Math 310 **Final, Fall 2007, Due Dec. 18**

You can quote all the theorems proven in class, homework and midterm problems to solve the problems below.

(1a) Show that for any two *real* numbers x and y with $x < y$, there are two *rational* numbers r and s such that $x < r < s < y$. (5 points).
(1b) We have shown in class that there are uncountably many real numbers in the open interval $(0, 1)$. Show that there are uncountably many *irrational numbers* in the interval $(0, 1)$. (5 points)
(1c) For any two *rational* numbers r and s with $r < s$, show that there are uncountably many *irrational* numbers in the open interval (r, s). (10 points)
(Hint: (1b) is useful.)
(1d) Show that for any two real numbers x and y with $x < y$, there are uncountably many *irrational* numbers in the interval (x, y). (5 points)

(2a) Prove that a rational number r with $0 < r < 1$ is either a finite decimal of the form

$$0.a_1a_2a_3\cdots a_k$$

for some k, or an infinite periodical decimal of the form

$$0.a_1a_2\cdots a_sb_1b_2\cdots b_tb_1b_2\cdots b_tb_1b_2\cdots b_t\cdots,$$

where the b-digits are periodic. (10 points)
(Hint: It is not as dreadful as it looks. What does the Euclidean algorithm say about the remainder? Check online about something called the *pigeonhole principle*.)
(2b) Conversely, a finite decimal or an infinite periodical decimal is a rational number. Do not give the general proof. You are only asked to express the periodical

$$0.034123123123123\cdots,$$

where the digits 123 repeat, as a fraction q/p in lowest terms, where q and p are natural numbers. (5 points)

(3a) Euler's theorem says that $a^{\phi(m)} \equiv 1(\mod m)$ if $\gcd(a, m) = 1$. Let s be the smallest natural number for which $a^s \equiv 1(\mod m)$. prove that s divides $\phi(m)$. (5 points)
(3b) Give an example where $\phi(m)$ is the smallest s. (5 points)

(4) Consider the sequence

$$1, -1/2, -1/2^2, 1/2^3, -1/2^4, -1/2^5, 1/2^6, -1/2^7, -1/2^8, 1/2^9, \cdots,$$

where the n^{th} term is $a_n = \pm 1/2^n, n = 0, 1, 2, 3, \cdots$, with one positive sign followed by two negative signs. Define a new sequence $s_1, s_2, s_3, \cdots$, where

$$s_n = a_0 + a_1 + a_2 + \cdots + a_n.$$

Show that $\lim_{n\to\infty} s_n$ exists. (10 points)

(Hint: Show that the sequence (s_n) is Cauchy.)