

IDEAL THEORY AND CLASSIFICATION OF ISOPARAMETRIC HYPERSURFACES

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ABSTRACT. The classification of isoparametric hypersurfaces with four principal curvatures in the sphere interplays in a deep fashion with commutative algebra, whose abstract and comprehensive nature might obscure a differential geometer's insight into the classification problem that encompasses a wide spectrum of geometry and topology. In this paper, we make an effort to bridge the gap by walking through the important part of commutative algebra central to the classification of such hypersurfaces, such that all the essential ideal-theoretic ingredients are laid out in a way as much intuitive, motivating and geometric with rigor maintained as possible. We then explain how we developed the technical side of the entailed ideal theory, pertinent to isoparametric hypersurfaces with four principal curvatures, for the classification done in our papers [6], [8] and [10].

1. INTRODUCTION

An isoparametric hypersurface M in the sphere is one whose principal curvatures and their multiplicities are fixed constants. The classification of such hypersurfaces has been an outstanding problem in submanifold geometry, listed as Problem 34 in [29], as can be witnessed by its long history. See Section 3 for more background details.

The story started with Cartan's seminal investigation and complete classification when g , the number of principal curvatures, is ≤ 3 [2], [3], [4], [5], followed by Münzner's remarkable structure theory [24] to lay the groundwork for the two classes of first known inhomogeneous examples with $g = 4$ constructed by Ozeki and Takeuchi [25, I], which was then generalized to infinite classes of inhomogeneous examples by Ferus, Karcher and Münzner [15].

Among other things, Münzner [24, II] established that $g = 1, 2, 3, 4$ or 6 . Thanks to the subsequent work of Abresch [1], who identified the only two possible multiplicity pairs of the principal curvatures when $g = 6$, Dorfmeister and Neher [13] succeeded in the classification in the case of the smaller pair ($= (1, 1)$), and recently Miyaoka [22], [23] settled the case of the other pair ($= (2, 2)$); the isoparametric hypersurfaces are homogeneous.

It is worth pointing out that isoparametric submanifolds in the sphere were introduced by Terng [27] and later those of codimension ≥ 2 were all classified to be homogeneous by Thorbergsson [28]. Thorbergsson's method was to associate the submanifold with a Tits building to employ the rigidity

of Tits buildings of rank ≥ 3 in the classification. Though Immervoll [18] proved that an isoparametric hypersurface with four principal curvatures in the sphere also gives rise to an incidence structure which is a Tits building, it cannot be applied directly to the classification as in Thorbergsson's approach since there is no such classification of rank 2.

As of this writing, for $g = 4$, there remains the last unsettled case with multiplicity pair (7, 8). The classification enjoys a deep interaction with a major part of the ideal theory in commutative algebra, whose abstract and comprehensive nature might obscure a differential geometer when facing a classification problem of the sort such as isoparametric hypersurfaces, that encompasses a wide spectrum of geometry and topology.

The purpose of the paper is twofold. On the one hand, we will walk through the important part of commutative algebra central to the classification of isoparametric hypersurfaces with four principal curvatures, in as much intuitive, motivating and geometric a way with the rigor of the presentation maintained as possible. On the other hand, with a good look at the entailed ideal theory we will then explain its technical side we developed in [6], [8] and [10] on which the classification hinges.

We hope the paper can bring the reader to a further appreciation of the breadth and depth of the intriguing classification story of isoparametric hypersurfaces.

2. A WALK THROUGH SOME IDEAL THEORY

2.1. Codimension 1 estimate and reducedness. Let \mathbb{C}^n be parametrized by z_1, \dots, z_n , and let V be a *variety* in \mathbb{C}^n , i.e., a set defined by the common zeros of $m+1$ polynomials p_0, p_1, \dots, p_m in the polynomial ring $P[n]$ in the variables z_1, \dots, z_n . Hilbert's basis theorem [16, p. 13] implies that all ideals of $P[n]$ are finitely generated. Moreover, Hilbert's Nullstellensatz [16, p. 20] states that $f \in P[n]$ vanishes on V if and only if f^n , for some positive integer n , belongs to the ideal $I \subset P[n]$ generated by p_0, \dots, p_m , denoted by (p_0, \dots, p_m) henceforth. In particular, if we let $\mathcal{O}(V)$ be the ideal of $P[n]$ of all polynomial functions vanishing on V , also called the *coordinate ring* of V , then there is a one-to-one correspondence between a variety V and its coordinate ring $\mathcal{O}(V)$ in \mathbb{C}^n .

In general, V may have finitely many irreducible components V_1, \dots, V_s which cannot be further decomposed into unions of varieties, a consequence of Hilbert's basis theorem [16, pp. 15-16]. Dually, $\mathcal{O}(V)$ is the intersection of finitely many prime ideals

$$(1) \quad \mathcal{O}(V) = \bigcap_{j=1}^s \mathcal{P}_j,$$

where \mathcal{P}_j is the ideal of $f \in P[n]$ vanishing on V_j . (Recall an ideal \mathcal{P} is prime if $ab \in \mathcal{P}$ implies either a or b is in \mathcal{P} .) Each \mathcal{P}_j is a minimal prime ideal containing $\mathcal{O}(V)$ since V_j is contained in no other irreducible varieties contained in V . On the other hand, each V_j is a complex manifold away

from its *singular set*, which is itself a variety of a smaller dimension where V_j is not manifold-like. In addition, there is another type of singular points of V , namely, those which lie in the intersection of two irreducible components where V is not manifold-like. Together, the two types of points constitute the singular set $\mathcal{S}(V)$ of V . Explicitly,

$$(2) \quad \mathcal{S}(V) = (\cup_{i \neq j} (V_i \cap V_j)) \cup (\cup_j \mathcal{S}(V_j)),$$

where if the coordinate ring \mathcal{P}_j of V_j is generated by the polynomials q_1, \dots, q_l , we let

$$\text{edim}(z) := n - \text{rank}(\partial(q_1, \dots, q_l) / \partial(z_1, \dots, z_n)),$$

be the *embedding dimension* that is the natural dimension one expects from the implicit function theorem in calculus. Then

$$\dim(V_j) = \inf_{z \in V_j} \text{edim}(z), \quad \mathcal{S}(V_j) = \{z \in V_j : \text{edim}(z) > \dim(V_j)\}.$$

See [20, p. 170] for (2) that is even true on the ideal level.

Example 1. Consider the polynomial

$$p = (x - 1)(y^2 - x^2(x + 1))$$

over \mathbb{C}^2 . The variety $p = 0$ consists of two irreducible components V_1 and V_2 , which are respectively the zeros sets $x - 1 = 0$ and $y^2 - x^2(x + 1) = 0$. The singular set of V consists of the singular point of V_2 , which is $(0, 0)$, and $(1, \pm\sqrt{2})$, the two points of intersection of V_1 and V_2 .

Here comes the subtlety. In general $\mathcal{O}(V)$ properly contains I that defines the variety V .

Example 2. Consider $p_0(x, y) = y - x^2$ and $p_1(x, y) = y$. Their common zero set V is $\{(0, 0)\}$. The polynomial x vanishes on V , i.e., $x \in \mathcal{O}(V)$. However, x does not belong to the ideal $I = (p_0, p_1)$, as can be easily verified. Instead, x^2 lies in I .

For an ideal I , we denote by \sqrt{I} the *radical* of I consisting of $f \in P[n]$ such that $f^n \in I$ for some positive integer n . A fundamental question is:

Under what condition I , which defines V , is exactly $\mathcal{O}(V)$?

Clearly, a necessary and sufficient condition is that f vanishes on V implies f lies in I . Alternatively put, by Nullstellensatz, $f^n \in I$ for some n implies $f \in I$, i.e., $\sqrt{I} = I$, in which case I is called a *radical ideal* and $P[n]/I$ interchangeably is called a *reduced ring*, for reason that it thus has no nilpotent elements, i.e., no $r \neq 0$ for which $r^n = 0$ for some n .

Note that I is radical when I is a prime ideal, or equivalently, when the variety V defined by I is irreducible. The second fundamental question is:

Under what condition is I a prime ideal?

To answer the first question, let us observe that if I is radical, i.e., if $I = \mathcal{O}(V)$, then by (1), we must have

(†) I is the intersection of only those minimal prime ideals containing I .

Not all ideals are the intersection of only minimal prime ideals containing I .

Example 3. Consider $I = (x^2, xy)$ in $P[2]$. It is easily seen that

$$I = (x) \cap (x^2, y).$$

Since the variety defined by I is the y -axis, the only minimal ideal containing I is (x) . Note that $(x) = \sqrt{I}$.

In addition, we must also have

(‡) For each m in a minimal prime ideal \mathcal{P} containing I , there is an s of $P[n]$ not in \mathcal{P} such that $sm \in I$.

In fact, for each $m \in \mathcal{P}_1$, pick an $s \in (\cap_{j=2}^s \mathcal{P}_j) \setminus \mathcal{P}_1$. Then s is a polynomial vanishing on $\cup_{j=2}^s V_j$ but not on V_1 . We have $sm = 0$ on V , etc.

It turns out that (†) and (‡) are also sufficient to imply that I is radical, called Serre's (S_1, R_0) criterion. One needs to establish that $\sqrt{I} = I$. To this end, on the one hand (†) and (‡) ensure that I is the intersection of minimal prime ideals containing I [20, p. 181]. On the other hand, it is well known [14, p. 71] that \sqrt{I} is the intersection of all prime ideals, and so in particular, is the intersection of all minimal prime ideals. Thus $\sqrt{I} = I$.

So now it comes down to asking when (†) and (‡) hold true. A broad category in which (†) is valid is when the generators p_0, \dots, p_m of I form a *regular sequence*, a notion central in commutative algebra that generalizes that of smooth transversal intersections.

Recall that in a ring an element $a \neq 0$ is called a *zero divisor* if $ab = 0$ for some element $b \neq 0$. Otherwise, it is called a *non-zerodivisor*.

