Math 310 Class Notes 1

We are given a set $\mathbb{N}$ whose elements are called *natural numbers*. $\mathbb{N}$ is also equipped with an equivalence relation called *equality*, denoted by $=$, in the sense that

(I): For all $x \in \mathbb{N}$, we have $x = x$.
(II): For all $x, y \in \mathbb{N}$, if $x = y$, then we have $y = x$.
(III): For all $x, y, z \in \mathbb{N}$, if $x = y$ and $y = z$, then we have $x = z$.

Remark 1. *A good real-life example of an equivalence relation $\sim$ is $a \sim b$ if a and b share the same last name. Here, the underlying set is the collection of all human beings. A relation $\mathcal{R}$ that is* **NOT** *an equivalence relation is $a\mathcal{R}b$ if a is the father of b.*

In addition to $=$, the set $\mathbb{N}$ satisfies five axioms of Peano as follows.

Peano's Axioms

(1): $1 \in \mathbb{N}$.
(2): For any $n \in \mathbb{N}$, there is one and only one natural number, denoted by n', called the *successor* of n.
(3): If $m, n \in \mathbb{N}$ such that $m' = n'$, then $m = n$. (No two different natural numbers have the same successor.)
(4): For any $m \in \mathbb{N}$, we have $m' \neq 1$. (No natural number has 1 as a successor.)
(5): Let $S \subset \mathbb{N}$ have the following properties:
  (a): $1 \in S$.
  (b): $k \in S$ implies $k' \in S$.
Then we can conclude that $S = \mathbb{N}$. (This is Mathematical Induction.)

Next, we introduce the notion of addition.

Definition 2. *For all $n \in \mathbb{N}$, define*

$$1 + n = n'.$$

Moreover, if $m + n$ is defined for m and all $n \in \mathbb{N}$, define

$$m' + n = (m + n)'$$

for m' and all n.

Theorem 3. *$+$ in Definition 2 can be extended into one and only one binary operation, also denoted by $+$, which assigns m, n to another natural number $m + n$ satisfying the following two properties:*

(A): $m' = m + 1$.
(B): $(m + n)' = m + n'$.

Proof. **Existence of +.**
Step 1. By Definition 2,

$$1 + n = n' \tag{1}$$

for all n.
Step 2. Let $S \subset \mathbb{N}$ be the set of all m such that + can be defined for m and all n satisfying (A) and (B).
Step 3. We show $m = 1 \in \mathbb{N}$. That is, from (1) we must establish (A) and (B). Now, setting $n = 1$ in (1), we obtain

$$1 + 1 = 1' \tag{2}$$

It follows that, with $m = 1$,

$$\begin{aligned} m' &= 1' \qquad (m = 1) \\ &= 1 + 1 \qquad (\text{by } (2) \\ &= m + 1 \qquad (m = 1) \end{aligned}$$

so that (A) holds true for $m = 1$. To establish (B) for $m = 1$, observe that

$$\begin{aligned} (m + n)' &= (1 + n)' \qquad (m = 1) \\ &= (n')' \qquad (\text{by } (1)) \\ &= 1 + n' \qquad (\text{by } (1), \text{ replacing } n \text{ by } n' \text{ in it}) \\ &= m + n' \qquad (m = 1) \end{aligned}$$

Hence (B) holds true for $m = 1$. Therefore, $1 \in S$.
Step 4. We show that $m \in S$ implies $m' \in S$. To this end, observe that $m \in S$ means $m + n$ is already defined with (A) and (B) true for all n, i.e., with

$$\begin{aligned} m' &= m + 1 \\ (m + n)' &= m + n' \end{aligned} \tag{3}$$

To show that $m' \in S$, observe next that by Definition 2, we have

$$m' + n = (m + n)' \tag{4}$$

for all n. All we need to do to show $m' \in S$ is to ensure that (A) and (B) remain true when we replace m by m' in them. In other words, we want to show

$$\begin{aligned} (m')' &= m' + 1 \\ (m' + n)' &= m' + n' \end{aligned} \tag{5}$$

However,

$$\begin{aligned} m' + 1 &= (m+1)' \qquad \text{(by (4) with } n = 1\text{)} \\ &= (m')' \qquad \text{(by the first equation of (3))} \end{aligned}$$

Hence the first equation of (5) is true. On the other hand,

$$\begin{aligned} (m' + n)' &= ((m+n)')' \qquad \text{(by (4))} \\ &= (m+n')' \qquad \text{(by the second equation of (3))} \\ &= m' + n' \qquad \text{by (4) with } n \text{ replaced by } n' \end{aligned}$$

Hence, the second equation of (5) is verified.

Step 5 Steps 3 and 4 fulfill the fifth axiom of Peano. So, we conclude that $S = \mathbb{N}$.

In summary, we have shown that we can define $+$ for any pair m, n of natural numbers.

Uniqueness of +.

Suppose there is another binary operation $\oplus$ such that $\oplus$ is defined for any pair m, n of natural numbers satisfying (A) and (B) in Theorem 3. Fix m. Let S be the set of all n such that

$$m + n = m \oplus n$$

Now $n = 1 \in S$; this is because by (A)

$$m + 1 = m' = m \oplus 1$$

We next show that $n \in S$ implies $n' \in S$. Now, $n \in S$ means

$$m + n = m \oplus n \tag{6}$$

Therefore,

$$\begin{aligned} m + n' &= (m+n)' \qquad \text{(by (B))} \\ &= (m \oplus n)' \qquad \text{(by (6))} \\ &= m \oplus n' \qquad \text{(by (B))} \end{aligned}$$

Therefore, $n' \in S$. Thus, the mathematical induction implies that $S = \mathbb{N}$. In other words, we have shown that

$$m + n = m \oplus n$$

for any pair $m, n \in \mathbb{N}$. That is, there is only one such binary operation.

□