(1) Prove:
   (i): $mn = nm$ for all natural numbers $m$ and $n$. (10 points)
   (ii): $(mn)p = m(np)$ for all natural numbers $m, n,$ and $p$. (10 points)

(2) Prove that for every natural number $n$, no natural number $m$ exists such that $n < m$ and $m < n + 1$. (10 points)

(3) Let $N$ be the set of natural numbers and let $S = N \times N$. Define a relation $\sim$ on $S$ by setting $(x, y) \sim (u, v)$ when $xv(y + u) = yu(x + v)$.
   (i): Show that $\sim$ is an equivalence relation. (10 points)
      (Hint: The transitivity of $\sim$ is harder to handle. Observe, however, that $xv(y + u) = yu(x + v)$ is essentially

      \[
      \frac{1}{u} + \frac{1}{y} = \frac{1}{v} + \frac{1}{x},
      \]

      provided we know rational numbers; transitivity is easy to see in this setup, though it is not allowed at this point of our course.
      Can you make sense out of this in terms of natural numbers?)
   (ii): List ten elements of the equivalence set of which $(1, 1)$ is a representative. (5 points)

(4)
   (i): Prove the commutative law for $+$ in $\mathbb{Z}$. (5 points)
   (ii): Prove the commutative law for $\cdot$ in $\mathbb{Z}$. (5 points)

(5)
   (i): Prove the associative law for $+$ in $\mathbb{Z}$. (5 points)
   (ii): Prove the associative law for $\cdot$ in $\mathbb{Z}$. (5 points)

(6) Prove the distributive law in $\mathbb{Z}$. That is, show that for all integers $a, b, c \in \mathbb{Z}$, we have $a \cdot (b + c) = a \cdot b + a \cdot c$. (10 points)