Math 310 Midterm, Fall 2008, Due Nov. 7

(1a) Use the mathematical induction to show that for all \( n \in \mathbb{N} \),
\[
1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2}.
\]
(5 points)

(1b) Use \((n + 1)^3 - n^3 = 3n^2 + 3n + 1\) and (1a) to show
\[
1 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n + 1)(2n + 1)}{6}
\]
(5 points)

(1c) Use the method in (1b) to find
\[
1 + 2^3 + 3^3 + \cdots + n^3.
\]
(5 points)

(Remark: Of course, (1a) can also be done by the method of (1b) by quoting the identity \((n + 1)^2 - n^2 = 2n + 1\). However, you must use the mathematical induction to solve (1a).)

(2a) Use the Euclidean division algorithm to show that for any two natural numbers \( m \) and \( n \), there is a natural number \( q \) such that \( mq > n \). (10 points)

(Hint: Divide \( n \) by \( m \).)

(2b) Conclude that for any two positive rational numbers \( r \) and \( s \), there is a natural number \( n \) such that \( nr > s \). (5 points)

(Remark: (2b) is the reason why you can pick the first natural number \( N \geq \) a given positive rational number, say, \( 10/\epsilon \), in all of our \( \epsilon - N \) proofs.)

(3) Show that the definition of the product of two real numbers is well-defined independent of the representatives chosen. (10 points)

(4a) Let \( I_n = \{ k \in \mathbb{N} : k \leq n \} \). Suppose \( f : I_n \to \mathbb{N} \) is a one-to-one function, that is \( f(x) = f(y) \) implies \( x = y \). Let \( Im(f) \) be the image of \( f \) in \( \mathbb{N} \). Use the mathematical induction to show that there is another one-to-one function \( g : I_n \to Im(f) \) such that \( g \) is an onto map, that is, each element in \( Im(f) \) is the image of some element in \( I_n \) via \( g \), and moreover \( g(x) < g(y) \) if \( x < y \). (10 points)

(Remark: In spite of its dry statement, the problem simply says we can reshuffle the image elements so that the new map is order-preserving. For instance, if
\[
f : 1 \mapsto 10, \quad 2 \mapsto 28, \quad 3 \mapsto 6,
\]
then
\[
g : 1 \mapsto 6, \quad 2 \mapsto 10, \quad 3 \mapsto 28.
\]
Here, \( I_n = \{1, 2, 3\} \) and \( Im(f) = \{6, 10, 28\}. \)
(4b) Conclude that if $m < n$ then a function $f : I_n \to I_m$ cannot be one-to-one. (5 points)

(Remark: This is the Pigeon Hole Principle that says that if the number of pigeons is greater than the number of pigeon holes, then there must be a pigeon hole with at least two pigeons.)

(5) Show that between any two real numbers $x$ and $y$ satisfying $x + 1 < y$, there is an integer $n$ such that $x < n < y$. (10 points)

(Hint: Without loss of generality, we may assume $0 \leq x \leq y$ (why?). Consider the nonempty set of natural numbers $\geq y$ (why is this set nonempty?). Apply the well-ordering principle to this set.)

(6) Show that $n^n/n!$ is not Cauchy. (10 points)