Definition 1. A *regular sequence* in the polynomial ring $P[n]$ is a sequence p_0, \dots, p_k in $P[n]$ such that firstly the variety defined by $p_0 = \dots = p_k = 0$ in \mathbb{C}^n is not empty. Moreover, p_i is a non-zerodivisor in the quotient ring $P[n]/(p_0, \dots, p_{i-1})$ for $1 \leq i \leq k$; in other words, any relation

$$p_1 f_1 + \dots + p_{i-1} f_{i-1} + p_i f_i = 0$$

will result in f_i being in the form

$$f_i = p_0 h_0^i + \dots + p_{i-1} h_{i-1}^i$$

for some $h_0^i, \dots, h_{i-1}^i \in P[n]$ for $1 \leq i \leq k$.

Thus a regular sequence imposes strong algebraic independence amongst its elements. We shall return to this later.

Example 4. A single nonconstant $p \in P[n]$ forms a regular sequence, because by Nullstellensatz $p = 0$ is nonempty, which is the only non-void condition in the definition of a regular sequence.

Two homogeneous and relatively prime polynomials p and q of degree ≥ 1 form a regular sequence. Firstly, $p = q = 0$ is nonempty since 0 is clearly a

solution. Secondly, $pf_1 + qf_2 = 0$ implies $f_2 = ph$ since p and q are relatively prime.

More generally, any two relatively prime polynomials p and q with a nonempty common zero set form a regular sequence.

Example 5. The first k coordinates z_1, \dots, z_k of \mathbb{C}^n form a regular sequence for any k . To see this, first of all $z_1 = \dots = z_k = 0$ is not empty. Next, if

$$z_1f_1 + z_2f_2 = 0, \quad \text{or } z_2f_2 = -z_1f_1,$$

then since z_2 does not vanish identically on the hyperplane $z_1 = 0$, it must be that f_2 does, so that $f_2 = z_1g_1$. Similarly, if

$$z_3f_3 = -z_1f_1 - z_2f_2,$$

then f_3 must vanish identically on the linear subspace $z_1 = z_2 = 0$, which ensures that $f_3 = z_1h_1 + z_2h_2$, etc.

Example 6. $p = xz$ and $q = yz$ in \mathbb{C}^3 do not form a regular sequence. This is because $py - qx = 0$ and x is not a multiple of p . Note that $p = q = 0$ is the variety V consisting of the plane $z = 0$ and the line $x = y = 0$. V is not of pure dimension.

It is a deep fact that the variety V defined by a regular sequence p_0, \dots, p_m in \mathbb{C}^n is of pure dimension $n - m - 1$. It is not just that the manifold part of each irreducible component of V is of the right dimension $n - m - 1$. What is remarkable is that it is the right dimension at each singular point as well, more generally so on the ideal level! The technical and deep concept entailed here is *Cohen-Macaulayness*. That (\dagger) holds for an ideal $I \subset P[n]$ generated by a regular sequence is a consequence of this property of pure dimension, on the ideal level, in the context of Macaulay Unmixedness Theorem [20, p. 187].

Having set aside (\dagger) , let us turn to (\ddagger) . We now express it in terms of the ring $R := P[n]/I$ itself to make the statement intrinsic.

(\ddagger') For each m in a minimal prime ideal \mathcal{P} in R , there is an $s \in R \setminus \mathcal{P}$ such that $sm = 0$.

Before proceeding further, let us look at Example 2 once more. In the example, since the ideal I is generated by $y - x^2$ and y , or equivalently by x^2 and y , the quotient ring is thus

$$R = P[2]/I = \{a + bx : a, b \in \mathbb{C}, x^2 = 0\}.$$

The minimal prime ideal $\mathcal{P} := (x)$ in R fails to satisfy (\ddagger') . Indeed, the only $s \in R \setminus \mathcal{P}$ is a nonzero constant in \mathbb{C} , whose product with x can never be zero. Note that this example satisfies (\dagger) as p_0 and p_1 in the example form a regular sequence.

Now, it is a pleasant fact that the implicit function theorem comes to the rescue to resolve (\ddagger') . This is known as Serre's criterion of reducedness [14, p. 462].

Theorem 1. (Serre) *Let I be the ideal generated by a regular sequence $p_0, \dots, p_m, m+1 \leq n$, in $P[n]$ that define the variety V . Let J be the subvariety of V consisting of all points of V where the Jacobian matrix*

$$\partial(p_0, \dots, p_m) / \partial(z_1, \dots, z_n)$$

is not of full rank $m+1$. Suppose the codimension of J is ≥ 1 in V . Then $R := P[n]/I$ is reduced.

Before we outline the idea of the proof of Serre's criterion of reducedness, let us first remark that (\ddagger') can be further transformed into a statement in terms of the important concept of *localization* in commutative algebra.

Definition 2. *Let R be a commutative ring with identity, and let $S \setminus \{0\}$ be a multiplicatively closed subset of R in the sense that $ab \in S$ for a and b in S . We define R_S to be the ring*

$$R_S := \{r/s : s \in S\}.$$

Here, r/s is the equivalence class of pairs (r, s) subject to the relation $(r_1, s_1) \sim (r_2, s_2)$ if there is an $t \in S$ such that $t(r_1s_2 - r_2s_1) = 0$.

The extra t in the definition is to ensure $r_1/s_1 = tr_2/ts_2$ if $r_1/s_1 = r_2/s_2$.

Example 7. *When $S = R \setminus \mathcal{P}$ for a prime ideal \mathcal{P} , the ring R_S is denoted instead by $R_{\mathcal{P}}$.*

$R_{\mathcal{P}}$ is a local ring in the sense that $\mathcal{P}_{\mathcal{P}}$ is its unique maximal ideal. To see this, we observe that $r \in R_{\mathcal{P}} \setminus \mathcal{P}_{\mathcal{P}}$ if and only if r is a unit in $R_{\mathcal{P}}$ (a unit a is one such that $ab = 1$ for some b). (Reason: $r = a/b$ with $a, b \in R \setminus \mathcal{P}$ so that $(a/b)(b/a) = 1$, and vice versa.) Moreover, any proper ideal I in $R_{\mathcal{P}}$ can never admit any unit, and so I must be contained in the ideal $\mathcal{P}_{\mathcal{P}}$.

r/s is regarded as a "rational function" of r divided by s , where s does not vanish on the irreducible variety defined by \mathcal{P} .

Example 8. *Recall that a commutative ring R with identity is a domain if it has no zero divisors. For an ideal I of R , the ring R/I is a domain if and only if I is a prime ideal.*

Assume R is a domain. Let $S := R \setminus \{0\}$. Then S is multiplicatively closed. R_S is a field called the quotient field of R .

Note that $R_{\mathcal{P}}/\mathcal{P}_{\mathcal{P}}$ is exactly the quotient field $\kappa(\mathcal{P})$ of the domain R/\mathcal{P} via the map

$$r/s \in R_{\mathcal{P}} \longmapsto (r + \mathcal{P}) / (s + \mathcal{P}) \in \kappa(\mathcal{P}).$$

Example 9. *More generally, let R be a commutative ring with identity, and let S be its subset of non-zero-divisors. Then S is multiplicatively closed. R_S is called the quotient ring of R , denoted by $Q(R)$.*

With Example 7, (\ddagger') can be rephrased as

- (•) The maximal ideal $\mathcal{P}_{\mathcal{P}} = 0$.

Example 10. Let us look at Example (5). The prime ideal $\mathcal{P} = (z_1, \dots, z_k)$ define the linear subspace $z_1 = \dots = z_k = 0$. Let $x = (z_{k+1}, \dots, z_n)$ and $y = (z_1, \dots, z_k)$. Any polynomial f can be Taylor expanded as

$$(3) \quad f(x, y) = f_0(x) + f_1(x)y + f_2(x)y^2 + \dots$$

with the obvious shorthand notation. Now, $P[n]_{\mathcal{P}}$ is the set of all rational functions f/g with f and g given as in (3) and $g_0 \neq 0$, while $\mathcal{P}_{\mathcal{P}}$ consists of f/g in $P[n]_{\mathcal{P}}$ with $f_0 = 0$. $P[n]_{\mathcal{P}}/\mathcal{P}_{\mathcal{P}}$ is the quotient field $\kappa(\mathcal{P})$ consisting of rational functions of the form $f_0(x)/g_0(x)$ with $g_0 \neq 0$.

Example 11. Continuing with the preceding example, for $f/g \in \mathcal{P}_{\mathcal{P}}$ with $f_0 = 0$, let us take the first differential restricted to $y = 0$ to obtain

$$(4) \quad \mathcal{P}_{\mathcal{P}} \longrightarrow \Omega^1(P[n]_{\mathcal{P}})|_{y=0}, \quad \frac{f}{g} \longmapsto d\left(\frac{f}{g}\right)|_{y=0} = \frac{f_1 dy}{g_0},$$

whose kernel consists of

$$\frac{f}{g}, \quad f = f_2 y^2 + f_3 y^3 + \dots = y^2 h \text{ for some } h \text{ so that } \frac{f}{g} \in (\mathcal{P}_{\mathcal{P}})^2.$$

Therefore, we have the injection

$$0 \longrightarrow \mathcal{P}_{\mathcal{P}}/(\mathcal{P}_{\mathcal{P}})^2 \xrightarrow{D} \Omega^1(P[n]_{\mathcal{P}})|_{y=0},$$

where D is induced by d . On the other hand, We have the natural projection

$$(5) \quad \begin{aligned} \Omega^1(P[n]_{\mathcal{P}})|_{y=0} &\xrightarrow{\pi} \Omega^1(\kappa(\mathcal{P})) \longrightarrow 0, \\ d\left(\frac{f}{g}\right)|_{y=0} &= d\left(\frac{f_0}{g_0}\right) + \frac{g_0 f_1 - f_0 g_1}{g_0^2} dy \longmapsto d\left(\frac{f_0}{g_0}\right), \end{aligned}$$

so that in fact we arrive at the exact sequence (called the conormal sequence)

$$(6) \quad 0 \longrightarrow \mathcal{P}_{\mathcal{P}}/(\mathcal{P}_{\mathcal{P}})^2 \xrightarrow{D} \Omega^1(P[n]_{\mathcal{P}})|_{y=0} \xrightarrow{\pi} \Omega^1(\kappa(\mathcal{P})) \longrightarrow 0$$

considered as vector spaces over the field $\kappa(\mathcal{P})$.

