm{PC}(X)$. Then there exists a birational morphism $f \colon X \to Y$ such that

- $P_Y := \{x, -x\} \in \mathrm{PC}(Y)$,
- $V(\mathcal{E}_{P_Y})$ has codimension ≥ 2 in Y,
- f is a composition of at most two blow-downs with disjoint centers and smooth target:

$$\mathrm{Exc}(f) = \bigcup_{\substack{Q \in \mathrm{PC}_x(X)\setminus\{P\}:\\ \mathrm{ord}(Q)=2}} V(\sigma(Q)) \subset X,$$

$$f(\mathrm{Exc}(f)) = \bigcup_{\substack{Q \in \mathrm{PC}_x(X)\setminus\{P\}:\\ \mathrm{ord}(Q)=2}} V(Q) \subset Y. \tag{5}$$

Proof. If $V(\mathcal{E}_P)$ has codimension ≥ 2 in X, i.e., if P is the unique primitive collection of order 2 containing x, then the statement holds with $f = \mathrm{Id}$.

Assume now that $V(\mathcal{E}_P)$ has 2 components of codimension 1, i.e. we have primitive relations

$$r(P)\colon x + (-x) = 0, \qquad r(Q_1)\colon x + y = z, \qquad r(Q_1')\colon -x + z = y,$$

and, for any other $R \in \mathrm{PC}_x(X) \cup \mathrm{PC}_{-x}(X)$, one has $\mathrm{ord}(R \setminus \{\pm x\}) \geq 2$. Let $f_1 \colon X \to Y$ be the smooth blow-down induced by the extremal ray of $\mathrm{NE}(X)$ corresponding to $r(Q_1)$. By Proposition 2.16 and Proposition 3.1, $\mathrm{PC}_x(Y) \cup \mathrm{PC}_{-x}(Y)$ consists of

- $P_Y = \{x, -x\}$;
- $R_Y = R_X$ for some $R_X \in \mathrm{PC}_x(X) \cup \mathrm{PC}_{-x}(X) \setminus \{Q_1\}$ such that $z \notin R_X$. In particular we have $\mathrm{ord}(R_Y \setminus \{\pm x\}) \geq 2$;
- $R_Y = (R_X \setminus \{z\}) \cup \{x, y\}$ for some $R_X \in \mathrm{PC}_z(X)$ such that $(R_X \setminus \{z\}) \cup \{x\} \notin \mathrm{PC}(X)$ and $(R_X \setminus \{z\}) \cup \{y\} \notin \mathrm{PC}(X)$. In particular, we have $\langle R_Y \setminus \{x\}\rangle = \langle R_X \setminus \{z\}, y\rangle$, so $\mathrm{ord}(R_Y \setminus \{x\}) \geq 2$.

It follows that $V(\mathcal{E}_{P_Y})$ has codimension ≥ 2 in Y, so the proposition holds with $f = f_1$.

Assume now that $V(\mathcal{E}_P)$ has 4 components of codimension 1, i.e., we have

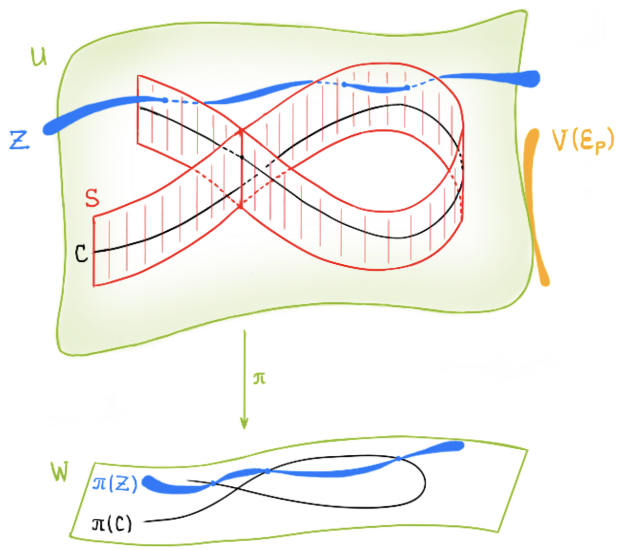
$$\begin{array}{rrr} r(P)\colon x + (-x) = 0 & r(Q_1)\colon x + y = -w & r(Q_1')\colon -x + (-w) = y \\ y + (-y) = 0 & r(Q_2)\colon x + w = -y & r(Q_2')\colon -x + (-y) = w \\ w + (-w) = 0 & w + y = -x & -y + (-w) = x \end{array}$$

by Proposition 2.21. By [Cas03b, p.1487, Case (3)], any other primitive collection $R \in \mathrm{PC}(X)$ is disjoint from $\{x, -x, y, -y, w, -w\}$. Let $f_1 \colon X \to X_1$ be the smooth blow-down induced by the extremal ray of $\mathrm{NE}(X)$ corresponding to $r(Q_1)$. By Proposition 2.16 the primitive collections of X_1 containing x or $-x$ are only

$$P_{X_1} = P \qquad (Q_2)_{X_1} = Q_2 \qquad (Q_2')_{X_1} = Q_2'.$$

It follows that $r(Q_2)$ corresponds to an extremal curve class in $\mathrm{NE}(X_1)$. Let $f_2 \colon X_1 \to X_2$ be the smooth blow-down induced by the extremal ray of $\mathrm{NE}(X)$ corresponding to $r(Q_2)$. By Proposition 2.16 $P_{X_2} = \{x, -x\}$ is the only primitive collection in $\mathrm{PC}_x(X_2) \cup \mathrm{PC}_{-x}(X_2)$. This implies that $V(\mathcal{E}_{P_{X_2}}) = \varnothing$, so the proposition holds with $Y = X_2$ and $f = f_2 \circ f_1$. □

Construction 3.4. Let Y be a projective toric manifold of dimension ≥ 3, and $P = \{x, -x\} \in \mathrm{PC}(Y)$ a centrally symmetric primitive collection of order 2. Let $V(\mathcal{E}_P) \subset Y$ be the closed subset defined in Notation 2.12, set $U := Y \setminus V(\mathcal{E}_P)$, and let $\pi : U \to W$ be the $\mathbb{P}^1$-bundle given by Proposition 2.13.



Assume that $V(\mathcal{E}_P)$ has codimension ≥ 2 in Y, and let $Z \subset Y$ be any given closed subset of codimension ≥ 2 in Y. Note that a toric manifold is rational, hence rationally connected, so by [Kol96, Proposition II.3.7], there is a smooth (very free) rational curve $C \subset U \setminus Z$. Consider the surface $S := \pi^{-1}(\pi(C)) \subset U$, and let $n \colon \tilde{S} \to S$ be its normalization. Then $\tilde{S}$ is a Hirzebruch surface with $\mathbb{P}^1$-bundle structure $\tilde{\pi} : \tilde{S} \to \mathbb{P}^1$ induced by π. The curve class of the image of a fiber of $\tilde{\pi}$ on Y corresponds to the centrally symmetric relation $x + (-x) = 0$. By taking C general, we may assume that $\pi(C)$ and $\pi(Z)$ meet transversely in at most finitely many general points. Hence S and Z meet transversely in at most finitely many points.

Proposition 3.5. Let Y be a projective toric manifold of dimension ≥ 3, and $P_Y = \{x, -x\} \in \mathrm{PC}(Y)$ a centrally symmetric primitive collection of order 2. Assume that $V(\mathcal{E}_P)$ has codimension ≥ 2 in Y, and let $S \subset Y$ be as in Construction 3.4. Then $S \cdot \mathrm{ch}_2(Y) = 0$.

