TAUT SUBMANIFOLDS ARE ALGEBRAIC

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ABSTRACT. We prove that every (compact) taut submanifold in Euclidean space is real algebraic, i.e., is a connected component of a real irreducible algebraic variety in the same ambient space. This answers affirmatively a question of Nicolaas Kuiper raised in the 1980s.

1. INTRODUCTION

An embedding f of a compact, connected manifold M into Euclidean space $\mathbb{R}^n$ is *taut* if every nondegenerate (Morse) Euclidean distance function,

$$L_p : M \to \mathbb{R}, \quad L_p(z) = d(f(z), p)^2, \quad p \in \mathbb{R}^n,$$

has $\beta(M, \mathbb{Z}_2)$ critical points on M, where $\beta(M, \mathbb{Z}_2)$ is the sum of the $\mathbb{Z}_2$-Betti numbers of M. That is, L_p is a perfect Morse function on M.

A slight variation of Kuiper's observation in [7] gives that tautness can be rephrased by the property that

$$H_j(M \cap B, \mathbb{Z}_2) \to H_j(M, \mathbb{Z}_2) \tag{1.1}$$

is injective for all closed disks $B \subset \mathbb{R}^n$ and all $0 \leq j \leq \dim(M)$. As a result, tautness is a conformal invariant, so that via stereographic projection we can reformulate the notion of tautness in the sphere S^n using the spherical distance functions. Another immediate consequence is that if $B_1 \subset B_2$, then

$$H_j(M \cap B_1) \to H_j(M \cap B_2) \tag{1.2}$$

is injective for all j.

Kuiper in [8] raised the question whether all taut submanifolds in $\mathbb{R}^n$ are real algebraic. We established in [4] that a taut submanifold in $\mathbb{R}^n$ is real algebraic in the sense that, it is a connected component of a real irreducible algebraic variety in the same ambient space, provided the submanifold is of dimension no greater than 4.

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In this paper, we prove that all taut submanifolds in $\mathbb{R}^n$ are real algebraic in the above sense, so that each is a connected component of a real irreducible algebraic variety in the same ambient space. In particular, any taut hypersurface in $\mathbb{R}^n$ is described as $p(t) = 0$ by a single irreducible polynomial $p(t)$ over $\mathbb{R}^n$. Moreover, since a tube with a small radius of a taut submanifold in $\mathbb{R}^n$ is a taut hypersurface [12], which recovers the taut submanifold along its normals, understanding a taut submanifold, in principle, comes down to understanding the hypersurface case defined by a single algebraic equation.

It is more convenient to prove that a taut submanifold in the sphere is real algebraic, though occasionally we will switch back to Euclidean space when it is more convenient for the argument. Since a spherical distance function $d_p(q) = \cos^{-1}(p \cdot q)$ has the same critical points as the Euclidean height function $\ell_p(q) = p \cdot q$, for $p, q \in S^n$, a compact submanifold $M \subset S^n$ is taut if and only if it is *tight*, i.e., every nondegenerate height function ℓ_p has the total Betti number $\beta(M, \mathbb{Z}_2)$ of critical points on M. We will use both d_p and ℓ_p interchangeably, whichever is more convenient for our argument.

Our proof is based on the *local finiteness property* [4, Definition 7] that played the decisive role for a taut submanifold to be algebraic when its dimension is ≤ 4. This property is parallel in spirit to the Riemann extension theorem in complex variables. Namely, let $\mathcal{G}$ (i.e., the first letter of the word "good") be the subset of a taut hypersurface M where the principal multiplicities are locally constant, and let $\mathcal{G}^c$ be its complement in M. Let $S \subset \mathcal{G}^c$ be closed in M; S is typically a set difficult to manage. If $M \setminus S$ is well-behaved to have finitely many connected components, and moreover, each $x \in \mathcal{G}^c \setminus S$ has a small neighborhood U in M for which $\mathcal{G} \cap U$ is also well-behaved to have finitely many connected components, then M is algebraic.

As is pointed out and manifested in [15], focal sets play an important role in the study of submanifolds. In general, however, they are nonsmooth, where, for instance, the best one can expect of that of a hypersurface in S^n is that it is at least of Hausdorff codimension 2 [4]. One thus expects that the (unit) tangent cones of a focal set can be a useful tool for understanding such nonsmooth objects, though in general the tangent cones of a focal set themselves are also rather untamed. In the case of taut submanifolds, nonetheless, we can tame a focal set when we use the mathematical induction on the dimension of the ambient sphere S^n for which all taut submanifolds are algebraic. Since a curvature surface Z, which is taut by Ozawa theorem [10], lies in a curvature sphere, the induction hypothesis implies that Z is algebraic, which leads us to the free access of parametrizing the focal set of Z

and its unit tangent cone at a point by semialgebraic sets. (For the reader's convenience, we include a section on semialgebraic sets and some of their important properties.) From this we can show, by conducting certain dimension estimates, facilitated and made precise by the introduction of unit tangent cones, that barring a closed set $S \subset \mathcal{G}^c$ of Hausdorff $(\dim(M) - 1)$-measure zero that, hence by [14], does not disconnect the taut hyperusrface $M \subset S^n$, the set $\mathcal{G}^c$ is essentially a manifold of codimension 1 in M, as expected. The local finiteness property then results in the hypersurface case, whence follows the algebraicity of a taut submanifold.

2. PRELIMINARIES

2.1. **The Ozawa theorem.** A fundamental result on taut submanifolds is due to Ozawa [10] (see also [16] for its generalization to the Riemannian case).

Theorem 1 (Ozawa)**.** *Let M be a taut submanifold in S^n, and let $\ell_p, p \in S^n$, be a linear height function on M. Let $x \in M$ be a critical point of ℓ_p, and let Z be the connected component of the critical set of ℓ_p that contains x. Then Z is*
(a) *a smooth compact manifold of dimension equal to the nullity of the Hessian of ℓ_p at x;*
(b) *nondegenerate as a critical manifold;*
(c) *taut in S^n.*

In particular, ℓ_p is perfect Morse-Bott [3]. We call such a connected component of a critical set of ℓ_p a *critical submanifold* of ℓ_p.

An important consequence of Ozawa's theorem is the following [5].

Corollary 2. *Let M be a taut submanifold in S^n. Then given any principal space T of any shape operator S_ζ at any point $x \in M$, there exists a submanifold Z (called a curvature surface) through x whose tangent space at x is T. That is, M is Dupin* [11], [12].

Let us remark on a few important points in the corollary. It is convenient to work in the ambient Euclidean space $\mathbb{R}^n$. Let μ be the principal value associated with T. Consider the focal point $p = x+\zeta/\mu$. Then the critical submanifold Z of the (Euclidean) distance function L_p through x is exactly the desired curvature surface through x. The unit vector field

$$\zeta(y) := \mu(p - y) \tag{2.1}$$

for $y \in Z$ extends ζ at x and is normal to and parallel along Z. The $(n-1)$-sphere of radius $1/\mu$ centered at p is called the *curvature sphere* of Z. In particular, two different focal points cannot have the same critical submanifold.

2.2. **A brief review on semialgebraic sets.** A *semialgebraic subset* of $\mathbb{R}^n$ is one which is a finite union of sets of the form

$$\cap_j \{x \in \mathbb{R}^n : F_j(x) * 0\},$$

where * is either $<$ or $=$, $F_j \in \mathbb{R}[X_1, \ldots, X_n]$, the polynomial ring in $X_1, \cdots, X_n$, and the intersection is finite. An *algebraic subset* is one when $*$ is $=$ for all j without taking the finite union operation. Clearly, an algebraic set is semialgebraic.

