Classifications of Dupin Hypersurfaces

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Abstract

A hypersurface M^{n-1} immersed in Euclidean space $\mathbf{R}^n$ or the sphere S^n is said to be Dupin if along each curvature surface, the corresponding principal curvature is constant. A Dupin hypersurface M is called proper Dupin if the number of distinct principal curvatures is constant on M. Pinkall proved that these properties are invariant under Lie sphere transformations. In this paper, we survey the known classification results for proper Dupin hypersurfaces that have been obtained in the context of Lie sphere geometry.

Let M be a hypersurface immersed in Euclidean space $\mathbf{R}^n$ or the unit sphere $S^n \subset \mathbf{R}^{n+1}$. A *curvature surface* of M is a smooth connected submanifold S such that for each point $x \in S$, the tangent space T_xS is equal to a principal space of the shape operator A of M at x. The hypersurface M is said to be *Dupin* if along each curvature surface, the corresponding principal curvature is constant. A Dupin hypersurface M is called *proper Dupin* if the number of distinct principal curvatures is constant on M. Dupin hypersurfaces have been studied extensively since the introduction of the cyclides of Dupin [19] in 1822, and great progress has been made over the past 25 years in their classification. Dupin hypersurfaces have played a major role in the theory of taut embeddings (see, for example, [2], [9], [28], [41], [48], [49]), and they recently appeared in the work of Ferapontov on Hamiltonian systems of hydrodynamic type [20]–[22], and in the work of Riveros and Tenenblat [43] on higher-dimensional Laplace invariants.

A major step forward in the theory of Dupin hypersurfaces was the work of Pinkall [39]–[40] (see also [10]–[11]) in the early 1980's which placed the study of Dupin hypersurfaces in the context of Lie sphere geometry. A Lie sphere transformation on S^n is a transformation on the space of oriented hyperspheres in S^n that preserves oriented contact of spheres. The Lie sphere group G contains the group of Möbius (conformal) transformations of S^n as a subgroup, and G is generated by the Möbius group and the set of parallel transformations $P_t, t \in \mathbf{R}$, which map an oriented hypersurface M in S^n to a parallel hypersurface M_t at oriented distance t from M in S^n (see, for example, [8, pp. 60-64]). Pinkall proved that both the Dupin and proper Dupin properties are invariant under Lie sphere transformations, and Lie sphere geometry has proven

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to be the appropriate context for subsequent major results in the theory.

An important special class of Dupin hypersurfaces is the class of isoparametric (constant principal curvatures) hypersurfaces in the sphere S^n. In a series of four papers in the late 1930's, E. Cartan [3]–[6] initiated the study of isoparametric hypersurfaces in spheres, and many mathematicians have made significant contributions to this beautiful theory. Of primary importance is the result of Münzner [35] which states that the number g of distinct principal curvatures of an isoparametric hypersurface in S^n must be $1, 2, 3, 4$ or 6. Cartan himself classified isoparametric hypersurfaces with $1, 2$ or 3 principal curvatures, but the classification problem for isoparametric hypersurfaces with 4 or 6 principal curvatures has not been completely solved, although much progress has been made in recent years, as will be described below. In this paper, we survey the known classification results for proper Dupin hypersurfaces that have been obtained in the context of Lie sphere geometry.

1 Local Constructions

Pinkall [40] demonstrated the existence of a proper Dupin hypersurface having an arbitrary number of distinct principal curvatures with any prescribed multiplicities by using the following four basic constructions.

Start with a Dupin hypersurface W^{n-1} in $\mathbf{R}^n$ and then consider $\mathbf{R}^n$ as the linear subspace $\mathbf{R}^n \times \{0\}$ in $\mathbf{R}^{n+1}$. The following constructions yield a Dupin hypersurface M^n in $\mathbf{R}^{n+1}$.

(1) Let M^n be the cylinder $W^{n-1} \times \mathbf{R}$ in $\mathbf{R}^{n+1}$.

(2) Let M^n be the hypersurface in $\mathbf{R}^{n+1}$ obtained by rotating W^{n-1} around an axis $\mathbf{R}^{n-1} \subset \mathbf{R}^n$.