Proof. Let $S = \pi^{-1}(\pi(C))$ be the surface from Construction 3.4, $n \colon \tilde{S} \to S$ its normalization, $\tilde{\pi} \colon \tilde{S} \to \mathbb{P}^1$ the $\mathbb{P}^1$-bundle structure induced by π, and F a fiber of $\tilde{\pi}$. Our goal is to compute

$$S \cdot \mathrm{ch}_2(Y) = \frac{1}{2} \sum_{v \in G(\Sigma_Y)} S \cdot V(v)^2 = \frac{1}{2} \sum_{v \in G(\Sigma_Y)} \big(n^*V(v)\big)^2.$$

Recall that the curve class of the image of F in Y is associated to the relation $x+(-x)=0$. By restricting the divisors $n^*V(v)$ to F, we have:

$$\begin{cases} n^*V(v) \cdot F = 0 & \text{if } v \neq x, -x, \\ n^*V(x) \cdot F = 1, \\ n^*V(-x) \cdot F = 1. \end{cases}$$

Hence there are sections σ, σ' of $\tilde{\pi} \colon \tilde{S} \to \mathbb{P}^1$, and $\alpha, \beta, \gamma \in \mathbb{Z}$ such that, on $\tilde{S}$,

$$\begin{cases} n^*V(v) = \alpha F & \text{if } v \neq x, -x, \\ n^*V(x) = \sigma + \beta F, \\ n^*V(-x) = \sigma' + \gamma F. \end{cases}$$

Therefore

$$\begin{aligned} \sum_{v \in G(\Sigma_Y)} \big(n^*V(v)\big)^2 &= \sum_{v \neq x, -x} \cancel{\big(n^*V(v)\big)^2} + \big(n^*V(x)\big)^2 + \big(n^*V(-x)\big)^2 \\ &= \sigma^2 + 2\beta + \sigma'^2 + 2\gamma \\ &= \sigma^2 - 2(\sigma \cdot \sigma') + \sigma'^2 \qquad \text{as } \sigma \cdot \sigma' + \beta + \gamma = n^*V(x) \cdot n^*V(-x) = 0 \\ &= (\sigma - \sigma')^2 = 0 \qquad \text{as } \sigma - \sigma' \text{ is a multiple of } F, \end{aligned}$$

and this concludes the proof. □

Lemma 3.6. ([dS06a, Lemma 5.1]) Consider the blowup diagram

$$\begin{array}{ccc} E & \overset{j}{\hookrightarrow} & X := \mathrm{Bl}_Z Y \\ {\scriptstyle \pi := f_{|E}} \downarrow & & \downarrow {\scriptstyle f} \\ Z & \hookrightarrow & Y \end{array}$$

where both Y and Z are smooth projective varieties and $\mathrm{codim}_Y Z = c \geq 2$. Then we have the following relation between the 2nd Chern characters of X and Y:

$$\mathrm{ch}_2(X) = f^* \mathrm{ch}_2(Y) + \frac{c+1}{2} E^2 - j_* \pi^* \mathrm{c}_1(\mathcal{N}_{Z/Y}).$$

Corollary 3.7. In the setting of Lemma 3.6, assume that $c = 2$. Let $S \subset Y$ be a surface that intersects Z at most transversely at $k \geq 0$ points, and let $S_X \subset X$ be its strict transform. Then

$$\mathrm{ch}_2(X) \cdot S_X = \mathrm{ch}_2(Y) \cdot S - \frac{3}{2} \cdot k.$$

Proof. Write $S \cap Z = \{p_1, \ldots, p_k\}$. Then S_X is isomorphic to the blowup of S at $p_1, \ldots, p_k$, $S_X \cap E = \cup_{i=1}^k e_i$, where $e_i \simeq \mathbb{P}^1$ is the exceptional curve over p_i, and $(e_i^2)_{S_X} = -1$. By

Lemma 3.6,

$$\begin{aligned}
\mathrm{ch}_2(X)\cdot S_X &= f^*\,\mathrm{ch}_2(Y)\cdot S_X + \frac{c+1}{2}E^2\cdot S_X - j_*\pi^*\,\mathrm{c}_1(\mathcal{N}_{Z/Y})\cdot S_X\\
&= \mathrm{ch}_2(Y)\cdot S + \frac{3}{2}(E_{|S_X})^2 - \pi^*\,\mathrm{c}_1(\mathcal{N}_{Z/Y})\cdot S_{X|E}\\
&= \mathrm{ch}_2(Y)\cdot S + \frac{3}{2}\sum_{i=1}^{k}(e_i)^2 - \pi^*\,\mathrm{c}_1(\mathcal{N}_{Z/Y})\cdot\sum_{i=1}^{k}e_i\\
&= \mathrm{ch}_2(Y)\cdot S - \frac{3}{2}k - \sum_{i=1}^{k}\cancel{\mathrm{c}_1(\mathcal{N}_{Z/Y})\cdot p_i}.
\end{aligned}$$

□

We are ready to prove Theorem 1.2: a toric Fano manifold X with $m(X)=1$ is not 2-Fano.

Proof of Theorem 1.2. Let X be a toric Fano manifold with $m(X)=1$, and fix a centrally symmetric primitive relation $r(P)\colon x+(-x)=0$. Let $f\colon X\to Y$ be as in Proposition 3.3, and $\pi\colon U=Y\setminus V(\mathcal{E}_{P_Y})\to W$ the $\mathbb{P}^1$-bundle structure induced by $r(P_Y)\colon x+(-x)=0$ (see Proposition 2.13). By Equation (5), $Z\coloneqq f(\mathrm{Exc}(f))$ has codimension ≥ 2 in Y. Let $S\subset Y$ be the surface given by Construction 3.4. Then S and Z meet transversely in at most finitely many points. It follows from Proposition 3.5 that $\mathrm{ch}_2(Y)\cdot S=0$. Let S_X be the strict transform of S in X. By Corollary 3.7, $\mathrm{ch}_2(X)\cdot S_X\leq \mathrm{ch}_2(Y)\cdot S=0$, and so X is not 2-Fano. □

4. Proof of Theorem 1.3

In this section, we work in the setting of Theorem 1.3:

(I) X is a Fano manifold of dimension $n\geq 6$ with fan $\Sigma=\Sigma_X$ in $N_{\mathbb{Q}}$;
(II) $m(X)\geq 3$;
(III) $r(P)\colon x_0+\cdots+x_{n-2}=0$ is a centrally symmetric primitive relation in Σ.

The equality $m(X)=n-2$, as well as uniqueness of the centrally symmetric primitive collection, follows immediately from Lemma 2.5. Now the goal is to prove that $\rho(X)\leq 3$, and this will follow from Proposition 4.11, Proposition 4.16 and Proposition 4.17. By Remark 2.18, we conclude that the projective space $\mathbb{P}^n$ is the only smooth n-dimensional toric 2-Fano variety with $m(X)\geq n-2$ (Corollary 1.5).

Let $P\coloneqq\{x_0,\dots,x_{n-2}\}$ and set $\Gamma=\mathrm{Span}\,P\subset N_{\mathbb{Q}}$. Consider the quotient

$$\pi\colon N_{\mathbb{Q}}\simeq\mathbb{Q}^n\to\mathbb{Q}^n/\Gamma\simeq\mathbb{Q}^2.$$

Since X is complete, the support of Σ is equal to $N_{\mathbb{Q}}$, and hence we can find generators

(IV) $x,y,z\in G(\Sigma)\setminus P$, for which
(V) $0\in\mathrm{Conv}(\pi(x),\pi(y),\pi(z))$.

Lemma 4.1. The nonnegative span $\langle x,y,z\rangle$ is not a cone in Σ.