More generally, for $R = P[n]/I$ with $I = (p_0, \dots, p_m)$, consider the first differential

$$I \xrightarrow{d} R \otimes_{P[n]} \Omega^1(P[n]),$$

where

$$d : p_i \longmapsto 1 \otimes dp_i = 1 \otimes \sum_j \frac{\partial p_i}{\partial z_j} dz_j = \sum_j \frac{\partial p_i}{\partial z_j} (\text{mod } I) \otimes dz_j.$$

Since $dp_i^2 = 0$, we see d induces a map

$$I/I^2 \xrightarrow{D} R \otimes_{P[n]} \Omega^1(P[n]).$$

We wish to define the projection from $R \otimes_{P[n]} \Omega^1(P[n])$ to $\Omega^1(R)$. But what is the R -module $\Omega^1(R)$ of first differentials (officially called *Kaehler differentials*) for R , when the corresponding variety may have singularities?

The "quick-and-dirty" way, for our expository purpose, is just to define $\Omega^1(R)$ to be the cokernel of D (see [21, p. 180] for a formal definition). In accordance, we have thus the natural projection

$$(7) \quad R \otimes_{P[n]} \Omega^1(P[n]) \xrightarrow{\pi} \Omega^1(R)$$

given by

$$(8) \quad 1 \otimes dz_j \mapsto d(z_j + I) := dz_j + (\text{mod } dp_0, \dots, dp_m).$$

Hence we obtain

$$(9) \quad I/I^2 \xrightarrow{D} R \otimes_{P[n]} \Omega^1(P[n]) \xrightarrow{\pi} \Omega^1(R) \longrightarrow 0.$$

The sequence cannot be made left exact in general:

Example 12. Consider $I = (x^2, xy)$. We know $x^3 \in I$ and

$$D(x^3 + I^2) = 3x^2(\text{mod } I) \otimes dx = 0.$$

However, it can be easily checked that $x^3 \notin I^2$.

The striking fact is that (9) can be made exact if we localize, as in (6), when we replace $P[n]$ by $R_{\mathcal{P}}$, I by the maximal ideal $\mathcal{P}_{\mathcal{P}}$ of $R_{\mathcal{P}}$, and $R = P[n]/I$ by $R_{\mathcal{P}}/\mathcal{P}_{\mathcal{P}} = \kappa(\mathcal{P})$, the quotient field of the domain R/\mathcal{P} :

$$(10) \quad 0 \longrightarrow \mathcal{P}_{\mathcal{P}}/(\mathcal{P}_{\mathcal{P}})^2 \xrightarrow{D} \kappa(\mathcal{P}) \otimes_{R_{\mathcal{P}}} \Omega^1(R_{\mathcal{P}}) \xrightarrow{\pi} \Omega^1(\kappa(\mathcal{P})) \longrightarrow 0,$$

considered as vector spaces over $\kappa(\mathcal{P})$. Here,

$$(11) \quad \Omega^1(R_{\mathcal{P}}) := R_{\mathcal{P}} \otimes_R \Omega^1(R)$$

given by

$$d(r/s) := -\frac{r}{s^2} \otimes ds + \frac{1}{s} \otimes dr$$

with $\Omega^1(R)$ defined in (7). (In fact, the equality in (11) can be derived as a consequence of the formal definition of Kaehler differentials [21, p. 187]. We introduce it as a definition for the sake of expository convenience.)

The underlying idea for the validity of (10) is hidden in (6). Namely, as long as we have a left inverse

$$D^{-1} : \kappa(\mathcal{P}) \otimes_{R_{\mathcal{P}}} \Omega^1(R_{\mathcal{P}}) \longrightarrow \mathcal{P}_{\mathcal{P}}/(\mathcal{P}_{\mathcal{P}})^2$$

such that

$$D^{-1} \circ D = id,$$

then D is injective. Accordingly, given D^{-1} , one can define a morphism

$$(12) \quad \nabla : h \in R_{\mathcal{P}} \mapsto D^{-1}(1 \otimes dh) \in \mathcal{P}_{\mathcal{P}}/(\mathcal{P}_{\mathcal{P}})^2.$$

Intuitively, ∇ picks up the first order term of the Taylor expansion of h , which can be seen by looking at (5), where

$$D^{-1} : 1 \otimes d\left(\frac{f}{g}\right)\Big|_{y=0} \mapsto \frac{g_0 f_1 - f_0 g_1}{g_0^2} y \quad (\text{modulo higher order terms}),$$

so that

$$\nabla : \frac{f}{g} \longmapsto \frac{g_0 f_1 - f_0 g_1}{g_0^2} y \text{ (modulo higher order terms),}$$

where the right hand side is exactly the first order term of f/g when we expand it as

$$\frac{f}{g}(x, y) = \frac{f_0}{g_0} + \frac{g_0 f_1 - f_0 g_1}{g_0^2} y + \dots$$

With the intuitive interpretation in mind, it is clear that

$$(13) \quad h - \nabla(h) = 0 \in \mathcal{P}_{\mathcal{P}}/(\mathcal{P}_{\mathcal{P}})^2, \quad h \in \mathcal{P}_{\mathcal{P}}.$$

Returning to (12), therefore, the map

$$\iota : R_{\mathcal{P}} \longrightarrow R_{\mathcal{P}}/(\mathcal{P}_{\mathcal{P}})^2, \quad h \longmapsto h - \nabla(h)$$

intuitively picks up the 0th order term of h . Moreover, since $\iota(\mathcal{P}_{\mathcal{P}}) = 0$ by (13), it follows that ι descends to a map

$$\iota : R_{\mathcal{P}}/\mathcal{P}_{\mathcal{P}} \longrightarrow R_{\mathcal{P}}/(\mathcal{P}_{\mathcal{P}})^2, \quad h \longmapsto h - \nabla(h).$$

In other words, the exact sequence

$$(14) \quad 0 \longrightarrow \mathcal{P}_{\mathcal{P}}/(\mathcal{P}_{\mathcal{P}})^2 \longrightarrow R_{\mathcal{P}}/(\mathcal{P}_{\mathcal{P}})^2 \longrightarrow R_{\mathcal{P}}/\mathcal{P}_{\mathcal{P}} \longrightarrow 0$$

splits by ι as \mathbb{C} -algebras. Conversely, the splitting of the sequence establishes the existence of D^{-1} , so that (10) is true. We refer the reader to [21, p. 204] for a proof of (14).

We are now ready to see why (\bullet) holds true. Indeed, it suffices to verify, via (10), that, as vector spaces over $\kappa(\mathcal{P})$, the dimension of $\kappa(\mathcal{P}) \otimes_{R_{\mathcal{P}}} \Omega^1(R_{\mathcal{P}})$ equals that of $\Omega^1(\kappa(\mathcal{P}))$. Now, since $\kappa(\mathcal{P})$ is the quotient field of the domain R/\mathcal{P} , or rather, the rational function field of the underlying irreducible variety W , the Kähler module $\Omega^1(\kappa(\mathcal{P}))$ must be of the same dimension as that of W , which is $n - m - 1$, by the fact that p_0, \dots, p_m defining the variety V form a regular sequence so that V is of pure dimension $n - m - 1$. (See [21, p. 191] for a formal proof.) On the other hand, by (8), the image of D in (9) is of dimension $m + 1$, the generic rank of the Jacobian matrix J by assumption, as a vector space over $\kappa(\mathcal{P})$, so that $\Omega^1(R)$, the cokernel of D , is of dimension $n - m - 1$ as a vector space over $\kappa(\mathcal{P})$. Consequently, by (11), the dimension of $\kappa(\mathcal{P}) \otimes_{R_{\mathcal{P}}} \Omega^1(R_{\mathcal{P}})$ is $n - m - 1$.

We have thus arrived at Serre's criterion of reducedness.

2.2. Codimension 2 estimate and normality. We now turn to the second question as to under what condition a reduced ideal I generated by p_0, \dots, p_m in $P[n]$ is prime. Clearly, a necessary condition is that the variety V defined by I is connected. It turns out that the remaining condition sufficient for the primeness of I is the codimension 2 Jacobian condition [14, p. 462].

Theorem 2. (Serre) *Let I be the ideal generated by a regular sequence $p_0, \dots, p_m, m+1 \leq n$, in $P[n]$ that define a connected variety V . Let J be the subvariety of V consisting of all points of V where the Jacobian matrix*

$$\partial(p_0, \dots, p_m) / \partial(z_1, \dots, z_n)$$

is not of full rank $m+1$. Suppose the codimension of J is ≥ 2 in V . Then I is a prime ideal.

To outline the proof, note that V is reduced by Theorem 1. Let $p \in P[n]/I$ be a non-zero-divisor. Then p_0, \dots, p_m, p form a regular sequence, so that the ideal I^* generated by p_0, \dots, p_m, p , in view of (†), is the intersection of minimal primes Q_1, \dots, Q_t containing I^* ,

$$I^* = \bigcap_{j=1}^t Q_j,$$

and the algebraic set V^* defined by I^* is of pure dimension $n - m - 2$. Put intrinsically, this says that the (principal) ideal (p) generated by p in $R = P[n]/I$ is the intersection of minimal primes $P_j := Q_j/I$ containing p in R :

$$(15) \quad (p) = \bigcap_{j=1}^t P_j.$$

For ease of notation, let us denote any of the prime ideals P_j by P .