(3) Let M^n be a tube in $\mathbf{R}^{n+1}$ around W^{n-1}.

(4) Project W^{n-1} stereographically onto a hypersurface $V^{n-1} \subset S^n \subset \mathbf{R}^{n+1}$. Let M^n be the cone over V^{n-1} in $\mathbf{R}^{n+1}$.

These constructions introduce a new principal curvature of multiplicity one which is constant along its lines of curvature. The other principal curvatures are determined by the principal curvatures of W^{n-1}, and the Dupin property is preserved for these principal curvatures. These constructions can be generalized to produce a new principal curvature of multiplicity m by considering $\mathbf{R}^n$ as a subset of $\mathbf{R}^n \times \mathbf{R}^m$ rather than $\mathbf{R}^n \times \mathbf{R}$.

Pinkall also showed that the cone construction (4) is equivalent by a Lie sphere transformation to the tube construction (3), and thus one often works with only the first three constructions. A proper Dupin hypersurface which is locally equivalent by a Lie sphere transformation to a hypersurface M^n obtained by one of these constructions is said to be *reducible*. Otherwise, the proper Dupin hypersurface is said to be *irreducible*.

By iterative use of these constructions, Pinkall proved the following basic existence theorem.

Theorem 1.1 (Pinkall). *Given positive integers $m_1, \ldots, m_g$ with $m_1 + \cdots + m_g = n - 1$, there exists a proper Dupin hypersurface M^{n-1} in $\mathbf{R}^n$ with g distinct principal curvatures having respective multiplicities $m_1, \ldots, m_g$.*

Of course, all of the hypersurfaces that Pinkall constructed in the proof of this theorem are reducible. Pinkall's constructions only yield a compact proper Dupin hypersurface if the original manifold W^{n-1} is itself a sphere [7]. Otherwise, the number of distinct principal curvatures is not constant on a compact manifold M^n obtained in this way, because there are points where the new principal curvature is equal to a principal curvature determined by one of the original principal curvatures of W^{n-1}. Thus, compact proper Dupin hypersurfaces are much more rare, as we describe in the following section.

2 Compact Proper Dupin Hypersurfaces

Since isoparametric hypersurfaces play such an important role in the known classification results for compact proper Dupin hypersurfaces, we begin by briefly describing the major results in that theory. An isoparametric hypersurface M in $\mathbf{R}^n$ can have at most two distinct principal curvatures, and M must be an open subset of a hyperplane, hypersphere or a spherical cylinder $S^k \times \mathbf{R}^{n-k-1}$ (Levi-Civita [26] for $n = 3$ and B. Segre [45] for arbitrary n).

In the sphere S^n, however, examples are far more abundant. First note that it suffices to classify compact, connected isoparametric hypersurfaces, since Münzner [35] showed that every connected isoparametric hypersurface lies in a unique compact, connected isoparametric hypersurface. Cartan classified isoparametric hypersurfaces M in S^n with $g \leq 3$ principal curvatures as follows. If $g = 1$, then M is totally umbilic, and it must be a great or small sphere. If $g = 2$, then M must be a standard product of two spheres, $S^k(r) \times S^{n-k-1}(s) \subset S^n$, $r^2 + s^2 = 1$. In the case $g = 3$, Cartan [4] showed that all the principal curvatures must have the same multiplicity $m = 1, 2, 4$ or 8, and M must be a tube of constant radius over a standard embedding of a projective plane $\mathbf{F}P^2$ into S^{3m+1}, where $\mathbf{F}$ is the division algebra $\mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$ (quaternions), $\mathbf{O}$ (Cayley numbers) for $m = 1, 2, 4, 8$, respectively. Thus, up to congruence, there is only one such family for each value of m.