It follows from the definition that any finite union or intersection of semialgebraic sets is semialgebraic, the complement of a semialgebraic set is semialgebraic, and hence a semialgebraic set taking away another semialgebraic set leaves a semialgebraic set. Moreover, the projection $\pi : \mathbb{R}^n \to \mathbb{R}^k$ sending $x \in \mathbb{R}^n$ to its first k coordinates maps a semialgebraic set to a semialgebraic set. In particular, the topological closure and interior of a semialgebraic set are semialgebraic.

Example 3. Let L be a k-by-l matrix each of whose entries is a polynomial over $\mathbb{R}^s$. Then the subset of $\mathbb{R}^s$ where L is of rank t is semialgebraic.

Proof. Let the j-by-j minors of L be $F_{j,1}, \cdots, F_{j,i_j}, 1 \leq j \leq \min(k, l)$. The set $R_j \subset \mathbb{R}^s$ where L is of rank $\leq j - 1$ is given by setting $F_{j,1} = \cdots = F_{j,i_j} = 0$, which is an algebraic set. The subset of $\mathbb{R}^s$ for which L is of rank t is the semialgebraic set $R_{t+1} \setminus R_t$. □

Example 4. Let $A \subset \mathbb{R}^m$ be a semialgebraic set. Consider the set

$$B := \{(x, y) \in (A \times A) : x \neq y\}.$$

B is semialgebraic since it is the complement of the diagonal of A in $A \times A$. Consider the set

$$C := \{(x, y, z) \in B \times S^{m-1} : (x, y) \in B, z = (x - y)/|x - y|\}.$$

C is semialgebraic since it is defined by the polynomial equations

$$(z_i)^2 |x - y|^2 = (x_i - y_i)^2$$

where x_i, y_i, z_i are the coordinates of x, y, z. Let D be the topological closure of C in $\mathbb{R}^m \times \mathbb{R}^m \times S^{m-1}$, and let

$$proj : \mathbb{R}^m \times \mathbb{R}^m \times S^{m-1} \to \mathbb{R}^m \times \mathbb{R}^m$$

be the standard projection. The preimage

$$E := proj^{-1}((p, p)), \quad p \in A$$

is also semialgebraic.

In fact, E is obtained by taking the limits of converging subsequences of $(q_n - p)/|q_n - p|$ for all converging sequences (q_n) to p.

Definition 5. We call the set E in Example 4 the *unit tangent cone* of the set A at p, remarking that the process of defining the unit tangent cone can be done for any set. We say a sequence (q_n) in A *converges to a unit tangent cone vector* e *at* p if q_n converges to p and $(q_n-p)/|q_n-p|$ converges to e .

Definition 6. For e in Definition 5 and any $\delta > 0$, we let $\mathcal{O}_e$ be the semialgebraic open set

$$\mathcal{O}_e(\delta) := \{q \in \mathbb{R}^m : |e - (q-p)/|q-p|| < \delta.$$

$\mathcal{O}_e(\delta)$ is a generalized cone with vertex p and axis e.

A map $f : S \subset \mathbb{R}^n \to \mathbb{R}^k$ over a semialgebraic S is *semialgebraic* if its graph in $\mathbb{R}^n \times \mathbb{R}^k$ is a semialgebraic set. It follows that the image of a semialgebraic map $f : S \subset \mathbb{R}^n \to \mathbb{R}^k$ is semialgebraic, via the composition $\mathrm{graph}(f) \subset \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^k$, where the last map is the projection onto the second summand.

Example 7. The conclusion in Example 3 continues to hold if the entries of the matrix involved consist of semialgebraic functions.

A *Nash function* is a C^∞ semialgebraic map from an open semialgebraic subset of $\mathbb{R}^n$ to $\mathbb{R}$. A real analytic function f defined on an open semialgebraic subset U of $\mathbb{R}^n$ is *analytic algebraic* if it is a solution of a polynomial equation on U of the form,

$$a_0(x)f^s(x) + a_1(x)f^{s-1}(x) + \cdots + a_s(x) = 0, \tag{2.2}$$

where $a_0(x) \neq 0, a_1(x), \cdots, a_s(x)$ are polynomials over $\mathbb{R}^n$. These two concepts are in fact equivalent[2, p. 165], that a function is Nash if and only if it is analytic algebraic.

The following example is instructive.

Example 8. For any number ϵ satisfying $0 < \epsilon < 1$, the open disk

$$B^n(\epsilon) = \{s = (s^1, \ldots, s^n) \in \mathbb{R}^n : |s| < \epsilon\} \tag{2.3}$$

is an open semi-algebraic subset of $\mathbb{R}^n$. The function

$$s^0 = \sqrt{1 - |s|^2} \tag{2.4}$$

on $B^n(\epsilon)$ is analytic algebraic, since $(s^0(x))^2 + a_0(x) = 0$ on $B^n(\epsilon)$, where $a_0(x)$ is the polynomial $|s|^2 - 1$ on $\mathbb{R}^n$. Partial derivatives of all orders of s^0 are analytic algebraic. In fact, an elementary calculation and induction argument shows that if D_i denotes the partial derivative with respect to s^i, then

$$D_{i_1 \dots i_k} s^0 = \frac{a_k(s)}{(s^0)^m}$$

where $a_k(s)$ is a polynomial on $\mathbb{R}^n$ and m is a positive integer. Therefore,

$$(s^0)^{2m}(D_{i_1 \dots i_k} s^0)^2 - a_k(s)^2 = 0$$

is an equation of the form (2.2), since $(s^0)^2$ is a polynomial on $\mathbb{R}^n$.

A slight generalization of the single-variable case in [2, p. 54], shows that the partial derivatives of any Nash function are again Nash functions.

Let S be a semialgebraic subset of $\mathbb{R}^n$. The *dimension* of S, denoted $\dim(S)$, is the dimension of the ring $R = \mathbb{R}[x^1, \cdots, x^n]/\mathcal{I}(S)$, where $\mathcal{I}(S)$ the ideal of all polynomials vanishing on S, which is the maximal length of chains of prime ideals of R. As usual, it is proved that if S is a semialgebraic subset of $\mathbb{R}^n$ that is a C^∞ submanifold of $\mathbb{R}^n$ of dimension d, then $\dim(S) = d$.

A semialgebraic subset M of $\mathbb{R}^m$ is a *Nash submanifold* of $\mathbb{R}^m$ of dimension n if for every point p of M, there exists a Nash diffeomorphism ψ from an open semialgebraic neighborhood U of the origin in $\mathbb{R}^m$ into an open semialgebraic neighborhood V of p in $\mathbb{R}^m$ such that $\psi(0) = p$ and $\psi((\mathbb{R}^n \times \{0\}) \cap U) = M \cap V$. Here, by a *Nash diffeomorphism* ψ we mean the coordinate functions of ψ and ψ^{-1} are Nash functions.

Let M be a Nash submanifold of $\mathbb{R}^m$. A mapping $f : M \to \mathbb{R}$ is a *Nash mapping* if it is semialgebraic, and for every ψ in the preceding definition, $f \circ \psi|_{\mathbb{R}^n \cap U}$ is a Nash function.

As in the C^∞ case, the semialgebraic version of the inverse and implicit function theorems also hold [2, p. 56]. Moreover, the semialgebraic version of the (Nash) tubular neighborhood theorem over Nash manifolds is true [2, p. 199].

Of special importance to us is the slicing theorem [2, p. 30], for which we only give the special version we need for the sake of clarity.