Proof. We will argue by contradiction and assume that $\langle x, y, z\rangle \in \Sigma$. By (V), we can find a nonnegative triple of constants (c_1, c_2, c_3) such that $c_1\pi(x) + c_2\pi(y) + c_3\pi(z) = 0$, or in other words

$$v \coloneqq c_1 x + c_2 y + c_3 z = a_0 x_0 + \cdots + a_{n-2} x_{n-2}$$

for some constants a_i. Note that by adding some multiple of $r(P)$ (III) to the right hand side, we can assume the a_i's are nonnegative and such that at least one of them, say a_j, is 0. It follows that v lies in two cones of Σ, namely

$$v \in \langle x, y, z\rangle \cap \langle x_0, \dots, \check{x_j}, \dots, x_{n-2}\rangle,$$

which is impossible since $\{x, y, z\} \cap S = \varnothing$. □

Since $\{x, y, z\}$ does not span a cone, we conclude that

(1) either it is a primitive collection,
(2) or two of these vectors do not form a cone.

The former case is the more technical one, and we start with it in Section 4.1. After we are done analyzing it, we can assume that none of the triples $\{x, y, z\}$ as in (IV) and (V) form a primitive collection, and this case will be treated in Section 4.2.

4.1. **First case: $\{x, y, z\}$ is a primitive collection.** We recall that if X is a projective toric Fano manifold and $m(X) > 1$, then any $x \in G(\Sigma)$ has at most one opponent by Corollary 2.22, where the opponent of x is an element $x' \in G(\Sigma)$ such that $\{x, x'\} \in \mathrm{PC}(X)$. When we write $\{x, x'\}$, we mean either the set of two elements if x' exists, or the singleton $\{x\}$ if x' does not exist.

Remark 4.2. In the setting (I)—(V), pick a generator $u \in G(\Sigma)$. Then (V) and plane geometry imply that the convex hull of $\pi(u)$ together with two of the vectors $\pi(x)$, $\pi(y)$, $\pi(z)$ contains 0.

Lemma 4.3. In the setting (I)—(V), assume in addition that $Q \coloneqq \{x, y, z\}$ is a primitive collection. Then the corresponding primitive relation is

- either $r(Q)\colon\ x + y + z = x_i + x_j$ for possibly equal $x_i, x_j \in P$,
- or $r(Q)\colon\ x + y + z = v$ for some $v \in G(\Sigma)$.

Proof. Since X is Fano (I), the degree of the primitive relation $x + y + z = A$ is positive, so we can have only three possibilities for A. The first one with $A = 0$ is actually not possible by our assumption (II). The second is $A = v$, and we cannot say much about v at the moment. The last possibility is

$$r(Q)\colon\ x + y + z = u + v$$

for some, possibly equal, $u, v \in G(\Sigma)$. By Proposition 2.20, this is an extremal primitive relation, so by Proposition 2.8 applied to $r(Q)$ and $\tau = \{0\}$, we get that $\langle u, x, y\rangle$, $\langle u, y, z\rangle$, $\langle u, x, z\rangle \in \Sigma$. By Remark 4.2, we may assume without loss of generality that $0 \in \mathrm{Conv}(\pi(u), \pi(x), \pi(y))$, hence by Lemma 4.1, we get $u \in P$. The same argument applies to conclude $v \in P$. □

Hence we have two cases to consider: when $\deg(Q)$ is 1 and when it is 2.

4.1.1. *Degree one.* In this case, by Lemma 4.3, we have

(VI) a primitive relation $r(Q)\colon\ x+y+z = x_i + x_j$ for possibly equal $x_i, x_j \in P$.

Lemma 4.4. In the setting (I)—(VI), assume in addition that $G(\Sigma)$ is contained in $P \cup \{x, y, z, x', y', z'\}$. Then x', y', z' do not exist. Consequently, $\rho(X) \le 2$.

Proof. By contradiction, assume that $x' \in G(\Sigma)$ exists, so $\{x, x'\}$ is a primitive collection, and let $x + x' = \alpha$ be the corresponding primitive relation. Clearly $\alpha \neq x$, x'. The relation $r(Q)$ (VI) gives

$$\alpha + y + z = x_i + x_j + x',$$

which shows $\alpha \notin P \cup \{y, z\}$ since otherwise the left hand side (LHS) forms a cone and we get an effective class of degree zero. Indeed, if $\alpha \in \{y, z\}$ it is clear that the LHS would form a cone; if $\alpha \in P$ applying Proposition 2.8 to $r(Q)$ and $\tau = \langle \alpha \rangle$ would follow that the LHS is a cone.

So without loss of generality, we can assume $x + x' = y'$. Applying the same argument to y', we get that $y + y' = x'$ or $y + y' = z'$. The former would imply $x + y = 0$, which is not possible by (II), so $y + y' = z'$. Again, applying the same argument to z', we get $z + z' = x'$. Summing the three primitive relations, we obtain $x + y + z = 0$, which contradicts (VI). □

This leaves us with the case when

(VII) there exists $u \in G(\Sigma) \setminus (P \cup \{x, y, z, x', y', z'\})$.

By Remark 4.2, we can assume without loss of generality that

(VIII) $0 \in \mathrm{Conv}(\pi(x), \pi(y), \pi(u))$.

Lemma 4.5. Assume (I)—(VIII), then $R \coloneqq \{x, y, u\}$ is a primitive collection with primitive relation $r(R)\colon\ x + y + u = v$ for some $v \in G(\Sigma)$.

Proof. By Lemma 4.1, we have $\langle x, y, u\rangle \notin \Sigma$, and since $u \neq x', y'$, we conclude that $\{x, y, u\}$ is a primitive collection. By Lemma 4.3, we can have either $r(R)\colon\ x + y + u = x_k + x_l$ or $r(R)\colon\ x + y + u = v$. In the former case, combining $r(R)$ with $r(Q)$ (VI) provides us with a relation

$$u + x_i + x_j = z + x_k + x_l.$$

By applying Proposition 2.8 to $r(Q)$ (VI) and $\tau = \langle x_k, x_l\rangle$, we get $\langle z, x_k, x_l\rangle \in \Sigma$ (here we are using the assumption $\dim(X) = n \ge 6$ (I)). Hence, by Proposition 2.6, we get an effective curve class of degree 0, contradicting the Fano assumption (I). □

We will write down the result of Lemma 4.5 as an additional assumption, remembering that it is implied by the previous assumptions:

(IX) We have a primitive relation $r(R)\colon\ x + y + u = v$ for some $v \in G(\Sigma)$.

Lemma 4.6. Assume (I)—(IX), then

$$G(\Sigma) \subset P \cup \{x, y, z, u, x', y', u', v'\}.$$

In particular, $z' \in P \cup \{x, y, z, u, x', y', u', v'\}$, and since $v \neq x, y, u, v'$, we have that $v \in P$ or $v \in \{z, x', y', u'\}$.

Proof. Take any $w \in G(\Sigma) \setminus (P \cup \{x, y, z, u, x', y', u'\})$. By Remark 4.2, the convex hull of $\pi(w)$ together with two of $\pi(x)$, $\pi(y)$, $\pi(u)$ contains 0, yielding an analog of (VIII). By

Lemma 4.1 and from $w \neq x', y', u'$, it follows that one of $\{w, x, u\}$, $\{w, y, u\}$, $\{w, x, y\}$ is a primitive collection.