For isoparametric hypersurfaces with four principal curvatures, Münzner proved that the principal curvatures can have at most two distinct multiplicities m_1, m_2. Then, using representations of Clifford algebras, Ferus, Karcher and Münzner [23] constructed for any positive integer m_1 an infinite series of isoparametric hypersurfaces with four principal curvatures having respective multiplicities (m_1, m_2), where m_2 is nondecreasing and unbounded in each series. This class of *FKM-type* isoparametric hypersurfaces contains all known examples of isoparametric hypersurfaces with four principal curvatures with the exception of two homogeneous examples, having multiplicities $(2, 2)$ and $(4, 5)$. Stolz [46] then proved that the multiplicities of the principal curvatures of an isoparametric hypersurface with four principal curvatures must be the same as those in the known examples of Ferus, Karcher and Münzner or the two homogeneous exceptions. In a recent paper [12], we proved that if the multiplicities satisfy $m_2 \geq 2m_1 - 1$, then the isoparametric hypersurface must be of FKM-type (a different proof of this result, using isoparametric triple systems, was given by Immervoll [25]). When this result is combined with the known classifications of Takagi [47] for $m_1 = 1$ and Ozeki and Takeuchi [38] for $m_1 = 2$, it yields a complete classification of isoparametric hypersurfaces with $g = 4$ with the exception of four pairs of multiplicities, $(3, 4), (6, 9), (7, 8)$ and $(4, 5)$, for which the classification problem remains open.

The classification of isoparametric hypersurfaces with six distinct principal curvatures also remains as an open problem. In this case, there exists one homogeneous family with six principal curvatures of multiplicity one in S^7, and one homogeneous family with six principal curvatures of multiplicity two in S^{13} (see Miyaoka [31] for a description). These are the only known examples up to congruence. Münzner showed that for $g = 6$, all of the principal curvatures must have the same multiplicity m, and then Abresch [1] showed that m must be 1 or 2. In the case $m = 1$, Dorfmeister and Neher [18] (see also [32]–[33]) showed in 1985 that an isoparametric hypersurface must be homogeneous. It remains an open question whether this is also true in the case $m = 2$, although that is widely conjectured to be the case.

In contrast to Pinkall's local Theorem 1.1 above, Thorbergsson [49] proved that for a compact Dupin hypersurface embedded in $\mathbf{R}^n$ or S^n, the number g of distinct principal curvatures must be $1, 2, 3, 4$ or 6, the same as for an isoparametric hypersurface in a sphere [35]. If $g \geq 3$, a compact proper Dupin hypersurface must be irreducible [13]. Furthermore, the restrictions on the multiplicities of the principal curvatures of isoparametric hypersurfaces are still valid for compact proper Dupin hypersurfaces (see Stolz [46] and Grove-Halperin [24]).

Some classifications have been obtained for compact proper Dupin hypersurfaces embedded in S^n. Of course, in the case of $g = 1$ principal curvature, the hypersurface must be a great or small hypersphere. In the case $g = 2$, Cecil and Ryan [16] showed that a compact proper Dupin hypersurface must be Möbius equivalent to an isoparametric hypersurface, and thus be a generalized cyclide of Dupin. Next Miyaoka [27] showed that a compact proper Dupin hypersurface with $g = 3$ principal curvatures must be equivalent by a Lie sphere transformation to an isoparametric hypersurface. Later Cecil and Jensen [14] gave a different proof of Miyaoka's result as a consequence of their classification of irreducible proper Dupin hypersurfaces with three principal curvatures.

Given these results, Cecil and Ryan [17, p.184] conjectured in 1985 that every compact proper Dupin hypersurface is equivalent by a Lie sphere transformation to an isoparametric hypersurface in a sphere. This conjecture was shown to be false by Pinkall and Thorbergsson [42], and independently by Miyaoka and Ozawa [34], who gave different constructions of counterexamples to the conjecture in the case $g = 4$. The method of Miyaoka of Ozawa also yields a counterexample in the case $g = 6$.

A key ingredient in the construction of these counterexamples is the notion of the *Lie curvatures* of the hypersurface introduced by Miyaoka [29], [30]. These are the cross-ratios of the principal curvatures taken four at a time, and they are easily seen to be invariant under Lie sphere transformations, which are projective transformations. In the counterexamples to the conjecture mentioned above, it was shown that the hypersurfaces constructed do not have constant Lie curvatures, and therefore they cannot be Lie equivalent to an isoparametric hypersurface, which obviously has constant Lie curvatures.