Theorem 9. *(Slicing Theorem) Let*

$$p_j(z, \lambda) := \lambda^{s_j} + a^j_{s_j - 1}(z)\lambda^{s_j - 1} + \cdots + a^j_1(z)\lambda + a^j_0(z), \quad 1 \le j \le a,$$

be real polynomials in $m+1$ variables $(z, \lambda) \in \mathbb{R}^m \times \mathbb{R}$ with degree s_j in λ. Then there is a partition of $\mathbb{R}^m$ into a finite number of (disjoint) semialgebraic sets $A_1, \cdots, A_l$, and for each $i = 1, \cdots, l$, a finite number (possibly zero) of semialgebraic functions

$$\zeta_{i,1} < \cdots < \zeta_{i,s_i} : A_i \to \mathbb{R},$$

such that for every $z \in A_i$, $\lambda = \zeta_{i,1}, \cdots, \zeta_{i,s_i}$ *are the distinct roots of* $p_j(z,\lambda) = 0, 1 \leq j \leq a$.

The number of these semialgebraic functions may be zero because a real polynomial may have no real roots. More importantly, the slicing theorem encodes the multiplicities of the roots into account. To see this for our later application, we start with a single polynomial

$$p(z,\lambda) := \lambda^s + a_{s-1}(z)\lambda^{s-1} + \cdots + a_1(z)\lambda + a_0(z).$$

The slicing theorem provides us with root functions $\zeta_{i,1}, \cdots, \zeta_{i,s_i}$ over $A_i, 1 \leq i \leq l$. The polynomial $p(z,\lambda)$ in the variable λ has repeated roots if and only if $s_i < s$, in which case the largest $j+1$ for which

$$\partial^j p/\partial\lambda^j = 0$$

evaluated at $\zeta_{i,1}(z), \cdots, \zeta_{i,s_i}(z)$ is the multiplicity of the respective root.

Therefore, to find the root functions of $p(z,\lambda)$ with multiplicities, we solve, for each k,

$$\partial^j p/\partial\lambda^j = 0, \quad 0 \leq j \leq k,$$

with the solution sets $A_{k,1}, \cdots, A_{k,l_k}$ and root functions

$$\xi_{k,i,1} < \cdots < \xi_{k,i,s_{k,i}} : A_{k,i} \to \mathbb{R}, \quad 1 \leq i \leq l_k.$$

Employ the two operations of taking the complement and finite intersection of sets to perform on $A_{k,1}, \cdots, A_{k,l_k}$ for $k = 0, \cdots, s$, knowing $A \cup B = (A^c \cap B^c)^c$ and $A \setminus B = A \cap B^c$. For instance,

$$\cup_i A_{0,i} \setminus \cup_i A_{1,i}$$

is the semialgebraic subset where $p(z,\lambda)$ has only simple roots with the root functions $\xi_{0,i,j}$ that carry multiplicity 1. Similarly,

$$((\cup_i A_{1,i}) \cap (\cup_i A_{2,i})) \setminus \cup_i A_{3,i}$$

is the semialgebraic subset where $p(z,\lambda)$ has only double and triple roots, with the root functions $\xi_{k,i,j}, k = 1, 2$, that carry multiplicity $k+1$, etc. Eventually, we end up with semialgebraic sets $V_1, \cdots, V_\tau$ that give rise to the root functions, with multiplicities. The set T where the closures of $V_1, \cdots, V_\tau$ intersect, which is also semialgebraic, is where the multiplicities of the roots of $p(z,\lambda)$ are not locally constant. It follows that $\dim(T) < k$, the ambient Euclidean dimension, because of the following dimension property [2, p. 53].

Proposition 10. *Let* $X \subset \mathbb{R}^s$ *be a semialgebraic set and let* $\overline{X}$ *be its topological closure. Then* $\dim(\overline{X} \setminus X) < \dim(X)$.

A weaker smooth version [13] than $\dim(T) < k$ above says that the set where the principal multiplicities of the shape operator S_ξ is not locally constant, as ξ varies on the unit normal bundle of a smooth submanifold, is nowhere dense.

Lastly, we record the important decomposition property [2, p. 57].

Proposition 11. *A semialgebraic set is the disjoint union of a finite number of Nash submanifolds N_j, each Nash diffeomorphic to an open cube* $(0,1)^{\dim(N_j)}$.

We call these Nash submanifolds in the decomposition *open cells* henceforth.

2.3. **The local finiteness property.** Recall the local finiteness property in [4] that holds the key for proving that a taut hypersurface M is real algebraic when its dimension is ≤ 4. We denote by $\mathcal{G}$ the subset of M where the multiplicities of principal curvatures are locally constant, and by $\mathcal{G}^c$ the complement of $\mathcal{G}$ in M.

Definition 12. A connected Dupin hypersurface M of S^n has the *local finiteness property* if there is a set $S \subset \mathcal{G}^c$, closed in M, such that S disconnects M in finitely many connected components, and for each point $x \in \mathcal{G}^c \setminus S$ there is an open neighborhood U of x in M such that $\mathcal{G} \cap U$ contains finitely many connected open sets (whose union is dense in U).

The technical advantage of excising a set S in the definition is that, we can remove certain types of points whose principal multiplicities are not locally constant and hard to handle without affecting establishing the algebraicity of the taut submanifold, as will become evident later.

3. Some local analysis and its implication

We first handle the case when M is a hypersurface. Fix a unit normal field $\mathbf{n}$ over M once and for all. We label the principal curvatures of M by $\lambda_1 \leq \cdots \leq \lambda_{n-1}$, which are Lipschitz-continuous functions on M because the principal curvature functions on the linear space $\mathcal{L}$ of all symmetric matrices are Lipschitz-continuous by general matrix theory [1, p. 64], and the Hessian of of M is a smooth function from M into $\mathcal{L}$. Let $\lambda_j = \cot(t_j)$ for $0 < t_j < \pi$. We have the Lipschitz-continuous focal maps

$$f_j(x) = \cos(t_j)x + \sin(t_j)\mathbf{n}. \tag{3.1}$$

In fact, the l-th focal point $f_l(x)$, counting multiplicities, along $\mathbf{n}$ emanating from x is antipodally symmetric to the $(n-l)$-th focal point, counting multiplicities, along $-\mathbf{n}$ emanating from x. The spherical distance functions $d_{f_l(x)}$ tracing backward following $-\mathbf{n}$ thus assumes the same critical point x as the distance function $d_{-f_l(x)}$ tracing backward following $\mathbf{n}$; thus we may just consider the former case without loss of generality. Accordingly, henceforth we refer to a focal point p as being $f_j(x)$ for some x and j with an obvious modification if necessary.

Notation 13. *Let p be the focal point of a fixed curvature surface Z and let $q \neq p$ be the focal point of a nearby curvature surface. Set*

$$g := \ell_q - \ell_p$$

considered as the perturbation of the height function ℓ_p to the nearby ℓ_q by g.

We assume

$$\ell_p(Z) = 0$$

without loss of generality. Let $W \subset M$ be a tubular neighborhood of Z so small that Z is the only critical submanifold of ℓ_p in W and let

$$\pi : W \to Z \tag{3.2}$$

be the projection.

Definition 14. Notation as above, we will call such a W a *neck* around Z.