We will prove that the corresponding primitive relation has the form $w + x + u = b$ or $w+y+u = b$ or $w+x+y = b$ for some $b \in G(\Sigma)$. Assume for a contradiction that this is not the case. By Lemma 4.3, we have that one of $w+x+u$, $w+y+u$, or $w+x+y$ equals x_k+x_l, for possibly equal $x_k, x_l \in P$. Hence there exist $a, b \in G(\Sigma)$ (one of which is $w \neq z$) such that either $x+a+b$ or $y+a+b$ equals x_k+x_l. Assume $x+a+b = x_k+x_l$. Combining this with $r(Q)$ (VI), it follows that $y+z+x_k+x_l = a+b+x_i+x_j$. By applying Proposition 2.8 to $r(Q)$ (VI) and $\tau = \langle x_k, x_l\rangle$, we get $\langle y, z, x_k, x_l\rangle \in \Sigma$ (here we are using the assumption $\dim(X) = n \geq 6$ (I)). Hence, by Proposition 2.6, we get an effective curve class of degree 0. The class is non-trivial since $w \neq y, z, x_k, x_l$. This contradicts the assumption that X is Fano. The case $y + a + b = x_k + x_l$ is similar.

So we have a primitive relation of the form $w+x+u = b$ or $w+y+u = b$ or $w+x+y = b$ for some $b \in G(\Sigma)$. Combining this with $r(R)$ (IX), we get $b + y = w + v$ or $b + x = w + v$ or $b + u = w + v$. All possibilities imply that $w = v'$, as otherwise, by Proposition 2.6, we obtain an effective class of degree 0 (non-trivial since $w, v \neq y, u$), which contradicts the Fano assumption (I). ☐

Lemma 4.7. Assume (I)—(IX). Then x, y don't have opponents,

$$G(\Sigma) \subset P \cup \{x, y, z, u, u', v'\},$$

and we have a trichotomy: $v \in P$ or $v = z$ or $v = u'$.

Proof. We will only show that x' does not exist, and the argument for y' is symmetric. Suppose to the contrary that $x' \in G(\Sigma)$, and let $x + x' = \alpha$ be the corresponding primitive relation, so $\alpha \neq x, x'$. Then we can substitute $x = \alpha - x'$ into $r(Q)$ (VI) to get

$$\alpha + y + z = x' + x_i + x_j,$$

which shows $\alpha \notin P \cup \{y, z\}$, as otherwise we would get an effective curve class of degree 0. Substituting $x = \alpha - x'$ into $r(R)$ (IX) gives the relation

$$\alpha + y + u = v + x'. \tag{6}$$

Note that $\langle v, x'\rangle \in \Sigma$ by Corollary 2.22. We claim that Equation (6) is an extremal primitive relation by Lemma 2.26. Assume not, then Lemma 2.26 implies that one of $\{\alpha, y, u\}$ is in $\{v, x'\}$ and the remaining two vectors are opponents. Since $u \neq y'$, then either α, y are opponents and $u \in \{v, x'\}$, or α, u are opponents and $y \in \{v, x'\}$. From (IX), we have $u \neq v$ and $y \neq v$; (VII) means $u \neq x'$; and (VI) implies $y \neq x'$. So both cases are impossible.

So Equation (6) is an extremal primitive relation. In particular, $\alpha \neq y', u, u'$. By Proposition 2.8, v forms a cone with α, hence $\alpha \neq v'$, which contradicts Lemma 4.6. ☐

Lemma 4.8. Assume (I)—(IX). Then we have

$$G(\Sigma) \subset P \cup \{x, y, z, u, v'\}$$

and a dichotomy: $v \in P$ or $v = z$. If moreover u' exists, we have $u + u' = z$, $v = x_i$ and $u = x'_j$.

Proof. Assume that u' exists and let $u + u' = \alpha$ be the corresponding primitive relation. Then substituting it into $r(R)$ (IX) gives a relation

$$x + y + \alpha = u' + v. \tag{7}$$

By $r(R)$ (IX), $v \neq u$, so $\langle u', v\rangle \in \Sigma$. Since there are no x' and y', it follows from Lemma 2.26 that Equation (7) is an extremal primitive relation. It follows that $\alpha \notin \{u, u', x, y, v'\}$. Moreover, $\alpha \notin P$, as otherwise applying Proposition 2.8 to $r(Q)$ and $\tau = \langle\alpha\rangle$ would imply $\langle x, y, \alpha\rangle \in \Sigma$. So by Lemma 4.7, the only possibility is that $\alpha = z$. Therefore $u' + v = x_i + x_j$ by $r(Q)$ (VI), so we have, after possibly relabeling, that $v = x_i$, $u' = x_j$, which by Lemma 4.7 proves the statement.

If instead u' does not exist, the statement follows directly from Lemma 4.7. □

Lemma 4.9. Assume (I)—(IX). Then z doesn't have an opponent.

Proof. We argue by contradiction and assume that z' exists. Clearly $z' \neq x, y, z$ by $r(Q)$ (VI) and $z' \neq u$ by (VII). Furthermore, $z' \neq u'$, otherwise we have $z = u$ by Corollary 2.22, which contradicts (VII). Finally, $z' \notin P$, otherwise $z' = x_k$ implies $z = x_k'$, and $r(Q)$ (VI) becomes

$$x + y + x_k' = x_i + x_j,$$

but applying Proposition 2.8 to $r(Q)$ and $\tau = \langle x_k\rangle$, we get $\langle x_k, x_k'\rangle \in \Sigma$, a contradiction. Thus by Lemma 4.8, the only possibility is $z' = v'$, so $z = v$ by Corollary 2.22. By Lemma 4.8, this implies $G(\Sigma) \subset P \cup \{x, y, z, u, z'\}$.

Consider the primitive relation $z + z' = \beta \in G(\Sigma)$. We will show that $\beta = u$. Indeed, it is clear that $\beta \neq z, z'$. Combining $z + z' = \beta$ with $r(Q)$ (VI), we have

$$x + y + \beta = x_i + x_j + z'.$$

If $\beta \in \{x, y\}$, the left hand side is a cone, and we get a non-trivial effective relation of degree 0, which is impossible by (I). If $\beta \in P$, applying Proposition 2.8 to $r(Q)$ and $\tau = \langle\beta\rangle$ implies that $\langle x, y, \beta\rangle \in \Sigma$, and we obtain a contradiction as before.

But now, substituting $z + z' = u$ into $r(R)$ (IX) yields $x + y + z' = 0$, hence contradicting $m(X) \geq 3$ (II). □

Lemma 4.10. Assume (I)—(VII). Then $\rho(X) \leq 3$.

Proof. Let us summarize the consequences of (I)—(VII):

(VIII) $0 \in \mathrm{Conv}(\pi(x), \pi(y), \pi(u))$ after possibly relabeling x, y, z;
(IX) we have a primitive relation $r(R)$: $x + y + u = v$ (Lemma 4.5);

- x, y, z don't have opponents (Lemma 4.7, Lemma 4.9);
- $G(\Sigma) \subset P \cup \{x, y, z, u, v'\}$ (Lemma 4.8), so $\rho(X) \leq 4$;
- we have $v \in P$ or $v = z$ (Lemma 4.8), so we can consider two cases.

Case $v = z$. By Lemma 4.9, $v' = z'$ does not exist, hence it follows from Lemma 4.8 that $G(\Sigma) \subset P \cup \{x, y, z, u\}$, so $\rho(X) \leq 3$.

Case $v \in P$, say $v = x_l$. If $v = x_l$ doesn't have an opponent, then $G(\Sigma) \subset P \cup \{x, y, z, u\}$ and $\rho(X) \leq 3$. So the tricky case is when $v = x_l$ has an opponent $x_l' \notin P$.