We claim that P_P is also generated by a single element by (10). The proof proceeds in a way entirely similar to the one given in the preceding section. First of all, $\Omega^1(\kappa(P)) = n - m - 2$ because the variety V^* is of pure dimension $n - m - 2$. Moreover, the middle space in (10), as a vector space over $\kappa(P)$, has the same dimension $n - m - 1$ as in the case of reducedness, because the codimension 2 condition and the fact that P defines an variety of codimension 1 imply that the image of D in (9) is still of dimension $m+1$; therefore, the dimension of $P_P/(P_P)^2$ is of dimension 1 as a vector space over $\kappa(P)$. This is equivalent to saying that the *minimum* number of generators of P_P is 1, which is a consequence of the fundamental Nakayama lemma whose proof we leave to [20, p. 105]. The claim follows.

So now $P_P = (f)$ in R_P . It follows that any element $x \in R_P$ is of the form $x = uf^n$ for some integer $n \geq 0$ and some unit $u \in R_P$, i.e., f is a *local uniformizing parameter* for R_P . Indeed, since the units of R_P constitute $R_P \setminus P_P$, an element $x \in R_P$ is either a unit, in which case we are done, or $x \in P_P = (f)$, in which case $x = ff_1$ for some $f_1 \in R_P$. Either f_1 is a unit and we are done, or $f_1 = ff_2$ for some $f_2 \in R_P$ with $x = f^2f_2$, etc. It follows that we have an ascending chain of ideals

$$(f_1) \subset (f_2) \subset (f_3) \cdots,$$

so that it must stabilize at some smallest n (the *Noetherian* condition; a ring with the condition is called a *Noetherian ring*). We obtain $x = f^n u$ for some unit u . With this there comes the following simple but important observation.

Proposition 1. *Let $Q(R_P)$ be the quotient ring of R_P . Suppose $a/b \in Q(R_P)$ satisfies a monic polynomial*

$$(16) \quad t^k + c_{k-1}t^{k-1} + \cdots + c_1t + c_0$$

in t , where $c_0, \dots, c_{k-1} \in R_P$. Then $a/b \in R_P$.

To see this, write $a = f^l u$ and $b = f^m v$ for some units $u, v \in R_P$. If $a/b \notin R_P$, then we have $m > l$ so that $a/b = w/f^s$ with $s > 0$ and w a unit, which we substitute into (16) to obtain

$$(w/f^s)^k + c_{k-1}(w/f^s)^{k-1} + \cdots + c_1(w/f^s) + c_0 = 0;$$

multiplying both sides by f^{sk} we derive $w^k = fg$ for some $g \in R_P$. This forces $f \in P_P$ to be a unit, which is a contradiction. So, we conclude that $a/b \in R_P$.

Corollary 1. *It follows that R satisfies the same property, namely, that if $q/p \in Q(R)$, the quotient ring of R , satisfies a monic polynomial with coefficients in R , then $q/p \in R$.*

Indeed, if $q/p \in Q(R) \setminus R$ for a non-zerodivisor p , then $q \notin (p)$. Following (15), we see $q \notin P_j$ for some j ; call it P for convenience. It follows that $q \notin P_P$ since q is a unit. Consequently, $q/p \notin R_P$; for otherwise $q/p = a/b$ implies $q = pa/b \in P_P$, a contradiction. But then q/p satisfies a monic polynomial in $Q(R_P)$, which is induced by the polynomial that q/p satisfies in $Q(R)$; therefore, by the preceding proposition $q/p \in R_P$, a contradiction.

In accordance with the corollary, we make the following definition.

Definition 3. *A reduced commutative ring R with identity is normal if whenever $x \in Q(R)$ satisfies a monic polynomial with coefficients in R , there follows $x \in R$.*

Then Serre's criterion of primeness is a consequence of the following:

Theorem 3. *A Noetherian normal ring R is a direct product of normal domains.*

To see that Theorem 3 implies Serre's criterion of normality, note first that the ring $R = P[n]/I$ under consideration is normal by Corollary 1. Thus Theorem 3 concludes that the variety V defined by I is a disjoint union of irreducible varieties, so that V must be irreducible itself because it is connected. In other words, I is a prime ideal.

On the other hand, Theorem 3 is a standard exercise in commutative algebra. We refer the reader to [20, pp. 85-86] for a proof.

Alternatively, we can understand normality from the function-theoretic point of view. Recall that a function f is *weakly holomorphic* in an open set O of V if it is holomorphic on $O \setminus \mathcal{S}$ and is locally bounded in O . Passing to the limit as O shrinks to a point p , we can talk about the germs of weakly holomorphic functions at p . The variety is said to be *normal* at p if

the germs of weakly holomorphic functions at p coincide with the germs of holomorphic functions at p . That is, the Riemann extension theorem holds true in the germs of neighborhoods around p . V is said to be *normal* if it is normal at all its points.

If V is normal, then its irreducible components are disconnected; or else a constant function with different values on different local irreducible branches, which is not even continuous, would give rise to a weakly holomorphic function that could be extended to a holomorphic function, a piece of absurdity. This is the geometric meaning of Theorem 3. See [17, p. 191] for details.

2.3. Algebraic independence of regular sequences. The Taylor expansion of (3) can be viewed as follows. Let $I = (z_1, \dots, z_k)$ be the ideal generated by the regular sequence z_1, \dots, z_k . In (3), we can think of

$$f_0(x) \in P[n]/I, \quad f_1(x)y \in I/I^2, \quad f_2(x)y^2 \in I^2/I^3, \quad \text{etc.}$$

(Precisely, y should be replaced by $y + I^2$.) On the other hand, we can also think of z_{k+1}, \dots, z_n as generating $P[n]/I$, so that $f_0(x), f_1(x), \dots \in P[n]/I$. Hence, the polynomial $f(x, y) \in P[n]$ written in (3) can also be thought of as a polynomial in k formal variables t_1, \dots, t_k with coefficients in $P[n]/I$, for which the expansion (3) is the evaluation when we set $t_1 = z_1, t_2 = z_2, \dots, t_k = z_k \in I/I^2$. In other words, there is an isomorphism

$$(17) \quad \begin{aligned} P[n]/I[t_1, \dots, t_k] &\longrightarrow P[n]/I \oplus I/I^2 \oplus I^2/I^3 \oplus \dots, \\ t_i &\longmapsto z_i + I^2, \end{aligned}$$

where the left hand side is the polynomial ring with coefficients in $P[n]/I$ and the direct sum module on the right hand side consists of elements whose components are zero eventually.

It turns out (17) is true for any regular sequence $p_1, \dots, p_k \in P[n]$ and (17) continues to hold when we replace z_1, \dots, z_k by p_1, \dots, p_k , respectively, with the evaluation map

$$(18) \quad t_i \longmapsto p_i + I^2, 1 \leq i \leq k.$$

Note that the evaluation in (18) is clearly surjective. Since a polynomial is the sum of its homogeneous terms, the injectivity of the evaluation comes down to proving the following:

Proposition 2. *Let $F(t_1, \dots, t_k)$ be a homogeneous polynomial of degree d in k variables t_1, \dots, t_k with coefficients in $P[n]$. Suppose the evaluation results in $F(p_1, \dots, p_k) \in I^{d+1}$. Then all the coefficients of F belong to $I = (p_1, \dots, p_k)$.*

Since any homogeneous element $f \in I^{d+1}$ can be written as a homogeneous $G(p_1, \dots, p_k)$ of degree d with coefficients in I , if we write $F \in I^{d+1}$ as a sum of homogeneous terms G_1, \dots, G_m of degrees $\geq d + 1$,

$$F(p_1, \dots, p_k) = G_1(p_1, \dots, p_k) + \dots + G_m(p_1, \dots, p_k),$$

and then regard each G_j as a homogeneous polynomial of degree d in t_1, \dots, t_k with coefficients in I , then

$$H(t_1, \dots, t_k) := F(t_1, \dots, t_k) - G_1(t_1, \dots, t_k) - \dots - G_m(t_1, \dots, t_k)$$

is homogeneous of degree d with $H(p_1, \dots, p_k) = 0$. If we can establish that this forces all the coefficients of H to be in I , it will follow that all the coefficients of F fall in I . Therefore, the above proposition is equivalent to:

Proposition 3. *Let p_1, \dots, p_k be a regular sequence in $P[n]$. Let $F(t_1, \dots, t_k)$ be a homogeneous polynomial of degree d in k variables with coefficients in $P[n]$. Suppose $F(p_1, \dots, p_k) = 0$. Then all the coefficients of F belong to $I = (p_1, \dots, p_k)$.*

We refer the reader to [20, p. 153] for a short proof. Let us look at an important application next.

2.4. Method for generating regular sequences. Granted Serre's criteria of reducedness and normality, checking that a sequence $p_0, \dots, p_m \in P[n]$ form a regular sequence is by no means easy. The first condition of forming a regular sequence is that $p_0 = \dots = p_m = 0$ defines a nonempty variety, or equivalently, that $(p_0, \dots, p_m) \neq P[n]$, which is already not that obvious to conclude. However, if we now stipulate that p_0, \dots, p_m all be homogeneous of degree ≥ 1 , then automatically $p_0 = \dots = p_m = 0$ defines a connected and nonempty variety V , because 0 clearly belongs to V and furthermore V is connected since it is a cone. Thus we can rephrase Serres's criterion of primeness in this case as follows:

Theorem 4. *Let I be the ideal generated by a regular sequence $p_0, \dots, p_m, m+1 \leq n$, of homogeneous polynomials of degree ≥ 1 in $P[n]$ that defines a variety V . Let J be the subvariety of V consisting of all points of V where the Jacobian matrix*

$$\partial(p_0, \dots, p_m) / \partial(z_1, \dots, z_n)$$

is not of full rank $m+1$. Suppose the codimension of J is ≥ 2 in V . Then I is a prime ideal.

From this we devised a criterion in [6] and developed it further in [8], [10] to construct regular sequences in $P[n]$ that fits perfectly in the classification scheme of isoparametric hypersurfaces.

Lemma 1. *Let $p_0, \dots, p_m \in P[n]$ be linearly independent homogeneous polynomials of equal degree ≥ 1 . For each $0 \leq k \leq m-1$, let V_k be the variety defined by $p_0 = \dots = p_k = 0$, and let J_k be the subvariety of V_k , where the Jacobian*

$$\partial(p_0, \dots, p_k) / \partial(z_1, \dots, z_n)$$

is not of full rank $k+1$. If the codimension of J_k in V_k is ≥ 2 for all $0 \leq k \leq m-1$, then p_0, \dots, p_m form a regular sequence.