Let us parametrize W by $v_1, \cdots, v_s, u_1, \cdots, u_{n-1-s}$ with $u_1 = \cdots = u_{n-1-s} = 0$ parametrizing Z such that

$$\pi : (v_1, \cdots, v_s, u_1, \cdots, u_{n-1-s}) \mapsto (v_1, \cdots, v_s). \tag{3.3}$$

By a linear change of $u_1, \cdots, u_{n-1-s}$, we may assume, around $0 \in Z$,

$$\begin{aligned} \ell_p &= \sum_{j=1}^{n-1-s} \alpha_j u_j^2 + O(3), \\ g &= h(u) + k(v) + \sum_{jk} \beta_{jk} v_j v_k + \sum_{jk} \gamma_{jk} u_j v_k + O(3), \end{aligned} \tag{3.4}$$

with

$$h(u) = \sum_i a_i u_i + \sum_{j,k} b_{jk} u_j u_k, \quad k(v) = \sum_l c_l v_l$$

for some small coefficients $a_i, \beta_{jk}, \gamma_{jk}, b_{jk}$, and c_l all in the magnitude of $|q - p|$ by the linear nature of g, where α_j are fixed nonzero constants. We obtain

$$F_j := \partial(\ell_p + g)/\partial u_j = a_j + 2\alpha_j u_j + 2\sum_l b_{jl} u_l + \sum_k \gamma_{jk} v_k + O(2)$$

for $1 \le j \le n-1-s$, and

$$G_i := \partial(\ell_p + g)/\partial v_i = c_i + 2\sum_k \beta_{ik} v_k + \sum_j \gamma_{ji} u_j + O(2)$$

for $1 \le i \le s$. We calculate

(3.5)

$$\partial(F_1, \cdots, F_{n-1-s}, G_1, \cdots, G_s)/\partial(u_1, \cdots, u_{n-1-s}, v_1, \cdots, v_s) = \begin{pmatrix} \Theta & \gamma \\ \gamma^{tr} & \Gamma \end{pmatrix},$$

where

$$\Theta := \left(2\alpha_j \delta_{jl} + 2b_{jl}\right), \quad \gamma := \left(\gamma_{jk}\right), \quad \Gamma := \left(\beta_{ik}\right),$$

at $u_1 = \cdots = u_{n-1-s} = v_1 = \cdots = v_s = 0$. Note that Θ is nonsingular when we let $|q-p|$ be much smaller than $\min_j |\alpha_j|$.

Due to the compactness of Z, we can cover W by a finite number of coordinate charts as above, so that there is an open disk $\mathcal{D}$ centered at p whose radius is so small that Θ is nonsingular in any of these coordinate charts for all $q \in \mathcal{D}$. In other words, since the left hand side of (3.5) is the Hessian of $\ell_q = \ell_p + g$, we may assume without loss of generality that the Hessian of ℓ_q is of rank $\ge n-1-s$ in W whenever $q \in \mathcal{D}$.

Recall Definition 5. Let us be given a vector e, which will in future applications be a unit tangent cone vector at p of an appropriately chosen set. Assume that Z is not on a level set of ℓ_e. Since Z is taut, the height function ℓ_e cuts Z in several critical submanifolds $Z_1, \cdots, Z_m$. Let us consider Z_1, for instance. Assume the codimension of Z_1 in Z is t.

Let T_1 be a neck of Z_1 in Z and let

$$N_1 := \pi^{-1}(T_1) \subset W$$

with π given in (3.2). Retaining the notation in (3.3), let us parametrize N_1 by the variables

$$v_1, \cdots, v_t, v_{t+1}, \cdots, v_s, u_1, \cdots, u_{n-1-s}$$

around 0, where $v_{t+1} \cdots, v_s$ parametrize Z_1 and $v_1, \cdots, v_s$ parametrize T_1 around Z_1 in Z. It is understood that 0 in the coordinate system corresponds to a point on Z_1.

Note that ℓ_e now assumes a simpler form

$$\ell_e = h(u) + \sum_{j=1}^{t} \beta_j v_j^2 + \sum_{jk} \gamma_{jk} u_j v_k + O(3), \tag{3.6}$$

where all β_i are nonzero since Z_1 is a critical submanifold of $\ell_e|_Z$ in Z.

By continuity, there is a small open set $\mathcal{O}_e$ given in Definition 6 (we ignore the radius δ in the definition) such that whenever $q \in \mathcal{O}_e$ the unit vector

$$e_q := (q-p)/|q-p|$$

with

$$\ell_{e_q} = h^q(u) + k^q(v) + \sum_{jk} \beta_{jk}^{e_q} v_j v_k + \sum_{jk} \gamma_{jk}^{e_q} u_j v_k + O(3), \tag{3.7}$$

where $k^q(v)$ is linear in v, satisfies that the upper t-by-t block of the matrix $\left(\beta_{jk}^{e_q}\right)$ is nonsingular, or equivalently, $\left(\beta_{jk}^{e_q}\right)$ is of rank $\geq t$ whenever $q \in \mathcal{O}_e$ (by shrinking the coordinates if necessary). It follows that, when we substitute

$$g_q := |q-p|\ell_{e_q}, \quad l = 1, 2, \cdots, \tag{3.8}$$

into (3.5), with

$$\ell_q = \ell_p + g_q, \tag{3.9}$$

we see the Hessian of ℓ_q are all of rank $\geq n-1-s+t$, or all of kernel dimension $\leq \dim(Z_1)$, whenever $q \in \mathcal{O}_e$. That is,

$$\dim(C_q) \leq \dim(Z_1) \tag{3.10}$$

whenever a critical submanifold C_q of ℓ_q lies in N_1. Similarly, the same holds for other N_j as well.

On the other hand, at a point $x \in Z$ away from $Z_1, \cdots, Z_m$, we can still parametrize W around x by $v_1, \cdots, v_s, u_1, \cdots, u_{n-1-s}$, where $v_1, \cdots, v_s$ parametrize Z around x identified with 0. Then slightly differently from the earlier expression we have

$$\begin{aligned}
\ell_p &= \sum_{j=1}^{n-1-s} \alpha_j u_j^2 + O(3), \\
\ell_e &= h(u) + \sum_{i=1}^{s} \gamma_i v_i + \sum_{i=1}^{s} \delta_i v_i^2 + \sum_{jk} \gamma_{jk} u_j v_k + O(3),
\end{aligned}$$

where at least one of γ_i is nonzero since x is not a critical point of ℓ_e on Z. As a consequence, following the argument below (3.6) we conclude that the gradient of ℓ_q is nonzero in a neighborhood of x in W, so that no critical submanifold C_{e_q} of ℓ_q passes through this neighborhood for $q \in \mathcal{O}_e$. In conclusion, we have the following.

Proposition 15. *Notations and conditions as above, let Z be a curvature surface with focal point p, and let e be a vector, which will be a unit tangent cone vector at p of an appropriately chosen set later,* *such that Z is not on any level set of ℓ_e. Let $Z_1, \cdots, Z_m$ be the critical submanifolds of $\ell_e|_Z$ in Z with small disjoint necks T_j of Z_j in Z and N_j of Z_j in $W, j = 1, \cdots, m$. Then there is an open set $\mathcal{O}_e$ such that for every $q \in \mathcal{O}_e$ a focal submanifold C_q of ℓ_q in W is contained in a unique N_s for some $s \leq m$ with*

$$\dim(C_q) \leq \dim(Z_s).$$

Proof. Since $Z_1, \cdots Z_m$ are disjoint in Z, as above we cover them by disjoint open necks T_j in Z and $N_j \supset T_j$ in $W, 1 \leq j \leq m$; then cover the complement of $\cup_{j=1}^m T_j$ in Z by finitely many small open balls B_j in Z and open balls $O_j \supset B_j$ in $W, 1 \leq j \leq a$, such that no critical submanifold passes through $O_i, \forall i$. Therefore, $C_q \subset N_s$ for some unique $s \leq m$.

The second statement is (3.10). □

Corollary 16. *Suppose all unit tangent cone vectors e at p of focal points q near p satisfy that Z is not on any level set of ℓ_e. Assume in any open ball $O_x(1/j)$ of radius $1/j$ centered at $x \in Z$, there is a point y_j at which the principal multiplicities are not locally constant, or equivalently, there is a curvature surface C_j through y_j whose dimension is not locally constant (so that it is not diffeomorphic to a sphere). Let e be the unit tangent cone vector at p of the focal points q_j of C_j and let $Z_1, \cdots, Z_m$ be the focal submanifolds of $\ell_e|_Z$ in Z. Then there is a unique Z_s through x that is not diffeomorphic to a sphere, or equivalently, whose dimension in Z is not locally constant.*

Proof. Notations are as in Proposition 15. That $x \in Z_s$ results when we let j get larger and larger while shrinking the necks $N_1, \cdots, N_m$ more and more.