Let $v + v' = \beta \in G(\Sigma)$. By $r(R)$, we have that $x + y + u + v' = v + v' = \beta$. Since $m(X) \geq 3$, $\beta \neq x, y, u, v'$. Hence $\beta \in P \cup \{z\}$. If $\beta \in P$, let $\beta = x_k$. Then $v' = x_k - x_l$ and $k \neq l$. Using $r(P)$, we obtain that $v' = 2x_k + \sum_{t \neq k,l} x_t$. As the vectors on the right hand side form a cone, we get an effective curve class of degree $2 - n$, which is impossible by (I).

So $\beta = z$. By $r(Q)$, we obtain that $x + y + v' = x_i + x_j - x_l$. If $l \neq i, j$, then again, by using $r(P)$ (III), we may write $x_i + x_j - x_l$ as $x_i + x_j + \sum_{t \neq l} x_t$. As the vectors on the right hand side form a cone, we obtain an effective curve class of degree $3 - n$, which is impossible by (I).

So $l = i$ or $l = j$. Up to symmetry, we may assume $l = i$, so $v = x_i$, $x_i + x'_i = z$. By $r(Q)$, we have that $x + y + x'_i = x_j$. Combining this with $r(R)$, we obtain that $x'_i + x_i = u + x_j$. Furthermore, $u \neq v' = x'_i$ and $u \neq x_i$. If u, x_j form a cone, we obtain an effective non-trivial curve class of degree 0, which is impossible by (I). So $u = x'_j$, $v = x_i$ and we have two primitive relations

$$x_i + x'_i = z \quad \text{and} \quad x_j + x'_j = z.$$

Then either $i = j$, in which case $\rho(X) \leq 3$, or $i \neq j$, and we notice that they are both degree 1 and hence extremal, so we can perform the contraction associated to one of them, say we contract the curve class $x_i + x'_i = z$:

$$X \longrightarrow Y,$$

$$V(z) \longrightarrow V(x_i, x'_i).$$

By Proposition 2.16,

$$\begin{aligned} r(P_Y)&\colon x_0 + \cdots + x_{n-2} = 0, \\ r(R_Y)&\colon x + y + x'_j = x_i, \\ r(Q')&\colon x + y + x'_i = x_j, \\ r(Q'')&\colon x_j + x'_j = x_i + x'_i \end{aligned}$$

are primitive relations in Y. Since $\rho(Y) = 3$, we apply Proposition 2.17. We adopt the same notation as in Proposition 2.17 and observe that we are in the case $l = 5$, hence

$$G(\Sigma) = \sqcup_{h=0}^{4} X_h = \{x_0, \ldots, \check{x}_j, \ldots, x_{n-2}\} \sqcup \{x_j\} \sqcup \{x, y\} \sqcup \{x'_j\} \sqcup \{x'_i\},$$

and either $X_2 \sqcup X_3 = \{x_0, \ldots, x_{n-2}\}$, or $\bar{c} = \bar{b} = \bar{0}$ and $X_4 \sqcup X_0 = \{x_0, \ldots, x_{n-2}\}$. However, all possibilities for $\{x_j\}$ lead to a contradiction. Indeed:

- if $X_3 = \{x_j\}$, then we have $r_3\colon\ \bar{1} \cdot X_3 + \bar{1} \cdot X_4 = x_j + x'_j = x_i + x'_i \neq \bar{1} \cdot X_h$ for all $h = 0, \ldots, 4$;
- if $X_2 = \{x_j\}$, then we have $r_1\colon\ \bar{1} \cdot X_1 + \bar{1} \cdot X_2 = x'_j + x_j = x_i + x'_i \neq \bar{1} \cdot X_h$ for all $h = 0, \ldots, 4$;
- if $X_4 = \{x_j\}$, then we have $r_3\colon\ \bar{1} \cdot X_3 + \bar{1} \cdot X_4 = x'_j + x_j = x_i + x'_i \neq \bar{1} \cdot X_h$ for all $h = 0, \ldots, 4$;
- if $X_0 = \{x_j\}$, then we have $r_0\colon\ \bar{1} \cdot X_0 + \bar{1} \cdot X_1 = x_j + x'_j = x_i + x'_i \neq \cancel{\bar{c} \cdot X_2} + (\cancel{\bar{b}} + \bar{1}) X_3 = \bar{1} \cdot X_3$.

This concludes the last case, and we get that $\rho(X) \leq 3$. $\square$

Proposition 4.11. Let X be a projective toric Fano manifold of $\dim(X) = n \geq 5$, $m(X) \geq 3$, which admits a primitive relation $r(P)\colon x_0 + x_1 + \cdots + x_{n-2} = 0$ (I)—(III). Let $x, y, z \in G(\Sigma) \setminus P$ be such that $0 \in \mathrm{Conv}(\pi(x), \pi(y), \pi(z))$ (IV)—(V). Assume in addition that $\{x, y, z\}$ is a primitive collection of degree 1 (VI). Then $\rho(X) \leq 3$.

Proof. We recall that assumptions (I)—(III) imply (IV)—(V). Then either $G(\Sigma) \subset P \cup \{x, y, z, x', y', z'\}$, in which case $\rho(X) \leq 2$ by Lemma 4.4, or there exists a vector $u \in G(\Sigma) \setminus (P \cup \{x, y, z, x', y', z'\})$ (VII), in which case Lemma 4.10 implies $\rho(X) \leq 3$, and we are done. □

4.1.2. *Degree two.* Still working in the setting (I)—(V), we assume that $\{x, y, z\}$ is a primitive collection whose primitive relation has degree 2, and by Proposition 4.11, we can exclude the case when degree one primitive collections as in (IV)—(VI) exist. In other words, these are the additional assumptions for this part of the proof:

(X) we have a primitive relation $r(Q)\colon\ x + y + z = v$ for some $v \in G(\Sigma)$;
(XI) there is no primitive collection $\{a, b, c\} \subset G(\Sigma) \setminus P$ with $0 \in \mathrm{Conv}(\pi(a), \pi(b), \pi(c))$ whose primitive relation has degree 1.

Lemma 4.12. In the setting (I)—(V) and (X)—(XI), we have

$$G(\Sigma) \subset P \cup \{x, y, z, x', y', z', v'\} \quad \text{and} \quad v \in P \cup \{x', y', z'\}.$$

Proof. If there is a generator $u \in G(\Sigma) \setminus (P \cup \{x, y, z, x', y', z'\})$, then by Remark 4.2 and (XI) we can assume that we have a primitive relation of the form

$$x + y + u = w.$$

Combining it with (X), we get $u + v = w + z$, so $u = v'$, otherwise $\langle u, v\rangle \in \Sigma$ and we get an effective non-trivial curve class of degree zero.

In particular, since $v \neq v'$, we have $v \in P \cup \{x', y', z'\}$. □

Lemma 4.13. In the setting (I)—(V) and (X)—(XI), assume that $v = x_m \in P$. Then x'_m does not exist.

Proof. The primitive relation $r(Q)$ (X) becomes $x + y + z = x_m$, and by Lemma 4.12, we have $G(\Sigma) \subset P \cup \{x, y, z, x', y', z', x'_m\}$.

Assume to the contrary that we have $x'_m \in G(\Sigma)$. Clearly, $x'_m \notin P$ because x_m forms a 2-dimensional cone in Σ with any other generator $x_i \in P$. By Remark 4.2, x'_m makes a primitive collection with two of x, y, z. Without loss of generality, assume we have a primitive relation of the form

$$x + y + x'_m = w.$$

Combining it with $r(Q)$ (X), we get $x'_m + x_m = z + w$, so $w = z'$ by Lemma 2.25. Let $\alpha = x_m + x'_m = z + z'$. Then $\alpha \notin P$ because $x'_m = \alpha - x_m$ is not in Span P. Now substituting $z = \alpha - z'$ into $r(Q)$ (X) yields

$$x + y + \alpha = x_m + z'. \tag{8}$$

Since $x_m \neq z$ by $r(Q)$ (X), we have $\langle x_m, z'\rangle \in \Sigma$, so Equation (8) is an extremal primitive relation. This implies that $\alpha \neq x, y, z, x', y', z', x'_m$, a contradiction with Lemma 4.12. □

Lemma 4.14. In the setting (I)—(V) and (X)—(XI), assume that $v = x_m \in P$. Then $\rho(X) \leq 3$.