Indeed, Theorem 4 applied to p_0 implies that p_0 is prime and clearly p_0 forms a regular sequence. So, the statement $S(k)$ that the ideal $I_k := (p_0, \dots, p_k)$ is prime and p_0, \dots, p_k form a regular sequence holds for $k = 0$.

Suppose the statement $S(k)$ holds. We show that p_{k+1} is not a zero divisor of $P[n]/I_k$. Let us assume

$$p_{k+1}f = p_0f_0 + p_1f_1 + \dots + p_kf_k$$

for some $f, f_0, \dots, f_k \in P[n]$. If p_{k+1} vanishes entirely on V_k , then $p_{k+1} \in I_k$ by Nullstellensatz as I_k is a prime ideal. But then

$$p_{k+1} = p_0g_0 + \dots + p_kg_k$$

for some $g_0, \dots, g_k \in P[n]$. However, since p_0, \dots, p_k, p_{k+1} are homogeneous of the same degree, we conclude that g_0, \dots, g_k are constants, which forces p_0, \dots, p_k, p_{k+1} to be linearly dependent. This is a contradiction. Thus p_{k+1} cannot vanish identically on V_k , which implies that f must vanish identically on V_k , so that $f \in I_k$. Now that p_{k+1} is not a zero divisor of $P[n]/I_k$, it follows that p_0, \dots, p_{k+1} form a regular sequence, which, together with the fact that J_{k+1} is of codimension 2 in V_{k+1} , make I_{k+1} a prime ideal by Theorem 4, so that the statement $S(k+1)$ is true, as long as $k \leq m-2$.

Lastly, when we reach that I_{m-1} is prime, the scheme results in the conclusion that p_0, \dots, p_m form a regular sequence.

2.5. The syzygy of a regular sequence. Let p_0, \dots, p_m be a sequence in $P[n]$. The ideal

$$\text{Syz} := \{(q_0, \dots, q_m) : p_0q_0 + \dots + p_mq_m = 0\}$$

is called the *first syzygy ideal* of p_0, \dots, p_m . Let $e_j := (0, \dots, 1, 0, \dots)$, where the only nonzero one ($= 1$) of the $m+1$ entries is at the j th slot. It is clear that $p_j e_i - p_i e_j \in \text{Syz}$. Syz is said to be *trivial* if it is generated by $p_j e_i - p_i e_j, i \neq j$, in which case all $(q_0, \dots, q_m) \in \text{Syz}$ are of the form

$$q_a = \sum_{b=0}^m r_{ab} q_b, \quad r_{ab} = -r_{ba}.$$

Proposition 4. *The first syzygy ideal generated by a regular sequence in $P[n]$ is trivial.*

Proof. Let p_0, \dots, p_m be a regular sequence. We do induction on m .

When $m = 1$, given $p_0f_0 + p_1f_1 = 0$, by the definition of a regular sequence, we know $f_1 = p_0h$ for some h . It follows that $f_0 = -p_1h$. The statement of the theorem is verified in this case.

Suppose the statement is true for $m = k$. For a regular sequence p_0, \dots, p_{k+1} ,

$$(19) \quad p_0f_0 + \dots + p_{k+1}f_{k+1} = 0$$

implies

$$(20) \quad f_{k+1} = r_{k+1,0}p_0 + \dots + r_{k+1,k}p_k$$

by the definition of a regular sequence. Substituting (20) into (19) we obtain

$$p_0(f_0 + p_{k+1} r_{k+1 0}) + \cdots + p_k(f_k + p_{k+1} r_{k+1 k}) = 0.$$

The induction hypothesis then ensures that

$$f_a + p_{k+1} r_{k+1 a} = \sum_{b=0}^k r_{ab} p_b, \quad r_{ab} = -r_{ba}, \quad 0 \leq a \leq k.$$

That is,

$$f_a = \sum_{b=0}^{k+1} r_{ab} p_b, \quad r_{ab} = -r_{ba}, \quad 0 \leq a \leq k+1,$$

where we define $r_{a k+1} := -r_{k+1 a}$, $0 \leq a \leq k$, with the latter defined in (20). \square

3. HOW THE IDEAL THEORY INTERACTS WITH ISOPARAMETRIC HYPERSURFACES

Through Münzner's work [24, II], we know the number g of principal curvatures of an isoparametric hypersurface M in the sphere is 1,2,3,4 or 6, and there are at most two multiplicities $\{m_1, m_2\}$ of the principal curvatures of M , occurring alternately when the principal curvatures are ordered, where $m_1 = m_2$ if g is odd. Over the ambient Euclidean space in which M sits there is a homogeneous polynomial F , called the Cartan-Münzner polynomial, of degree g that satisfies

$$|\nabla F|^2(x) = g^2 |x|^{2g-2}, \quad (\Delta F)(x) = (m_2 - m_1)g^2 |x|^{g-2}/2$$

whose restriction f to the sphere has image in $[-1, 1]$ with ± 1 the only critical values [24, I]. For any $c \in (-1, 1)$, the preimage $f^{-1}(c)$ is an isoparametric hypersurface with $f^{-1}(0) = M$. This 1-parameter of isoparametric hypersurfaces degenerates to the two submanifolds $f^{-1}(\pm 1)$ of codimension $m_1 + 1$ and $m_2 + 1$ in the sphere.

The isoparametric hypersurfaces with $g = 1, 2, 3$ were classified by Cartan to be homogeneous [3], [4]. For $g = 6$, it is known that $m_1 = m_2 = 1$ or 2 by Abresch [1]. Dorfmeister and Neher [13] showed that the isoparametric hypersurface is homogeneous in the former case and Miyaoka [22], [23] settled the latter.

For $g = 4$, there are infinite classes of inhomogeneous examples of isoparametric hypersurfaces, two of which were first constructed by Ozeki and Tackeuchi [25, I] to be generalized later by Ferus, Karcher and Münzner [15], referred to collectively as isoparametric hypersurfaces of OT-FKM type subsequently. We remark that the OT-FKM type includes all the homogeneous examples barring the two with multiplicities $\{2, 2\}$ and $\{4, 5\}$. To construct the OT-FKM type, let P_0, \dots, P_m be a Clifford system on \mathbb{R}^{2l} , which are orthogonal symmetric operators on \mathbb{R}^{2l} satisfying

$$P_i P_j + P_j P_i = 2\delta_{ij} I, \quad i, j = 0, \dots, m.$$

The 4th degree homogeneous polynomial

$$F(x) = |x|^4 - 2 \sum_{i=0}^m (\langle P_i(x), x \rangle)^2$$

is the Cartan-Münzner polynomial, where the angle brackets on the right hand side denote the Euclidean inner product. The two multiplicities of the OT-FKM type are m and $k\delta(m) - 1$ for any $k = 1, 2, 3, \dots$, where $\delta(m)$ is the dimension of an irreducible module of the Clifford algebra C_{m-1} with $l = k\delta(m)$. Stolz [26] showed that these multiplicity pairs and $\{2, 2\}$ and $\{4, 5\}$ are exactly the possible multiplicities of isoparametric hypersurfaces with four principal curvatures in the sphere.

To fix notation, we make the convention, by changing F to $-F$ if necessary, that its two focal manifolds are $M_+ := F^{-1}(1)$ and $M_- := F^{-1}(-1)$ with respective codimensions $m_1 + 1 \leq m_2 + 1$ in the ambient sphere $S^{2(m_1+m_2)+1}$. The principal curvatures of the shape operator S_n of M_+ (vs. M_-) with respect to any unit normal n are 0, 1 and -1 , whose multiplicities are, respectively, m_1, m_2 and m_2 (vs. m_2, m_1 and m_1).

The third fundamental form of M_+ is the symmetric tensor

$$q(X, Y, Z) := (\nabla_X^\perp S)(Y, Z)/3$$

where ∇^\perp is the normal connection. For a chosen normal frame n_0, \dots, n_{m_1} write

$$p_a(X, Y) := \langle S(X, Y), n_a \rangle, \quad q_a(X, Y, Z) = \langle q(X, Y, Z), n_a \rangle, \quad 0 \leq a \leq m_1.$$

The Cartan-Münzner polynomial F is related to p_a and q_a by the expansion formula of Ozeki and Takeuchi [25, I, p. 523]

$$(21) \quad \begin{aligned} F(tx + y + w) &= t^4 + (2|y|^2 - 6|w|^2)t^2 + 8\left(\sum_{i=0}^{m_1} p_i w_i\right)t \\ &+ |y|^4 - 6|y|^2|w|^2 + |w|^4 - 2\sum_{i=0}^{m_1} p_i^2 - 8\sum_{i=0}^{m_1} q_i w_i \\ &+ 2\sum_{i,j=0}^{m_1} \langle \nabla p_i, \nabla p_j \rangle w_i w_j \end{aligned}$$

where $w := \sum_{i=0}^{m_1} w_i n_i$, y is tangential to M_+ at x , $p_i := p_i(y, y)$, $q_i := q_i(y, y, y)$ and ∇ is the Euclidean gradient. Note that our definition of q_i differs from that of Ozeki and Takeuchi by a sign. An entirely similar formula holds when m_1 is replaced by m_2 .