We show Z_s is not diffeomorphic to a sphere. For a sufficiently large j, since C_j lies in the neck N_s of Z_s, tautness of C_j and Z_s imply that the topology of C_j embeds in the topology of Z_s (see the remark below). Therefore, Z_s cannot be a sphere. Otherwise C_j would be a sphere, which is not the case. □

Remark 17. We let

$$M_{p,\epsilon} := \ell_p^{-1}((-\infty, \epsilon]), \quad M_{p,\epsilon}^\circ := \ell_p^{-1}((-\infty, \epsilon)).$$

Fix noncritical values a, b of ℓ_p with $a < 0 < b$ and 0 the only critical value between them. Notation as in the preceding corollary. Under the negative gradient flow of ℓ_p, $W \supset Z$ is homotopic to the disk bundle B with base Z and fiber the unit disk $\mathbb{D}^\mu \subset \mathbb{R}^\mu$, where μ is the Morse-Bott index of Z. So, N_s is homotopic to the disk bundle B restricted to T_s, which is homotopic to the disk bundle B restricted to Z_s, denoted by B_s. The topology of N_s attached to $M_{p,a}$ is therefore

$$H_k(B_s, \partial B_s) = H_{k-\mu}(Z_s)$$

by Thom isomorphism. On the other hand, since ℓ_{q_j} assumes the critical submanifolds C_j and possibly other $Y_1, \cdots, Y_a$ in N_s with indexes $\sigma_j, \tau_1, \cdots, \tau_a$ and critical values $\alpha_j, \beta_1, \cdots, \beta_a$, respectively, we see there is an isomorphism

$$H_{k-\sigma_j}(C_j) \oplus_{b=1}^{a} H_{k-\tau_b}(Y_b) \simeq H_{k-\mu}(Z_s), \tag{3.11}$$

because we can find noncritical values a' and b', $a' < \alpha_j, \beta_1, \cdots, \beta_a < b'$, of ℓ_{q_j} such that

$$M_{p,a} \subset M^\circ_{q_j,a'} \subset M_{q_j,\alpha_j}, M_{q_j,\beta_1}, \cdots, M_{q_j,\beta_a}, M_{p,0} \subset M^\circ_{q_j,b'} \subset M_{p,b},$$

so that the topology of N_s attached to $M_{q_j,a'}$, which deformation retracts to $M_{p,a}$, is the left hand side of (3.11). Shifting indexes, $H_k(C_j)$ embeds in $H_{k+\sigma_i-\mu}(Z)$ for all k.

In conclusion, if Z_s is a sphere, then C_j must be either a point or a sphere of the same dimension as Z_s. In particular, if C_j is a curvature surface, then C_j must be a sphere of the same dimension as Z_s, so that the dimension of Z_s is locally constant.

Corollary 18. *Let (q_j) be a sequence of focal points converging to the unit tangent cone vector e at p. If Z is not on any level set of l_{q_j} for all j, then Z is not on any level set of l_e.*

Proof. Suppose Z is on a level set of ℓ_e. Then Z must be on a regular level set of ℓ_e; otherwise, Corollary 2 would imply that $p = e$, which is not the case ($p \perp e$ on S^n). Therefore, as in (3.8), $\ell_{q_j} = \ell_p + g_j$ would be regular over Z for $j \geq L$ for some L when $q_j \in \mathcal{O}_e$ given in Proposition 15. Now locally,

$$\ell_p = \sum_{j=1}^{n-1-s} \alpha_j u_j^2 + O(3),$$

$$\ell_{e_j} = h(u) + k(v) + \sum_{i=1}^{s} \delta_i v_i^2 + \sum_{ik} \gamma_{ik} u_i v_k + O(3),$$

where either $h(u)$ or $k(v)$ has a nontrivial linear term, since ℓ_{e_j} is regular on Z. It follows that the gradient of $\ell_{q_j}, j \geq L$, is nonzero in a tubular neighborhood of Z, which implies that q_j would not converge to p, a contradiction. $\square$

4. The proof

We do induction on n, the dimension of the ambient sphere, with the induction statement that all compact taut submanifolds in S^n are algebraic. The statement is clearly true for $n = 1$. Assume the statement is true for $n - 1$. Let us first handle a taut hypersurface M in S^n to show that it is algebraic. Since any curvature surface of M is contained in a curvature sphere of dimension $n - 1$, we know by the induction hypothesis that all curvature surfaces, being taut by Ozawa theorem, are algebraic. We then proceed to establish that M is algebraic by establishing the local finiteness property on M.

To this end, let x be a point in $\mathcal{G}^c$ and let Z through x be a curvature surface with focal point p. We stipulate that all the conditions on the neck

$$\pi : W \to Z$$

we encountered in the preceding section prevail.

Let J be the principal index range of the principal maps (3.1) satisfying

$$p = f_a(x), \quad \forall a \in J.$$

Since the principal multiplicities are not locally constant at x, given any neighborhood V_p of p and U_x of x, there is a point $y \in (\cup_{a\in J} f_a^{-1}(V_p)) \cap U_x$ and a curvature surface C_y through y such that $\dim(C_y) < \dim(Z)$. Furthermore, by the continuity of the focal maps there are open neighborhoods N_p of p and O_x of x so small that $f_b(O_x)$ is disjoint from N_p for all $b \notin J$. Set

$$O^* := (\cup_{a\in J} f_a^{-1}(N_p)) \cap O_x. \tag{4.1}$$

Then for any focal point $q \in N_p$, there is a curvature surface of ℓ_q through O^*; moreover, each curvature surfaces C_y through $y \in O^*$ has principal index range contained in J so that in particular $\dim(C_y) \leq \dim(Z)$, and moreover, there exist C_y with $\dim(C_y) < \dim(Z)$ in any neighborhood of x in O^* because $x \in \mathcal{G}^c$.

We further stipulate, by choosing O^* so small, that

Z be the only critical submanifold of ℓ_p contained in the topological closure of W, and,

any curvature surface passing through O^* with principal index range contained in J be entirely contained in W.

Since there are only a finite number of critical submanifolds with the focal point p and Z is the only critical submanifold of ℓ_p in W, all focal points q of curvature surfaces $\neq Z$ in W are different from p in any small neighborhood of p.

We have two cases to consider.

Category 1. One of the curvature spheres of a focal point $\ell_q, q \neq p$, contains Z.

This means that Z is contained in a level set of such a height function ℓ_q. Suppose Z is contained in a critical submanifold of ℓ_q. Then by Corollary 2, the height functions ℓ_p and ℓ_q share the same center of the curvature sphere through Z, so that it must be that $p = q$, which is not the case. Therefore, all points of Z are regular points of ℓ_q.

We can understand all these q explicitly. Let S^l be the smallest sphere containing Z. It is more convenient to view what goes on in R^n when we place the pole of the stereographic projection on Z. Then we are looking at an $\mathbb{R}^l$, which, by a conformal transformation of the sphere, we may assume is the standard one contained in $\mathbb{R}^n$, in which Z sits. Let $E \simeq \mathbb{R}^{n-l}$ be the orthogonal complement of the $\mathbb{R}^l$. Any $\mathbb{R}^{n-l-1}$ in E gives rise to an $\mathbb{R}^{n-1}$ containing Z, and vice versa. Back on the sphere, this means that we have an $(n-l-1)$-parameter family of S^{n-1} containing Z. The focal points f of these S^{n-1} form an S^{n-l-1} on the equator.