Proof. By Lemma 4.12 and Lemma 4.13, we have $G(\Sigma) \subset P \cup \{x, y, z, x', y', z'\}$, and, as before, the primitive relation (X) is $r(Q)\colon\ x + y + z = x_m$.

We show that at most one of x', y', z' exists. Suppose to the contrary that for example $x', y' \in G(\Sigma)$, and let $\alpha = x + x'$, $\beta = y + y' \in G(\Sigma)$. Since

$$\alpha + y + z = x_m + x' \quad \text{and} \quad \langle x_m, x' \rangle \in \Sigma,$$

applying Lemma 2.26 shows that this is an extremal primitive relation: indeed, either α and y are opponents and $z \in \{x_m, x'\}$, or α and z are opponents and $y \in \{x_m, x'\}$. But $y, z \notin P \cup \{x'\}$, so this is a contradiction. It follows that $\alpha \notin \{x, x', y, y', z, z'\}$, so $\alpha = x_l \in P$; similarly, $\beta = x_k \in P$. But we have

$$\alpha + \beta + z = x' + y' + x_m,$$

which is not possible since we show that the right hand side forms a cone. Indeed, applying Proposition 2.8 to $x + x' = x_l$ and $\tau = \langle x_m, x_k \rangle$ (we use here that $n \geq 5$), we obtain $\langle x_m, x_k, x' \rangle \in \Sigma$, and applying then Proposition 2.8 to $y + y' = x_k$ and $\tau = \langle x_m, x' \rangle$ we have that $\langle x_m, x', y' \rangle \in \Sigma$. So, after possibly relabelling x, y, z, we have $G(\Sigma) \subset P \cup \{x, y, z, x'\}$, hence $\rho(X) \leq 3$. □

Lemma 4.15. In the setting (I)—(V) and (X)—(XI), assume that $v = x'$. Then $\rho(X) \leq 3$.

Proof. We have $r(Q)\colon\ x + y + z = x'$. We know $G(\Sigma) \subset P \cup \{x, y, z, x', y', z'\}$ by Lemma 4.12. So it is enough to show y' and z' do not exist. Suppose to the contrary that, for example, $y' \in G(\Sigma)$, and let $\beta = y + y'$. Then

$$\beta + x + z = x' + y' \quad \text{and} \quad \langle x', y' \rangle \in \Sigma,$$

so again, by Lemma 2.26, this is an extremal primitive relation. Indeed, assume the relation is not primitive. By Lemma 2.26, either β and x are opponents and $z \in \{x', y'\}$, or β and z are opponents and $x \in \{x', y'\}$. But $z, x \notin \{x', y'\}$, since $\{x, y, z\}$ is a primitive collection. Hence $\beta + x + z = y' + x'$ is a primitive relation. It follows from Proposition 2.8 that $\langle x, x' \rangle \in \Sigma$, which is a contradiction. □

Proposition 4.16. Let X be a projective toric Fano manifold of $\dim(X) = n \geq 5$, $m(X) \geq 3$, which admits a primitive relation $r(P)\colon x_0 + x_1 + \cdots + x_{n-2} = 0$ (I)—(III). Assume that any primitive collection $\{x, y, z\}$ such that $x, y, z \in G(\Sigma) \setminus P$ and $0 \in \mathrm{Conv}(\pi(x), \pi(y), \pi(z))$ (IV) —(V) has degree 2 (XI), and that there exists such a triple x, y, z (X). Then $\rho(X) \leq 3$.

Proof. The statement follows from Lemma 4.12, Lemma 4.14 and Lemma 4.15. □

4.2. **Second case: none of $\{x, y, z\}$ form a primitive collection.**

Proposition 4.17. Let X be a toric Fano manifold of $\dim(X) = n \geq 5$, $m(X) \geq 3$, which admits a primitive relation $r(P)\colon x_0 + x_1 + \cdots + x_{n-2} = 0$ (I)—(III). Assume in addition that none of the triples $\{x, y, z\} \subseteq G(\Sigma) \setminus P$ such that $0 \in \mathrm{Conv}(\pi(x), \pi(y), \pi(z))$ (IV)—(V) form a primitive collection. Then $\rho(X) \leq 3$.

Before proving Proposition 4.17, we will formulate the following useful lemma.

Lemma 4.18. In the setting of Proposition 4.17, take any triple $\{x, y, z\} \subseteq G(\Sigma) \setminus P$ with $\langle x, y \rangle \in \Sigma$. Assume that $0 \in \mathrm{Conv}(\pi(x), \pi(y), \pi(z))$, or equivalently, $\pi(z) \in \langle -\pi(x), -\pi(y) \rangle$. Then $z = x'$ or $z = y'$.

Proof. By Lemma 4.1, x, y, z do not span a cone, and by assumption, they do not form a primitive collection. So two of the vectors must not form a cone. Since we assumed that $\langle x, y \rangle \in \Sigma$, we must have $z = x'$ or $z = y'$. □

Proof of Proposition 4.17. We select $x, y \in G(\Sigma)$ with $\langle x, y\rangle \in \Sigma$ and such that the cone generated by $\pi(x)$ and $\pi(y)$ is maximal among cones in $\mathbb{Q}^2 \simeq \mathbb{Q}^n/\Gamma$ coming from such pairs. If there is $z \in G(\Sigma)$ such that $\pi(z)$ is outside the cone $\langle \pi(x), \pi(y)\rangle$, then we show that $z = x'$ or $z = y'$. Indeed, the case $\pi(z) \in \langle -\pi(x), -\pi(y)\rangle$ is covered by Lemma 4.18; $\pi(z) \in \langle \pi(x)\rangle$ or $\pi(z) \in \langle \pi(y)\rangle$ cannot happen by an argument similar to Lemma 4.1; and in the remaining case, $\pi(z)$ is in $\langle \pi(x), -\pi(y)\rangle$ or $\langle -\pi(x), \pi(y)\rangle$, then we use maximality of the cone generated by $\pi(x)$ and $\pi(y)$.

Let v be such that $\pi(v)$ is in the open half plane determined by $\operatorname{Span} \pi(x)$ not containing $\pi(y)$. Up to relabelling of x, y, we can assume that such a v exists. If $v = x'$, then $x + x' = y'$, since $\pi(x + x')$ is non-zero and is outside of $\langle \pi(y), \pi(x)\rangle$. So

$$\langle \pi(x), \pi(y)\rangle \subset \langle -\pi(y), -\pi(x')\rangle \cup \langle -\pi(x'), -\pi(y')\rangle \cup \langle -\pi(y'), -\pi(x)\rangle.$$

It follows from Lemma 4.18 that $G(\Sigma) = P \cup \{x, y, x', y'\}$, which concludes this case.