In the expansion formula, the components of the second and third fundamental forms are intertwined in ten convoluted equations. The first three say that the shape operator S_n satisfies $(S_n)^3 = S_n$ for any normal direction n , which is agreeable with the fact that the eigenvalues of S_n are 0, 1, -1

with fixed multiplicities. Set

$$\langle p_a, q_b \rangle := \langle \nabla p_a, \nabla q_b \rangle, \quad 0 \leq a, b \leq m_1.$$

The fourth and fifth combined and the sixth are

$$\begin{aligned} \langle p_a, q_b \rangle + \langle p_b, q_a \rangle &= 0, \\ \langle \langle p_a, p_b \rangle, q_c \rangle + \langle \langle p_c, p_a \rangle, q_b \rangle + \langle \langle p_b, p_c \rangle, q_a \rangle &= 0, \quad a, b, c \text{ distinct.} \end{aligned}$$

The seventh is

$$(22) \quad p_0 q_0 + \cdots + p_{m_1} q_{m_1} = 0.$$

Set $G := \sum_{a=0}^{m_1} (p_a)^2$. The last three are

$$\begin{aligned} 16 \sum_{a=0}^{m_1} (q_a)^2 &= 16G |y|^2 - \langle G, G \rangle, \\ 8 \langle q_a, q_a \rangle &= 8(\langle p_a, p_a \rangle |y|^2 - (p_a)^2) + \langle \langle p_a, p_a \rangle, G \rangle - 24G \\ (23) \quad -2 \sum_{b=0}^{m_1} \langle p_a, p_b \rangle^2, \\ 8 \langle q_a, q_b \rangle &= 8(\langle p_a, p_b \rangle |y|^2 - p_a p_b) + \langle \langle p_a, p_b \rangle, G \rangle \\ -2 \sum_{c=0}^{m_1} \langle p_a, p_c \rangle \langle p_b, p_c \rangle, \quad a, b \text{ distinct.} \end{aligned}$$

It looks at the first glance that it is a rather daunting task to tackle the classification of isoparametric hypersurfaces with four principal curvatures in the sphere. However, (22), which appears to be the simplest of all the above equations, brings good tidings.

Let us bring Proposition 4 into perspective. Suppose now the components p_0, \dots, p_{m_1} of the second fundamental form constitute a regular sequence. Then Proposition 4 warrants that the components q_0, \dots, q_{m_1} of the third fundamental form satisfy

$$(24) \quad q_a = \sum_{b=0}^{m_1} r_{ab} p_b,$$

where $r_{ab} = -r_{ba}$ are homogeneous of degree 1.

Now let us introduce the Euclidean coordinates of the eigenspaces V_+, V_-, V_0 , with eigenvalues 1, -1, 0, respectively, of the shape operator S_{n_0} to be

$$\begin{aligned} z_p, \quad m_1 + 1 \leq p \leq 2m_1, \\ u_\alpha, \quad 2m_1 + 1 \leq \alpha \leq 2m_1 + m_2, \\ v_\mu, \quad 2m_1 + m_2 + 1 \leq \mu \leq 2m_1 + 2m_2, \end{aligned}$$

with respect to which we write

$$(25) \quad r_{ab} := \sum_{\alpha} T_{ab}^{\alpha} u_{\alpha} + \sum_{\mu} T_{ab}^{\mu} v_{\mu} + \sum_p T_{ab}^p z_p.$$

We have

$$(26) \quad \begin{aligned} p_0 &= \sum_{\alpha} (u_{\alpha})^2 - \sum_{\mu} (v_{\mu})^2, \\ p_a &= 2 \sum_{\alpha\mu} S_{\alpha\mu}^a u_{\alpha} v_{\mu} + 2 \sum_{\alpha p} S_{\alpha p}^a u_{\alpha} z_p + 2 \sum_{\mu p} S_{\mu p}^a v_{\mu} z_p, \end{aligned}$$

for $1 \leq a \leq m_1$, where we set

$$S_{\alpha\mu}^a := \langle S(X_{\alpha}, Y_{\mu}), n_a \rangle,$$

etc., with X_{α}, Y_{μ} , and Z_p the orthonormal bases for the coordinates u_{α}, v_{μ} , and w_p , respectively. We claim that

$$(27) \quad T_{a0}^{\alpha} = T_{a0}^{\mu} = 0,$$

for $1 \leq a \leq m_1$. To this end, we calculate q_a in two ways. On the one hand, substituting (25) and (26) into (24), we see that q_a has the term

$$\left(\sum_{\alpha} T_{a0}^{\alpha} u_{\alpha} \right) \left(\sum_{\beta} (u_{\beta})^2 \right) + \cdots,$$

so that the coefficient of $(u_{\alpha})^3$ in q_a , denoted by $q_a^{\alpha\alpha\alpha}$, is

$$q_a^{\alpha\alpha\alpha} = T_{a0}^{\alpha}.$$

On the other hand, by a direct inspection, the right hand side of the first identity of (23) has no $(u_{\alpha})^6$ -term, so that $q_a^{\alpha\alpha\alpha} = 0$.

Next, we calculate q_0 in two ways. On the one hand, we expand q_0 by (24), (25), (26), and (27), keeping in mind that q_0 is homogeneous of degree 1 in u_{α}, v_{μ} and z_p , by [25, I, p. 537], to obtain that the coefficient of the $u_{\alpha} v_{\mu} z_p$ -term of q_0 , denoted by $q_0^{\alpha\mu p}$, is

$$(28) \quad q_0^{\alpha\mu p} = 2 \sum_{b \geq 1} T_{0b}^p S_{\alpha\mu}^b.$$

On the other hand, traversing along the great circle spanned by x and n_0 by length $\pi/2$, we end up again on M_+ at n_0 with x as a normal vector. Accordingly, set $x^{\#} := n_0 \in M_+$ and $\mathbf{n}_0^{\#} := x$ normal to M_+ at $x^{\#}$.

At $x^{\#}$, set

$$\begin{aligned} t^{\#} &= w_0, & u_1^{\#} &= u_1, \cdots, u_{m_2}^{\#} = u_{m_2}, & v_1^{\#} &= v_1, \cdots, v_{m_2}^{\#} = v_{m_2}, \\ z_1^{\#} &= w_1, \cdots, z_{m_1}^{\#} = w_{m_1}, & w_0^{\#} &= t, w_1^{\#} = z_1, \cdots, w_{m_1}^{\#} = z_{m_1}. \end{aligned}$$

Then with $|y|^2 = |u|^2 + |v|^2 + |z|^2$, it is easily checked that F in (21) will be converted to

$$(t^\#)^4 + (2|y^\#|^2 - 6|w^\#|^2)(t^\#)^2 + |y^\#|^4 - 6|y^\#|^2|w^\#|^2 + |w^\#|^4 + \dots .$$

In other words, the eigenspaces $V_+^\#, V_-^\#, V_0^\#$ of $S_{n_0^\#}$ with eigenvalues $1, -1, 0$ are, respectively, $V_+, V_-, n_0^\perp := \text{span}(n_1, \dots, n_{m_1})$. Moreover, $\mathbb{R}x \oplus V_0$ is the normal space to M_+ at $x^\#$. (See [6, p. 15] for a geometric proof.)

Note that the third term of (21) at $x^\#$, which is

$$8\left(\sum_{a=0}^{m_1} p_a^\# w_a^\#\right) t^\#,$$

is what determines the second fundamental form $S^\#$ at $x^\#$; in fact, only $-8q_0w_0$ of (21) at x , when substituted by the $\#$ -quantities, contributes to the $u_\alpha v_\mu$ -components of $S^\#$. So, expanding $-8q_0w_0$ in z_1, \dots, z_{m_1} , we obtain

$$(29) \quad 8q_0w_0 = 8\left(\sum_p H^p z_p\right)w_0 = 8\left(\sum_p H^p w_p^\#\right)t^\#,$$

where

$$(30) \quad H^p := 2 \sum_{\alpha\mu} S_{\alpha\mu}^p u_\alpha v_\mu,$$

and S_{ij}^p denotes the tangential (ij) -component of the second fundamental form of M_+ in the normal p -direction at $x^\#$. Here, we invoke again the fact that q_0 is homogeneous of degree 1 in all x_α, y_μ, z_p .

Comparing (28), (29) and (30) we derive

$$S_{\alpha\mu}^p = \sum_b f_b^p S_{\alpha\mu}^b, \quad f_b^p = T_{0b}^p.$$

Therefore, we may assume, with the index range $m_1 + 1 \leq p \leq 2m_1$, that

$$(31) \quad S_{\alpha\mu}^{a+m_1} = S_{\alpha\mu}^a,$$

by an orthonormal frame change, so long as we can show that the matrix (f_b^p) is orthogonal. Remarkably, this is indeed true! The key is the second identity of (23), where we can employ the commutative algebra scheme Proposition 3 to rewrite it as a polynomial homogeneous in all $p_a p_b$ whose coefficients are homogeneous polynomials of degree 2, so that these coefficients are linear combinations of all p_a . Specifically, the coefficient of $(p_0)^2$ is

$$16 \sum_{a=1}^{m_1} (r_{0a})^2 - 16 \left(\sum_{\alpha} (u_\alpha)^2 + \sum_{\mu} (v_\mu)^2 + \sum_p (z_p)^2 \right) + 4 \langle p_0, p_0 \rangle,$$

which is a linear combination of p_0, p_1, \dots, p_{m_1} . Knowing that r_{0a} are functions of z_p alone by (25) and (27), we invoke (26) and compare variable types

to conclude that

$$(32) \quad \sum_{a=1}^{m_1} (r_{0a})^2 = \sum_{p=m_1+1}^{2m_1} (z_p)^2.$$

But then (25) for r_{0a} in terms of (32) says exactly that the matrix (f_b^p) is orthogonal.