Assume $\dim(Z) < l$ first. Notations as above, let us consider the incidence space

$$\mathcal{I} \subset S^{n-l-1} \times M$$

given by

$$\mathcal{I} := \{(t, z) : t \text{ is sufficiently close to } p \text{ and } z \text{ belongs to a critical submanifold of } \ell_t \text{ passing through } O^*\}.$$

Let $\mathcal{I}^\circ \subset \mathcal{I}$ be defined by

$$\mathcal{I}^\circ := \{(t, z) \in \mathcal{I} : z \text{ belongs to a critical submanifold with dimension } < \dim(Z)\}.$$

Let

$$\Pi_2 : \mathcal{I} \to M$$

be the projection from $\mathcal{I}$ onto its second summand.

We may assume O^* is so small that it is contained in the coordinate chart V employed in (3.5); we adopt the notations there. Around each $(t, z) \in \mathcal{I}^\circ$, choose a small neighborhood $V_{t,z} \subset \mathcal{D} \times V$ over which a certain $(n-s)$-by-$(n-s)$ minor of the Hessian matrix, given on the left hand side of (3.5), is nonsingular; here, $s = \dim(Z)$. Choose a countable refinement $V_1, V_2 \cdots$ of the open covering $\{V_{t,z}\}$ of $\mathcal{I}^\circ$. Fix a V_j, over which we may assume without loss of generality that the upper left $(n-s)$-by-$(n-s)$ minor of the Hessian matrix is nonsingular. Via the map

$$h := (q, z) \in V_j :\mapsto (F_1(q, z), \cdots, F_{n-s-1}(q, z), G_1(q, z)) \in \mathbb{R}^{n-s},$$

the implicit function theorem ensures that $h^{-1}(0)$ consists of countably many (disjoint) connected manifolds $V_{jk}, k = 1, 2, \cdots$, of dimension $n + s - 1$. Each V_{jk} is parametrized by $(q, v_2, \cdots, v_s)$ with the chart map

$$g_{jk} : (q, v_2, \cdots, v_s) \in \mathbb{R}^{n+s-1} \mapsto (q, u_1, \cdots, u_{n-s-1}, v_1, \cdots, v_s) \in V_{jk},$$

where $u_1, \cdots, u_{n-s-1}, v_1$ are functions of $q, v_2, \cdots, v_s$. Since $v_2, \cdots, v_s$ are coordinate functions over V_{jk} with respect to the chart, we can define the map

$$f_{jk} : (q, z) \in V_{jk} \mapsto (v_2(q, z), \cdots, v_s(q, z)) \in \mathbb{R}^{s-1}.$$

Consider the map

$$F_{jk} : V_{jk}|_{\mathcal{I}^\circ} \to \mathbb{R}^{\dim(M)-l} \times \mathbb{R}^{s-1}, \quad F_{jk} = (id, f_{jk}) : (t, z) \mapsto (t, f_{jk}(t, z)).$$

It is clear that $\mathbb{R}^{\dim(M)-l} \times \mathbb{R}^{s-1}$ is of Hausdorff dimension $= \dim(M)-2$ since $l > s$. Therefore, $V_{jk}|_{\mathcal{I}^\circ}$ is also of Hausdorff dimension at most $\dim(M)-2$ via the inverse Lipschitz-continuous map g_{jk}. It follows that each $V_j|_{\mathcal{I}^\circ}$ and thus $\mathcal{I}^\circ$, and its topological closure $\overline{\mathcal{I}^\circ}$, are of Hausdorff dimension at most $\dim(M) - 2$ as well. As a consequence, $\Pi_2(\overline{\mathcal{I}^\circ})$ is of Hausdorff dimension at most $\dim(M) - 2$, which therefore does not disconnect M [14, p. 269].

In other words, the set of points $z \in O^*$ belonging to the critical submanifolds of ℓ_t passing through O^* with dimension $< \dim(Z)$, for t sufficiently close to p, does not disconnect M and so does not contribute to the local finiteness property. Once such points are excised, $\mathcal{I} \setminus \overline{\mathcal{I}^\circ}$ is a manifold of dimension $= \dim(M) - l + \dim(Z)$, which can be seen by solving

$$F_1 = \cdots = F_{n-1-s} = G_1 = \cdots = G_s = 0$$

by the implicit function theorem for $u_1, \cdots, u_{n-1-s}$ in terms of $v_1, \cdots, v_s$ and t.

Lastly, observe that $\Pi_2 : \mathcal{I} \setminus \overline{\mathcal{I}^\circ} \to O^*$ is a finite map, because there are only at most $\dim(M)$ many curvature surfaces through $z \in O^*$. It is also an open map since it is the restriction to $\mathcal{I}$ of the standard projection from $S^n \times S^n$ to S^n. By Federer's version of Sard's theorem [6, p. 316], which states that the critical value set of a smooth map $f : \mathbb{R}^l \to \mathbb{R}^s$, at which the rank of the derivative is $\leq \nu$, is of Hausdorff ν-dimensional measure zero. Consequently, the critical value set of $\Pi_2|_{\mathcal{I} \setminus \overline{\mathcal{I}^\circ}}$ is of Hausdorff $(\dim(M) - l + \dim(Z))$-dimensional measure zero, and so in particular, of Hausdorff $(\dim(M)-1)$-dimensional measure zero since $l > \dim(Z)$. So by [14, p. 269] the critical value set of $\Pi_2(\mathcal{I} \setminus \overline{\mathcal{I}^\circ})$ does not disconnect M, which can thus be excised as well. What remains is thus the regular set $\mathcal{R}$ of $\mathcal{I} \setminus \overline{\mathcal{I}^\circ}$, over which Π_2 is a finite covering map onto its image. It follows that $\Pi_2(\mathcal{R})$ is an immersed manifold of dimension $= \dim(M) - l + \dim(Z) \leq \dim(M) - 1$, which thus disconnect M in only finitely many components.

If $\dim(Z) = l$, then $Z = S^l$. Remark 17 implies that all curvature surfaces passing through $\mathcal{O}^*$ (by shrinking it if necessary) are S^l, so that no principal index change occurs in $\mathcal{O}^*$. This is a contradiction.

In summary, barring a closed set of Hausdorff $(\dim(M)-1)$-dimensional measure zero in O^* that does not disconnect M, the union of the curvature surfaces in $\mathcal{G}^c$ in Category 1 is an immersed manifold of dimension $= \dim(M) - l + \dim(Z) \leq \dim(M) - 1$ and hence disconnects $\mathcal{G}$ in at most finitely many connected components.

Category 2. No curvature spheres of ℓ_q of a focal point $q, q \neq p$, contain Z.

Definition 19. Let $\mathcal{F}_Z$ be the set of all points q for which no curvature spheres of ℓ_q contain Z, and let $\mathcal{UC}_p$ be the set of the unit tangent cone vectors of $\mathcal{F}_Z$ at p.

By Corollary 2, $\mathcal{F}_Z$ is the image of the unit normal bundle UN of Z under the normal exponential map

$$Exp : ((x, n), t) \in UN \times (-\pi, \pi) \mapsto \cos(t)x + \sin(t)n \in \mathcal{F}_Z.$$

Hence, $\mathcal{F}_Z$ is semialgebraic, and so $\mathcal{UC}_p$ is semialgebraic by construction. Note that $\mathcal{UC}_p \subset \mathcal{F}_Z$ by Corollary 18.

By Corollary 16, for a sequence C_j of curvature surfaces through $y_j \in C_j$ converging to x, where the dimension of each C_j is not locally constant, we know a unit tangent cone vector e at p to which a subsequence of the focal points q_j of C_j converge has the property that, the dimension of the critical submanifold of ℓ_e through x in Z is not locally constant. Accordingly, we make the following definition.

Definition 20. We let $\mathcal{UC}_p^\circ \subset \mathcal{UC}_p$ be the set where the dimension of the critical submanifold of $\ell_e|_Z$ through x in Z is not locally constant.