If now $v = y'$, in case $\pi(v) \in \langle -\pi(x), -\pi(y)\rangle$, we notice that $y + y' = x'$ as above, and in case $\pi(v) \in \langle \pi(x), -\pi(y)\rangle$, by completeness, we find w such that $\pi(w) \in \langle -\pi(x), \pi(y)\rangle \cup \langle -\pi(x), -\pi(y)\rangle$ and notice $w = x'$. In either case, we get

$$\mathbb{Q}^2 = \langle -\pi(x), -\pi(y)\rangle \cup \langle -\pi(y), -\pi(x')\rangle \cup \langle -\pi(x'), -\pi(y')\rangle \cup \langle -\pi(y'), -\pi(x)\rangle,$$

and now Lemma 4.18 gives $G(\Sigma) = P \cup \{x, y, x', y'\}$. □

5. Toric Fano manifolds with $m(X) = n - 2$

In this section, we classify all n-dimensional toric Fano manifolds X with $m(X) = n-2$ and $n \geq 6$ (Theorem 1.4). By Theorem 1.3, we know that $\rho(X) \leq 3$. Theorem 5.1 and Theorem 5.2 classify n-dimensional toric Fano manifolds X with $m(X) = n - 2$, $n \geq 5$, and Picard rank $\rho(X) = 2$ and 3, respectively. Together, these results yield a classification of toric Fano manifolds with $m(X) = n - 2$ and $n \geq 6$.

Theorem 5.1. Let X be a toric Fano manifold of dimension $n \geq 5$, $m(X) = n - 2$ and $\rho(X) = 2$. Then $X \simeq \mathbb{P}_{\mathbb{P}^2}(\mathcal{E})$, where $\mathcal{E}$ is one of the following vector bundles on $\mathbb{P}^2$:

- $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus n-2}$,
- $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus n-3}$,
- $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus n-2}$.

Proof. Let X be an n-dimensional toric Fano manifold with $m(X) = n - 2$ and $\rho(X) = 2$. We recall that toric manifolds with Picard rank 2 are classified by [Kle88]: they are projective space bundles over projective spaces. The assumption $m(X) = n - 2$ implies that X is a $\mathbb{P}^{n-2}$-bundle over $\mathbb{P}^2$.

We can write $X = \mathbb{P}(\mathcal{E})$, where $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(a_0) \oplus \mathcal{O}_{\mathbb{P}^2}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^2}(a_{n-2})$ and $a_0 \geq a_1 \geq \cdots \geq a_{n-2} = 0$. The Fano assumption on X is equivalent to saying that $\sum_{i=0}^{n-2} a_i \leq 2$. Thus we have the following cases:

(1) $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}^{\oplus n-1}$ and $X \simeq \mathbb{P}^{n-2} \times \mathbb{P}^2$,
(2) $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus n-2}$,
(3) $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus n-3}$,
(4) $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus n-2}$.

In Case (1), we have $m(X) = 2$. Cases (2)—(4) provide the complete list of toric Fano manifolds of Picard rank 2 and $m(X) = n - 2$. □

Theorem 5.2. Let X be a toric Fano manifold of dimension $n \geq 5$, $m(X) = n-2$ and $\rho(X) = 3$. Then one of the following holds:

(1) $X = \mathbb{P}_S(\mathcal{E})$ is a $\mathbb{P}^{n-2}$-bundle over a toric surface S, where $(S, \mathcal{E})$ is one of the following:
- $S = \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1) \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}^{\oplus n-2}$,
- $S = \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,0) \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,1) \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}^{\oplus n-3}$,
- $S = \mathbb{F}_1$ and $\mathcal{E} = \mathcal{O}_{\mathbb{F}_1}(e+f) \oplus \mathcal{O}_{\mathbb{F}_1}^{\oplus n-2}$, where $e \subset \mathbb{F}_1$ is the -1-curve, and $f \subset \mathbb{F}_1$ is a fiber of $\mathbb{F}_1 \to \mathbb{P}^1$.

(2) Let $Y \simeq \mathbb{P}_{\mathbb{P}^2}\big(\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus n-2}\big)$ be the blowup of $\mathbb{P}^n$ along a linear subspace $L = \mathbb{P}^{n-3}$, and denote by $E \subset Y$ the exceptional divisor. Then X is the blowup of Y along a codimension 2 center $Z \subset Y$, where:
- Z is the intersection of E with the strict transform of a hyperplane of $\mathbb{P}^n$ containing the linear subspace L, or
- Z is the intersection of the strict transforms of two hyperplanes of $\mathbb{P}^n$, one containing the linear subspace L, and the other one not containing it.

Proof. We apply Batyrev's classification of toric manifolds with $\rho(X) = 3$, stated in Proposition 2.17. We adopt the same notation as in Proposition 2.17, and treat separately the cases when the number of primitive collections is $l = 3$ and $l = 5$.

Case $l = 3$. As $G(\Sigma) = X_0 \sqcup X_1 \sqcup X_2$ and $m(X) = n-2$, we have $X_0 = \{x_0, \dots, x_{n-2}\}$, $X_1 = \{v_1, v_2\}$ and $X_2 = \{z_1, z_2\}$. The corresponding primitive relations are all extremal by Proposition 2.17. By Proposition 2.13, X is a $\mathbb{P}^{n-2}$-bundle over a surface S. Up to relabelling in X_0, X_1 and X_2, three possible choices for the remaining two primitive relations are (noting that $m(X) = n-2 > 1$):

$$\begin{cases} v_1 + v_2 = x_0, \\ z_1 + z_2 = v_1; \end{cases} \qquad \begin{cases} v_1 + v_2 = x_0, \\ z_1 + z_2 = x_0; \end{cases} \qquad \begin{cases} v_1 + v_2 = x_0, \\ z_1 + z_2 = x_1. \end{cases}$$

It follows that S is isomorphic to $\mathbb{F}_1$ when $r(X_1)$ and $r(X_2)$ are as in the first column above, or to $\mathbb{P}^1 \times \mathbb{P}^1$ in two other cases.

Case $l = 5$. We denote by $l_i = |X_i| \geq 1$ the cardinality of X_i.

If $(\overline{c}, \overline{b}) = (\overline{0}, \overline{0})$, then both $r_2\colon \overline{1} \cdot X_2 + \overline{1} \cdot X_3 = 0$ and $r_4\colon \overline{1} \cdot X_4 + \overline{1} \cdot X_0 = 0$ are centrally symmetric primitive relations. By the assumption $m(X) = n-2$, we have $2n-2 \leq l_2 + l_3 + l_4 + l_0 \leq |G(\Sigma)| - 1 = n+2$, which implies $n \leq 4$, a contradiction.

So we have $(\overline{c}, \overline{b}) \neq (\overline{0}, \overline{0})$, and $P \coloneqq X_2 \sqcup X_3$ is the only centrally symmetric primitive collection, so $l_2 + l_3 = n-1$ and $l_0 + l_1 + l_4 = 4$, with $l_0, l_1, l_4 \in \{1, 2\}$. As $\deg(r_0) > 0$, we get the inequality

$$3 \geq l_0 + l_1 > \sum c_i + \sum b_j + l_3,$$

which is only satisfied when $l_3 = 1$, $l_0 + l_1 = 3$ and exactly one entry in

$$(c_1, c_2, \dots, c_{l_2}, b_1)$$

equals one, while the others are all zero. Up to relabelling, there are two cases: $c_1 = 1$ or $b_1 = 1$.

We then have $l_4 = 4 - (l_0 + l_1) = 1$. From $\deg(r_3) > 0$, we get

$$l_3 + l_4 > l_1,$$

which means $2 > l_1$, and this ensures $l_1 = 1$, hence $(l_0, l_1, l_4) = (2, 1, 1)$.