Now that

$$f_b^{a+m_1} = \delta_b^a$$

for (31) to hold, we deduce by (25) and (27)

$$r_{0b} = \sum_a \delta_b^a z_{a+m_1} = z_{b+m_1},$$

and, invoking the Einstein summation convention,

$$\begin{aligned} q_0 &= r_{0b} p_b \\ &= 2(\delta_b^a z_{a+m_1})(S_{\alpha\mu}^b u_\alpha v_\mu + S_{\alpha c+m_1}^b u_\alpha z_{c+m_1} + S_{\mu c+m_1}^b v_\mu z_{c+m_1}). \end{aligned}$$

Hence, we obtain

$$\sum_{abc\alpha} (\delta_b^a z_{a+m_1})(S_{\alpha c+m_1}^b u_\alpha z_{c+m_1}) = 0$$

or equivalently,

$$\sum_{ac} S_{\alpha c+m_1}^a z_{c+m_1} z_{a+m_1} = 0.$$

In other words, we have

$$(33) \quad S_{\alpha c+m_1}^a = -S_{\alpha a+m_1}^c.$$

Likewise, we have

$$(34) \quad S_{\mu c+m_1}^a = -S_{\mu a+m_1}^c.$$

It is evident now that (31), (33), and (34) enjoy a certain "Clifford" property. In fact, as shown in [7], the geometric meaning of these three equations is that they give rise to intrinsic isometries on M_+ that exactly form the *Spin*-action on M_+ in the case when the isoparametric hypersurface is of OT-FKM type. Moreover, we showed in [8], based on [6], [7], that if we assume the mild condition that $m_1 < m_2$, which essentially says that M_+ is sufficiently curved, then these intrinsic isometries extend to extrinsic isometries of the ambient sphere to yield the OT-FKM type:

Proposition 5. *Let $m_1 < m_2$. If (31), (33), and (34) hold, then the hypersurface is of OT-FKM type. In particular, if $m_1 < m_2$ and the components of the second fundamental form p_0, p_1, \dots, p_{m_1} of M_+ form a regular sequence, then the isoparametric hypersurface is of OT-FKM type.*

By this proposition, the classification of isoparametric hypersurfaces with four principal curvatures now boils down to exploring Lemma 1 to warrant that the components p_0, \dots, p_{m_1} of the second fundamental form of M_+ constitute a regular sequence. To this end, let us look at the $p_0, \dots, p_k, k \leq m_1 - 1$. Following Lemma 1 we must estimate the codimension of J_k in V_k by understanding the rank of the Jacobian matrix of p_0, \dots, p_k .

Let us parametrize $\mathbb{C}^{2m_2+m_1}$ by points (u, v, w) with coordinates u_α, v_μ , and w_p , where $1 \leq \alpha, \mu \leq m_2$, and $1 \leq p \leq m_1$. For $0 \leq k \leq m_1$, let

$$V_k := \{(u, v, w) \in \mathbb{C}^{2m_2+m_1} : p_0(u, v, w) = \dots = p_k(u, v, w) = 0\}.$$

We first estimate the dimension of the subvariety X_k of $\mathbb{C}^{2m_2+m_1}$, where

$$X_k := \{(u, v, w) \in \mathbb{C}^{2m_2+m_1} : \text{rank of the Jacobian of } p_0, \dots, p_k < k + 1\}.$$

This amounts to saying that dp_0, \dots, dp_k are linearly dependent, or, that there are constants c_0, \dots, c_k such that

$$(35) \quad c_0 dp_0 + \dots + c_k dp_k = 0.$$

Since $p_a = \langle S_a(x), x \rangle$, we see $dp_a = 2\langle S_a(x), dx \rangle$ for $x = (u, v, w)^{tr}$; therefore, by (35)

$$X_k = \{(u, v, w) : (c_0 S_0 + \dots + c_k S_k) \cdot (u, v, w)^{tr} = 0\}.$$

for $[c_0 : \dots : c_k] \in \mathbb{C}P^k$. Here, $\langle S_a(X), Y \rangle = \langle S(X, Y), n_a \rangle$ is the shape operator of the focal manifold M_+ in the normal direction n_a . By Lemma 1, we wish to establish

$$\dim(X_k \cap V_k) \leq \dim(V_k) - 2$$

for $k \leq m_1 - 1$ to verify that p_0, p_1, \dots, p_{m_1} form a regular sequence since

$$(36) \quad J_k = X_k \cap V_k.$$

Note that for a fixed $\lambda = [c_0 : \dots : c_k] \in \mathbb{C}P^k$, if we set

$$\mathcal{S}_\lambda := \{(u, v, w) : (c_0 S_0 + \dots + c_k S_k) \cdot (u, v, w)^{tr} = 0\},$$

then we have

$$(37) \quad X_k = \cup_{\lambda \in \mathbb{C}P^k} \mathcal{S}_\lambda.$$

Thus, it is fundamental to estimate the dimension of \mathcal{S}_λ .

We break it into two cases. If c_0, \dots, c_k are either all real or all purely imaginary, then

$$\dim(\mathcal{S}_\lambda) = m_1,$$

since $c_0 S_{n_0} + \dots + c_k S_{n_k} = c S_n$ for some unit normal vector n and some nonzero real or purely imaginary constant c , and we know that the null space of S_n is of dimension m_1 for all normal n .

On the other hand, if c_0, \dots, c_k are not all real and not all purely imaginary, then after a normal basis change, we can assume that

$$(38) \quad \mathcal{S}_\lambda = \{(u, v, w) : (S_{1^*} - \mu_\lambda S_{0^*}) \cdot (u, v, w)^{tr} = 0\}$$

for some complex number μ_λ relative to a new orthonormal normal basis $n_0^*, n_1^*, \dots, n_k^*$ in the linear span of n_0, n_1, \dots, n_k ; explicitly, n_0^* and n_1^* are obtained by decomposing $n := c_0 n_0 + \dots + c_k n_k$ into its real and imaginary parts $n = \alpha + \sqrt{-1}\beta$ and define n_0^* and n_1^* by performing the Gram-Schmidt process.

In matrix terms, the equation in (38) assumes the form

$$(39) \quad \begin{pmatrix} 0 & A & B \\ A^{tr} & 0 & C \\ B^{tr} & C^{tr} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mu_\lambda \begin{pmatrix} I & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

where x, y , and z are (complex) eigenvectors of S_{0^*} with eigenvalues 1, -1 , and 0, respectively.

Suffices it to say, leaving the details to [8], that [6, Lemma 49, p. 64] ensures that we can normalize the matrix on the left hand side of (39) to decompose x, y, z into $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2)$ with $x_2, y_2, z_2 \in \mathbb{C}^{r_\lambda}$, where r_λ is the rank of B , or intrinsically, $m_1 - r_\lambda$ is the dimension of the intersection of the kernels of S_{0^*} and S_{1^*} . With respect to this decomposition either $x_1 = y_1 = 0$ or both are nonzero with $\mu_\lambda = \pm\sqrt{-1}$. In both cases we have $x_2 = -y_2$ and can be solved in z_2 so that z can be chosen to be a free variable. Hence, either $x_1 = y_1 = 0$, in which case

$$\dim(\mathcal{S}_\lambda) = m_1,$$

or both x_1 and y_1 are nonzero, in which case $y_1 = \pm\sqrt{-1}x_1$ and so

$$(40) \quad \dim(\mathcal{S}_\lambda) = m_1 + m_2 - r_\lambda,$$

where x_1 contributes dimension $m_2 - r_\lambda$ while z does m_1 . Now since by (36) and (37)

$$(41) \quad J_k = X_k \cap V_k = \cup_{\lambda \in \mathbb{C}P^k} (\mathcal{S}_\lambda \cap V_k),$$

where V_k is defined by $p_0 = \dots = p_k = 0$ and also by $p_{0^*} = \dots = p_{k^*}$, let us cut \mathcal{S}_λ by

$$0 = p_{0^*} = \sum_{\alpha} (x_\alpha)^2 - \sum_{\mu} (y_\mu)^2$$

to achieve an initial estimate of $\dim(J_k)$.

Case 1: x_1 and y_1 are both nonzero. This is the case of nongeneric $\lambda \in \mathbb{C}P^k$. We substitute $y_1 = \pm\sqrt{-1}x_1$ and x_2 and y_2 in terms of z_2 into $p_{0^*} = 0$ to deduce that

$$0 = p_{0^*} = (x_1)^2 + \dots + (x_{m_2 - r_\lambda})^2 + z \text{ terms};$$

hence, $p_{0^*} = 0$ cuts \mathcal{S}_λ to reduce the dimension by 1. That is, now by (40),

$$(42) \quad \dim(V_k \cap \mathcal{S}_\lambda) \leq (m_1 + m_2 - r_\lambda) - 1 \leq m_1 + m_2 - 1,$$

noting that V_k is also cut out by $p_{0^*}, p_{1^*}, \dots, p_{k^*}$. Meanwhile, only a subvariety of λ of dimension $k-1$ in $\mathbb{C}P^k$ assumes $\mu_\lambda = \pm\sqrt{-1}$; in fact, this subvariety is a smooth hyperquadric \mathcal{Q}_{k-1} in $\mathbb{C}P^k$. This is because if we write $(c_0, \dots, c_k) = \alpha + \sqrt{-1}\beta$ where α and β are real vectors, then $\mu_\lambda = \pm\sqrt{-1}$ is equivalent to the conditions that $\langle \alpha, \beta \rangle = 0$ and $|\alpha|^2 = |\beta|^2$. That is, the nongeneric $\lambda \in \mathbb{C}P^k$ constitute the smooth hyperquadric. Therefore, by (41), an irreducible component \mathcal{W} of J_k over nongeneric λ will satisfy

$$\dim(\mathcal{W}) \leq \dim(V_k \cap \mathcal{S}_\lambda) + k - 1 \leq m_1 + m_2 + k - 2.$$

(Total dimension \leq base dimension + fiber dimension.)

Case 2: $x_1 = y_1 = 0$. This is the case of generic λ , where $\dim(\mathcal{S}_\lambda) = m_1$, so that an irreducible component \mathcal{V} of J_k over generic λ will satisfy

$$\dim(\mathcal{V}) \leq m_1 + k \leq m_1 + m_2 + k - 2,$$

as we may assume that $m_2 \geq 2$, noting that the case $m_1 = m_2 = 1$ is straightforward [6, p. 61].

Putting these two cases together, we conclude that

$$(43) \quad \dim(J_k) = \dim(X_k \cap V_k) \leq m_1 + m_2 + k - 2.$$

On the other hand, since V_k is cut out by $k+1$ equations $p_0 = \dots = p_k = 0$, we have

$$(44) \quad \dim(V_k) \geq m_1 + 2m_2 - k - 1.$$

Therefore,

$$(45) \quad \dim(J_k) \leq \dim(V_k) - 2$$

when $k \leq m_1 - 1$, taking $m_2 \geq 2m_1 - 1$ into account.