Lemma 21. *$\mathcal{UC}_p^\circ$ is semialgebraic.*

Proof. $\mathcal{UC}_p^\circ$ consists of those $e \in \mathcal{UC}_p$ for which the gradient of $\ell_e|_Z = 0$ at x and the kernel (or rank) of the Hessian of $\ell_e|_Z$ at x is not locally constant. Therefore, Example 7 and Proposition 10 give the desired conclusion. □

The nature of $\mathcal{F}_Z$ and $\mathcal{U}C_p^\circ$ motivates us to look into the following semialgebraic object.

Lemma 22. *The set UN^o of unit normals ξ of Z at which the shape operator S_ξ has multiplicity change is semialgebraic of dimension $\leq \dim(M) - 1$.*

Proof. Let $\dim(Z) = s$ and let $(y, \zeta) \in Z \times S^{n-s-1}$ parametrize the unit normal bundle UN of Z. The characteristic polynomial of S_ξ is of the form

$$\lambda^s + a_{s-1}\lambda^{s-1} + \cdots + a_1\lambda + a_0,$$

where $a_1, \cdots, a_{s-1}$ are polynomials in the zero jet of ζ and the second jets of y; hence they are Nash functions. By the discussion following Theorem 9 (the slicing theorem), $Z \times S^{n-s-1}$ is decomposed into finitely many disjoint semialgebraic sets $V_1, \cdots, V_\tau$, where each V_i is equipped with semialgebraic functions $\eta_{i,1} < \cdots < \eta_{i,l_i}$ that solve the characteristic polynomial, counting multiplicities; moreover, UN^o, where the principal multiplicities are not locally constant, is semialgebraic of a lower dimension by Proposition 10 and the discussion preceding it. So

$$\dim(UN^o) \leq n - 2 = \dim(M) - 1. \tag{4.2}$$

□

Now in view of Corollary 2, for a unit normal ξ to Z, we let $q_\xi^1, q_\xi^2, \cdots$, and $q_\xi^{\dim(Z)}$ be the focal point of the curvature surface through the base point of ξ corresponding to the principal curvature function $\lambda^1(\xi), \cdots$, and $\lambda^{\dim(Z)}(\xi)$ of S_ξ, respectively. The remark following Corollary 2 gives the focal maps $g^1, \cdots, g^{\dim(Z)}$ that send ξ to the respective focal points; by the algebraic nature of Z, all these maps are semialgebraic.

Consider the semialgebraic set $\mathcal{X} \subset UN^o \times S^n \times S^n$ defined by

$$\mathcal{X} := \{(\xi, q, r) : q = g^j(\xi) \text{ for some } j; r \text{ belongs a critical set of } Z \text{ of } \ell_q\}.$$

Due to the nature of all these defining functions, $\mathcal{X}$ is semialgebraic. (For instance, critical submanifolds are obtained by setting the first derivative of the height function equal to zero on Z, which is a semialgebraic process.) Let

$$pr : UN \times S^n \times S^n \to S^n \times S^n \tag{4.3}$$

be the standard projection, which is a Nash submanifold, and let

$$\mathcal{J} := pr(\mathcal{X}).$$

The set $\mathcal{J}$ is also semialgebraic.

Lemma 23. $\dim(\mathcal{J}) \leq \dim(M) - 1$.

Proof. Consider

$$\alpha : \mathcal{J} \to UN^o, \quad \alpha : (q, z) \mapsto \xi(q, z), \tag{4.4}$$

where $\xi(q, z)$ is the unit tangent vector, based at z, along the geodesic of S^n from q to z. Note that α is the restriction to $\mathcal{J}$ of the Nash map

$$\beta : S^n \times S^n \to S^n \times S^n, \quad \beta : (u, v) \mapsto \xi(u, v).$$

Note also that α is a finite map, since in general each $\xi \in (UN^o)_z$ gives rise to at most $\dim(Z)$ many curvature surfaces through z. Therefore, by (4.2) and (4.4), we have

$$\dim(\mathcal{J}) \leq \dim(UN^o) \leq \dim(M) - 1. \tag{4.5}$$

□

Now we let

$$\Pi^1, \Pi^2 : S^n \times S^n \to S^n \tag{4.6}$$

be the standard projections onto the first and second summands, respectively. Note that Π^2 is a finite map because through each point in M there are only at most $\dim(M)$ many critical submanifolds. Moreover,

$$\mathcal{U}C_p^\circ \subset \Pi^1(\mathcal{J})$$

by construction.

Corollary 24. *The set*

$$\mathcal{I} := (\Pi^1|_{\mathcal{J}})^{-1}(\mathcal{U}C_p^\circ)$$

is semialgebraic of dimension $\leq \dim(M) - 2$.

Proof. The dimension of $\mathcal{I}$ is 1 less than $\dim(\mathcal{J})$ given in Lemma 23 because $\mathcal{U}C_p^\circ$ consists of unit tangent cone vectors. (This can be seen most clearly in $\mathbb{R}^n$ in place of S^n.) □

Recall the open set $\mathcal{O}_e$ defined before (3.7), which is semialgebraic. We now stipulate that $\mathcal{O}_e \subset N_p$ defined in (4.1) and set

$$\mathcal{O} := \cup_{e \in \mathcal{U}C_p^\circ} \mathcal{O}_e.$$

Corollary 25. $\mathcal{O} \cap \mathcal{F}_Z$ *is σ-semialgebraic in the sense that it is a countable union of increasing compact semialgebraic sets* $X_1 \subset X_2 \subset X_3 \subset \cdots$, *because there is a compact exhaustion of* $\mathcal{O}$.

In view of Proposition 15, we let

$$\mathcal{K} \subset (\mathcal{O} \cap \mathcal{F}_Z) \times M \subset S^n \times S^n$$

be the incidence space

$\mathcal{K} := \{(q, z) :$ for $q \in \mathcal{O}_e, z \in$ a critical submanifold of $\ell_q \subset$ a neck $N_1 \supset$ the critical submanifold Z_1 of $\ell_e|_Z, x \in Z_1,$ as given in Proposition 15$\}$.

Proposition 26. *Away from a closed subset $\mathcal{N}$ of Hausdorff* $(\dim(M)-1)$*-measure zero,* $\Pi^2(\mathcal{K})$ *is a manifold of dimension at most* $\dim(M)-1$.

Proof. Let $X_1 \subset X_2, \subset X_3, \cdots$ be a countable collection of increasing compact semialgebraic sets whose union is $\mathcal{O} \cap \mathcal{F}_Z$. By Proposition 11, fix a semialgebraic open cell decomposition $\mathcal{T}_j$ of $X_j, j = 1, 2, \cdots,$ in such a way that $\mathcal{T}_j$ is a sub-decomposition of $\mathcal{T}_{j+1}$ for all j (by decomposing $X_{j+1} \setminus X_j$); let $F_k, k = 1, 2, \cdots, s$, be the open cells in the decomposition.

We now collect the unit tangent cone vectors of F_k at p and call the set UC_k (it is empty if p is not in the closure of F_k), which is semialgebraic with

$$\dim(UC_k) + 1 = \dim(F_k) \tag{4.7}$$

(because UC_k consists of unit tangent cone vectors) if UC_k is not empty.

Sublemma 27.

$$\mathcal{U}C_p^\circ = \cup_k UC_k.$$

Proof. $\mathcal{U}C_p^\circ$ is a closed set since those e for which the dimension of the critical submanifold of ℓ_e is locally constant constitute an open set. Thus for a sequence q_j of F_k converging to the unit tangent cone vector e at p, let $q_j \in O_{e_j}$ for some $e_j \in \mathcal{U}C_p^\circ$. Then a converging subsequence of e_j must converge to $e \in \mathcal{U}C_p^\circ$. So, $\cup_k UC_k \subset \mathcal{U}C_p^\circ$.