To sum up, we denote $X_0 = \{v_1, v_2\}$, $X_1 = \{y\}$, $X_2 = \{x_1, \dots, x_{n-2}\}$, $X_3 = \{x_0\}$ and $X_4 = \{z\}$, and get the following two possibilities for X:

- $b_1 = 1$ and $c_j = 0$ for every j:

$$\begin{aligned} r_0\colon &\quad v_1 + v_2 + y = 2x_0, \\ r_1\colon &\quad y + x_1 + \cdots + x_{n-2} = z, \\ r_2\colon &\quad x_0 + \cdots + x_{n-2} = 0, \\ r_3\colon &\quad x_0 + z = y, \\ r_4\colon &\quad z + v_1 + v_2 = x_0. \end{aligned}$$

- $c_1 = 1$, $b_1 = 0$ and $c_j = 0$ for every $j > 1$:

$$\begin{aligned} r_0\colon &\quad v_1 + v_2 + y = x_1 + x_0, \\ r_1\colon &\quad y + x_1 + \cdots + x_{n-2} = z, \\ r_2\colon &\quad x_0 + \cdots + x_{n-2} = 0, \\ r_3\colon &\quad x_0 + z = y, \\ r_4\colon &\quad z + v_1 + v_2 = x_1. \end{aligned}$$

By Proposition 2.13, the open $U = X \setminus \big(V(y) \cup V(z)\big)$ has a $\mathbb{P}^{n-2}$-bundle structure. The relation r_3 corresponds to an extremal curve class by Proposition 2.20, which induces a smooth blow-down $h\colon X \to Y$. By Proposition 2.16, the primitive relations in Y in cases $b_1 = 1$ and $c_1 = 1$ are, respectively:

$$\begin{cases} x_0 + \cdots + x_{n-2} = 0, \\ z + v_1 + v_2 = x_0; \end{cases} \qquad\qquad \begin{cases} x_0 + \cdots + x_{n-2} = 0, \\ z + v_1 + v_2 = x_1. \end{cases}$$

It follows that $Y \simeq \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus n-2}\right)$, i.e., Y is the blowup of $\mathbb{P}^n$ along a linear subspace $L = V(z, v_1, v_2) \simeq \mathbb{P}^{n-3} \subset \mathbb{P}^n$, with exceptional divisor $E = V(x_0) \subset Y$ in the first case, and $E = V(x_1) \subset Y$ in the second case. The center of the blowup $h\colon X \to Y$ is $V(x_0, z) \subset Y$, yielding the two varieties described in (2). $\square$

Appendix A. Code for computing primitive collections

by Will Reynolds

Let Σ be the fan of a toric manifold of dimension n. For each integer $k \in \{1, \dots, n\}$, we denote by $\Sigma(k)$ the subset of Σ consisting of the k-dimensional cones. To determine whether a given subset $P \subseteq G(\Sigma)$ is a primitive collection, we proceed in two steps. First make sure there does not exist $\sigma \in \Sigma(n)$ with $P \subseteq G(\sigma)$, and if there does, stop; otherwise, for each $v \in P$, make sure there exist $\sigma \in \Sigma(n)$ with $P \setminus \{v\} \subseteq G(\sigma)$. In the special case $|P| = 2$ it suffices to check that there does not exist $\sigma \in \Sigma(n)$ with $P \subseteq G(\sigma)$.

Note that, by definition, if P is a primitive collection and $P \subseteq Q$, then Q is not a primitive collection. Therefore, when looking for primitive collections, we go through subsets of $G(\Sigma)$ in increasing cardinality.

Assuming an implementation of the above basic algorithm, a reasonably efficient way to list all of the primitive collections of a fan is to arrive at such a list by eliminating $P \subseteq G(\Sigma)$

which are not primitive collections. The first step is to remove any P with $|P| = 1$. Then for each P we check whether it is a primitive collection. If it is, we keep it and remove all sets containing it. If it is not, we remove it. One way of implementing this method of listing primitive collections is implemented in pseudocode in Algorithm 1.

This algorithm is impractical if $G(\Sigma)$ is too large. In practice, on a modern laptop, it works reasonably well up to about $|G(\Sigma)| = 17$, partly because of the combination of the following factors: first, eliminating all of the supersets of any P with $|P| = 2$ cuts down the remaining search space significantly, and second, the relative abundance of primitive collections of size 2, at least among toric Fano varieties. For example, the 124 toric Fano 4-folds altogether have 785 primitive collections, of which 566 have cardinality 2.

This last factor makes the computation of the value of $m(X)$ for a given toric Fano variety X easier as well. Of the toric Fano varieties of a given dimension n (for $n \leq 6$) those X with $m(X) = 1$ make up an overwhelming majority. This means that computing $m(X)$ is usually extremely fast, even in the most straightforward way. The following table summarizes the data for $\dim(X) \in \{4, 5, 6\}$.

$\dim(X)$	# Fanos	#(m=1)	#(m=2)	#(m=3)	#(m=4)	#(m=5)	#(m=6)
4	124	107	15	1	1		
5	866	744	112	8	1	1	
6	7622	6333	1174	105	8	1	1

Algorithm 1: List primitive collections of a given fan

Input: fan Σ;
$n \leftarrow \dim \Sigma$;
$PC \leftarrow \{P \subseteq G(\Sigma) : |P| > 1\}$;
for $P \in PC$ *satisfying* $|P| = 2$ **do**
    if *there exists* $\sigma \in \Sigma(n)$ *such that* $P \subseteq G(\sigma)$ **then**
        $PC \leftarrow PC \setminus \{P\}$;
    else
        $PC \leftarrow PC \setminus \{Q \in G(\Sigma) : P \subsetneq Q\}$;
    end
end
for $i \in \{3, \ldots, n\}$ **do**
    for $P \in PC$ *satisfying* $|P| = i$ **do**
        if *there exists* $\sigma \in \Sigma(n)$ *such that* $P \subseteq G(\sigma)$ **then**
            $PC \leftarrow PC \setminus \{P\}$;
        else
            $b \leftarrow$ **True**;
            for $v \in P$ **do**
                if *there does not exist* $\sigma \in \Sigma(n)$ *with* $P \setminus \{v\} \subseteq G(\sigma)$ **then**
                    $b \leftarrow$ **False**;
                end
            end
            if b **then**
                $PC \leftarrow PC \setminus \{Q \subseteq G(\Sigma) : P \subsetneq Q\}$;
            else
                $PC \leftarrow PC \setminus \{P\}$;
            end
        end
    end
end
Output: PC;

For convenience, we also provide Macaulay2 code implementing the algorithm for computing primitive collections.

```
coneExistenceCheck = (S, fan) -> (
    for cone in fan do (
        if isSubset(S, cone) then (
            return true;
        );
    );
    return false;
);
```

```
properSubsetCheck = (S, fan) -> (
    for ray in S do (
        if coneExistenceCheck(S-set{ray}, fan) == false then (
            return false;
        );
    );
    return true;
);

isPrimitiveCollection = (P, Var) -> (
    if coneExistenceCheck(P, orbits(Var, 0)) then (
        return false)
    else (
        return properSubsetCheck(P, orbits(Var, 0)
    );
);

supsetsOfPrimColl = (E, B) -> (
    return set{for P in E-set{B} when isSubset(B, P) list P};
);

primitiveCollections = (Var) -> (
    n = length rays Var;
    primColls = select(subsets(toList(0..n-1)), x -> length x > 1);
    for P in subsets(toList(0..n-1), 2) do (
        if coneExistenceCheck(P, orbits(Var, 0)) == false then (
            primColls = primColls - supsetsOfPrimColl(primColls, P);)
        else (
            primColls = primColls - set{P};
        );
    );
    for i in toList(3..n) do (
        for P in subsets(toList(0..n-1), i) do (
            if member(P, primColls) == false then continue;
            if isPrimitiveCollection(P, Var) then (
                primColls = primColls - supsetsOfPrimColl(primColls, P);
            ) else (
                primColls = primColls - set{P};
            );
        );
    );
    return sort primColls;
);
```

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