In summary, we have established (45) for $k \leq m_1 - 1$, so that the ideal (p_0, p_1, \dots, p_k) is prime when $k \leq m_1 - 1$. Lemma 1 then implies that p_0, p_1, \dots, p_{m_1} form a regular sequence. It follows by Proposition 5 that the isoparametric hypersurface is of OT-FKM type. Thus, we derived in [8] the classification proven in [6] in a simpler fashion:

Theorem 5. *Assume $m_2 \geq 2m_1 - 1$. Then the isoparametric hypersurface with four principal curvatures is of OT-FKM type.*

By the multiplicity result of Stolz [26], which says that (m_1, m_2) is either $(2, 2)$, $(4, 5)$ or that of an isoparametric hypersurface of OT-FKM type, Theorem 5 finishes off all the isoparametric hypersurfaces with four principal curvatures, except when $(m_1, m_2) = (3, 4), (4, 5), (6, 9)$ or $(7, 8)$. The class of isoparametric hypersurfaces in the theorem are tied with *complete*

intersections, i.e., those polynomial ideals generated by regular sequences. In sharp contrast, the four remaining cases have the peculiar property, due to the fact that they are tied with quaternion and octonion algebras, that p_0, \dots, p_{m_1} fail to be regular sequences; for if they formed a regular sequence, Proposition 5 would imply that the isoparametric hypersurface was to be of OT-FKM type where the Clifford action acted on M_+ . However, such an isoparametric hypersurface can never be of OT-FKM type when $(m_1, m_2) = (4, 5)$, whereas for $(m_1, m_2) = (3, 4), (6, 9)$ or $(7, 8)$, there are examples in the same ambient sphere where the Clifford action acts on M_- . This is a contradiction. Thus, p_0, \dots, p_{m_1} cannot be regular. Irregular sequences, even over complex numbers, can be wildly untamed.

It turns out that *Condition A* of Ozeki and Takeuchi plays a decisive role in handling the exceptional cases when the multiplicity pair is $(m_1, m_2) = (3, 4), (4, 5)$ or $(6, 9)$.

Definition 4. *A point $p \in M_+$ is of Condition A if S_n at p share the same kernel in all normal directions n .*

Originally Ozeki and Takeuchi [25, I] introduced Conditions A and B in their construction of two families of inhomogeneous isoparametric hypersurfaces with four principal curvatures with multiplicity pair $(m_1, m_2) = (3, 4k)$ or $(7, 8k)$ for $k \geq 1$, where the Clifford action acts on M_+ . Later, Dorfmeister and Neher [12] showed that Condition A alone implies that the isoparametric hypersurface is of OT-FKM type (see also [9]); in particular, in the case when $(m_1, m_2) = (3, 4)$ or $(7, 8)$, either the Clifford action acts on M_+ , which are the ones constructed by Ozeki and Takeuchi, or it acts on M_- , which are the ones constructed by Ferus, Karcher and Münzner. In contrast, those isoparametric hypersurfaces with $(m_1, m_2) = (4, 5)$ or $(6, 9)$ do not admit any points of Condition A.

We must now come up with a finer estimate on the right hand side of (42) in which the quantity r_λ is entirely discarded. This is where Condition A comes in.

Note that if we stratify the above hyperquadric \mathcal{Q}_{k-1} of nongeneric $\lambda \in \mathbb{C}P^k$ into subvarieties \mathcal{L}_j over which $r_\lambda = j$, then by (42) an irreducible component \mathcal{W} of $V_k \cap (\cup_{\lambda \in \mathcal{L}_j} \mathcal{S}_\lambda)$ will satisfy

$$\dim(\mathcal{W}) \leq \dim(V_k \cap \mathcal{S}_\lambda) + k - 1 \leq m_1 + m_2 + k - 2 - j.$$

We run through the same arguments as that following (42) to deduce that the codimension 2 estimate (45) holds true over \mathcal{L}_j when

$$(46) \quad m_2 \geq 2k + 1 - j.$$

Let us look at the case when $(m_1, m_2) = (3, 4)$. Here, $0 \leq k \leq m_1 - 1 = 2$. First observe that (46) is automatically satisfied when $j \geq 1$. Assume $j = 0$ now; let λ_0 be an element in \mathcal{L}_0 and so $r_{\lambda_0} = j = 0$.

Suppose that M_+ is free of points of Condition A everywhere. Since $r_{\lambda_0} = 0$, the matrices $B = C = 0$ and $A = I$ in (39) for S_{1^*} . For notational

clarity, let us denote the associated B and C blocks of the shape operator matrices $S_{n_a^*}$ by B_{a^*} and C_{a^*} for the normal basis elements $n_1^*, \dots, n_{m_1}^*$. It follows that $p_{0^*} = 0$ and $p_{1^*} = 0$ cut \mathcal{S}_{λ_0} in the variety

$$\{(x, \pm\sqrt{-1}x, z) : \sum_{\alpha} (x_{\alpha})^2 = 0\}.$$

(B_{2^*}, C_{2^*}) or (B_{3^*}, C_{3^*}) must be nonzero since M_+ has no points of Condition A; assume it is the former. Since z is a free variable, $p_{2^*} = 0$ will have nontrivial z -terms

$$\begin{aligned} 0 = p_{2^*} &= \sum_{\alpha p} S_{\alpha p} x_{\alpha} z_p + \sum_{\mu p} T_{\mu p} y_{\mu} z_p + x_{\alpha} y_{\mu} \text{ terms} \\ &= \sum_{\alpha p} (S_{\alpha p} \pm \sqrt{-1} T_{\alpha p}) x_{\alpha} z_p + x_{\alpha} x_{\mu} \text{ terms,} \end{aligned}$$

taking $y = \pm\sqrt{-1}x$ into account, where $S_{\alpha p} := \langle S(X_{\alpha}^*, Z_p^*), n_2^* \rangle$ and $T_{\mu p} := \langle S(Y_{\mu}^*, Z_p^*), n_2^* \rangle$ are (real) entries of B_{2^*} and C_{2^*} , respectively, and $X_{\alpha}^*, 1 \leq \alpha \leq m_2, Y_{\mu}^*, 1 \leq \mu \leq m_2,$ and $Z_p^*, 1 \leq p \leq m_1,$ are orthonormal eigenvectors for the eigenspaces of $S_{n_0^*}$ with eigenvalues $1, -1,$ and $0,$ respectively; hence, the dimension of \mathcal{S}_{λ_0} will be cut down by 2 by $p_{0^*}, p_{1^*}, p_{2^*} = 0$. In conclusion, modifying (42) we have

$$\dim(V_2 \cap \mathcal{S}_{\lambda}) \leq m_1 + m_2 - 2,$$

for all $\lambda \in \mathcal{L}_0$. As a consequence, the right hand side of (46), which is 5 for $j = 0$, is now cut down to 4 with the additional $p_{2^*} = 0$ so that the codimension 2 estimate goes through for \mathcal{L}_0 as well. It follows that the isoparametric hypersurface is in fact the example constructed by Ozeki and Takeuchi of OT-FKM type, which thus has points of Condition A, a contradiction to the assumption that M_+ has no points of Condition A. Therefore, M_+ admits points of Condition A. But then the result of Dorfmeister and Neher implies the isoparametric hypersurface is of OT-FKM type [8]:

Theorem 6. *Let $(m_1, m_2) = (3, 4)$. Then the isoparametric hypersurface is either the homogeneous one, or is the inhomogeneous one constructed by Ozeki and Takeuchi.*

For $(m_1, m_2) = (4, 5)$ (vs. $(m_1, m_2) = (6, 9)$) and $0 \leq k \leq m_1 - 1 = 3$ (vs. $0 \leq k \leq m_1 - 1 = 5$), *a priori* (46) gives $5 \geq 7 - j \geq 2k + 1 - j$ (vs. $9 \geq 11 - j \geq 2k + 1 - j$). Therefore, the codimension 2 estimate goes through for $j \geq 2$ in both cases. Thus it looks hopeful that one will only have to handle $j \leq 1$ for the classification. Indeed, this is so. Employing the fact that M_+ admits no points of Condition A in the case of these two multiplicity pairs, a delicate analysis was performed in [10] to establish that either the isoparametric hypersurface is the inhomogeneous one constructed by Ferus, Karcher and Münzner in the $(6, 9)$ case where the Clifford action acts on M_+ , or the second fundamental form of M_+ is exactly that of the homogeneous example

in either case. The classification result follows by pinning down the third fundamental form to determine uniquely the Cartan-Münzner polynomial via the expansion formula of Ozeki and Takeuchi, where (23) plays a decisive role [10]:

Theorem 7. *Let $(m_1, m_2) = (4, 5)$ or $(6, 9)$. Then the isoparametric hypersurface with four principal curvatures is either homogeneous, or is the inhomogeneous one constructed by Ferus, Karcher and Münzner in the latter case.*

$(m_1, m_2) = (7, 8)$ appears to be the most subtle case of all. Unlike the other three cases where either the isoparametric hypersurface is homogeneous for $(m_1, m_2) = (4, 5)$, or one is homogeneous and the other is not for $(m_1, m_2) = (3, 4)$ or $(6, 9)$, the three known examples in this last case are all inhomogeneous and are intertwined with the nonassociativity of the octonion algebra. Meanwhile, with $0 \leq k \leq m_1 - 1 = 6$, *a priori* (46) gives $8 \geq 13 - j \geq 2k + 1 - j$; this becomes much more entangled than the previous cases, as we have $j \leq 4$ to handle. To be able to effectively handle the codimension 2 estimate, we may need to introduce a concept more general than Condition A. We have made progress in this direction and shall report on it in the future.

Lastly, we remark that Immervoll [19] gave a different proof of Theorem 5 by employing isoparametric triple systems Dorfmeister and Neher developed in [11]. It appears that the method has not been applicable to the four exceptional cases.

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