Conversely, each $\mathcal{O}_e$ contains a sequence q_j of points in $\mathcal{F}_Z$ converging to e at p. Now choose a small compact semialgebraic disk $\mathcal{B}$ around the focal point p of Z. $\mathcal{B}$ intersects only finitely many compact $X_1 \subset X_2 \subset \cdots$, and so finitely many $F_1, \cdots, F_b$. Thus, there is a subsequence of q_j falling in one of these $F_1, \cdots, F_b$, say, F_1; it follows that e is a unit tangent cone vector of F_1 at p. That is, $\cup_k U_k \supset \mathcal{U}C_p^\circ$. $\square$

Define the semialgebraic set

$$Q_k := (\Pi^1|_{\mathcal{J}})^{-1}(UC_k) \subset \mathcal{I};$$

clearly, we have

$$\dim(Q_k) \leq \dim(\mathcal{I}) \leq \dim(M) - 2.$$

Lastly, let

$$P_k := (\Pi^1|_{\mathcal{K}})^{-1}(F_k).$$

Sublemma 28. *Assume UC_k is not empty. P_k is then of Hausdorff $(\dim(M)-1)$-measure zero provided* $\dim(Q_k) \leq \dim(M)-3$.

Proof. Since each F_k is an open cube with p on its boundary, we can set up a Nash diffeomorphism ι_k between a cone extended out of UC_k, denoted by $\mathbb{R}^+UC_k$, and F_k,

$$\iota_k : \mathcal{R}^+UC_k \to F_k,$$

such that $\iota_k(te), e \in UC_k, 0 < t < \alpha_e$ for some α_e, all lie in $\mathcal{O}_e$. (Again, this is most clearly seen when viewed in $\mathbb{R}^n$ in place of S^n, where $\iota_k(te)$ is the axis of the cone $\mathcal{O}_e$.) Note that, in particular, the fiber over $\iota_k(te), 0 < t < \alpha_e$, is of dimension $\leq$ the dimension of the fiber over e by Proposition 15. Thus, by an analysis analogous to the one in Category 1, we obtain, by (4.7),

$$\text{Hausdorff } \dim(P_k) \leq \dim(Q_k) + 1 \leq \dim(M) - 2.$$

The countable union of all P_k, where $\dim(Q_k) \leq \dim(M)-3$, is thus also of Hausdorff $(\dim(M)-1)$-measure zero. □

We let

$$\mathcal{N}_1 := \cup_k \Pi^2(P_k), \quad \dim(Q_k) \leq \dim(M) - 3.$$

$\mathcal{N}_1$ is of Hausdorff $(\dim(M)-1)$-measure zero so that $\mathcal{N}_1$ does not disconnect M [14, p. 269], and so does not contribute to the local finiteness property.

Thus what is left now are P_k with

$$\dim(Q_k) = \dim(M) - 2$$

(assuming UC_k is not empty). Note that in the fibration of P_k over F_k we may ignore the fibers over those $\iota_k(te)$ of fiber dimension less than the fiber dimension d_e over e in the fibration of Q_k over UC_k. In fact, let C be the fiber over $\iota_k(te)$ of dimension $d < d_e$. Then the Hessian matrix on the left hand side of (3.5) is of kernel dimension d on C, so that around $\iota_k(te)$ in F_k there is a neighborhood N over which the Hessian matrix is of kernel dimension $\leq d$ around C, such that the neighboring critical submanifolds around C in the fibration of P_k are contained in a smooth family P_k^* parametrized by $q \in N$, each of whose fibers is of dimension $= d < d_e$. Therefore, P_k^* is a manifold of dimension $\leq \dim(M)-2$ and so P_k^* and its topological closure $\overline{P_k^*}$ are of Hausdorff $(\dim(M)-1)$-measure zero. We let

$$\mathcal{N}_2 := \text{ the union of such } \Pi^2(\overline{P_k^*}).$$

We denote by P_k° the remaining part of P_k away from the preceding two classes of sets we excised. P_k° is then a smooth manifold of dimension $\dim(M) - 1$. By Federer's version of Sard's theorem [6, p. 316], the critical value set $\mathcal{C}_k \subset M$ of

$$\Pi^2 : P_k^\circ \to M$$

is of rank $\leq \dim(M) - 1$ and so is of Hausdorff $(\dim(M) - 1)$-measure zero. This implies that C_k and hence the union of all C_k do not disconnect M. We let

$$\mathcal{N}_3 = \cup_k \mathcal{C}_k.$$

Now let $\mathcal{N}$ be the topological closure of the union of $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$; $\mathcal{N}$ is of Hausdorff $(\dim(M) - 1)$-measure zero. In particular, $\mathcal{N}$ is closed and does not disconnect M and so is nowhere dense in M. Hence,

$$\Pi^2 : P_k \setminus (\Pi^2)^{-1}(\mathcal{N}) \to S^n \setminus \mathcal{N} \tag{4.8}$$

is a locally diffeomorphic finite map from a manifold of dimension $\dim(M) - 1$ into M, whose image thus consists of at most $\beta(M)$ immersed manifolds of dimension $\dim(M) - 1$ in M.

Lastly, we need to take care of those F_k for which UC_k are empty, i.e., whose closures do not contain p. Choose a small compact semialgebraic disk $\mathcal{B}$ around the focal point p. $\mathcal{B}$ intersects only finitely many $X_1 \subset X_2 \subset \cdots$, and hence finitely many $F_1, \cdots, F_b$, from which we remove those whose closures do not contain p and call the remaining ones $F_{i_1}, \cdots, F_{i_a}$. We then go through the same arguments as above for each of $F_{i_1}, \cdots, F_{i_a}$ to conclude that $\Pi^2(P_{i_1}), \cdots, \Pi^2(P_{i_a})$ are immersed submanifolds of codimension 1 in the neighborhood $\mathcal{B}$ around p, away from a closed set that does not disconnect M. $\square$

In summary, barring a closed set of Hausdorff $(\dim(M)-1)$-dimensional measure zero in W that does not disconnect M, the union of the curvature surfaces in $\mathcal{G}^c$ in Category 2 is locally an immersed manifold of dimension $\leq \dim(M) - 1$, and hence locally disconnects $\mathcal{G}$ in at most finitely many connected components.

We have established the local finiteness property in the hypersurface case.

We now handle the case when M is a taut submanifold. It is more convenient to work in $\mathbb{R}^n$. Let M_ϵ be a tube over M of sufficiently small radius that M_ϵ is an embedded hypersurface in $\mathbb{R}^n$. Then M_ϵ is a taut hypersurface [12], so that by the above M_ϵ is algebraic. Consider the focal map $F_\epsilon : M_\epsilon \to M \subset \mathbb{R}^n$ given by

$$F_\epsilon(x) = x - \epsilon\xi, \tag{4.9}$$

where ξ is the outward field of unit normals to the tube M_ϵ. Any point of M_ϵ has an open neighborhood U parametrized by an analytic algebraic map. The first derivatives of this parametrization are also analytic algebraic [2, p. 54], and thus the Gram-Schmidt process, applied to these first derivatives and some constant non-tangential vector, produces the vector field ξ and shows that ξ is analytic algebraic on U. Hence F_ϵ is analytic algebraic on U and so the image $F_\epsilon(U) \subset M$ is a semialgebraic subset of $\mathbb{R}^n$. Covering M_ϵ by finitely many sets of the kind of U, we see that M, being the union of their images under F_ϵ, is a semialgebraic subset of $\mathbb{R}^n$. Then the Zariski closure $\overline{M}^{\text{zar}}$ of M is an irreducible algebraic variety of the same dimension as M and contains M (see [4] for more details).

The induction is thus completed.

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