This left open the possibility that a compact proper Dupin hypersurface with 4 or 6 six principal curvatures, and constant Lie curvatures must be Lie equivalent to an isoparametric hypersurface. In fact, Miyaoka [28] showed that this is true if the hypersurface also satisfies some additional assumptions on the intersections of the leaves of the various principal foliations. The goal of current research is this field is to prove that Miyaoka's additional assumptions are not necesary, and thus prove the following conjecture.

Conjecture 2.1. *Every compact, connected proper Dupin hypersurface in S^n with four or six principal curvatures and constant Lie curvatures is Lie equivalent to an isoparametric hypersurface.*

In [15], the conjecture was shown to be true for a compact proper Dupin hypersurface with four principal curvatures of multiplicity one. In a recent paper [13], we have verified the conjecture in the case of a compact proper Dupin hypersurface with four principal curvatures having multiplicities $m_1 = m_2 \geq 1$, $m_3 = m_4 = 1$ to obtain the following theorem.

Theorem 2.1. *Let M be a compact, connected proper Dupin hypersurface in S^n with four principal curvatures having multiplicities $m_1 = m_2 \geq 1$, $m_3 = m_4 = 1$, and constant Lie curvature. Then M is Lie equivalent to an isoparametric hypersurface.*

Note that since the multiplicities of a compact, connected proper Dupin hypersurface with four principal curvatures, must satisfy the conditions $m_1 = m_2$ and $m_3 = m_4$ when the principal curvatures are appropriately ordered, this means that the full conjecture would be proven if the assumption that $m_3 = m_4 = 1$ could be eliminated from the theorem above. In proving Theorem 2.1, we used a different approach than Miyaoka, and obtained the result as a consequence of our local classification of irreducible proper Dupin hypersurfaces with four principal curvatures having these multiplicities and constant Lie curvature, which we will discuss in the next section. In the case $g = 6$, we do not know of any results beyond those of Miyaoka towards proving the conjecture.

3 Local Classification Results

The primary idea in obtaining local classifications of proper Dupin hypersurfaces is to replace the assumption of compactness with the assumption of irreducibility in the sense of Pinkall defined above. As Pinkall's Theorem 1.1 shows, there is no restriction on the number of distinct principal curvatures of a non-compact proper Dupin hypersurface. However, classification results have been obtained for irreducible non-compact proper Dupin hypersurfaces M in S^n with a small number of principal curvatures. To be specific, in the case $g = 1$, M is totally umbilic and therefore is an open subset of a metric hypersphere in S^n. In the case $g = 2$, every connected proper Dupin hypersurface M with two principal curvatures is Möbius equivalent to an isoparametric hypersurface [40], [8, p.154], i.e., M is a generalized cyclide of Dupin. In the case of a connected proper Dupin hypersurface M in S^4 with three principal curvatures of multiplicity one, Pinkall [39] showed that if M is irreducible, then M is Lie equivalent to an isoparametric hypersurface with three principal curvatures. Niebergall [36] then showed that every proper Dupin hypersurface M^4 in S^5 with three principal curvatures is reducible. These results were generalized by Cecil and Jensen [14] who proved the following theorem.

Theorem 3.1. *Let M be an irreducible connected proper Dupin hypersurface in S^n with three principal curvatures having multiplicities m_1, m_2, m_3. Then $m_1 = m_2 = m_3$, and M is Lie equivalent to an isoparametric hypersurface.*

Note that this theorem was originally proven under the assumption that M is locally irreducible, i.e., it does not contain any reducible open subset. However, we showed in [13] that irreducibility implies local irreducibility.

In the case of four distinct principal curvatures, the examples of Pinkall and Thorbergsson, and Miyaoka and Ozawa show that one must assume constant Lie curvature in order to obtain Lie equivalence to an isoparametric hypersurface. Furthermore, one must also assume irreducibility, because of the examples of Cecil [8, p.106] of reducible proper Dupin hypersurfaces with four principal curvatures and constant Lie curvature. These are constructed by considering tubes in S^n over an isoparametric hypersurface $V \subset S^{n-m}$ with three principal curvatures, where S^{n-m} is embedded as a great sphere in S^n. These examples are not Lie equivalent to an isoparametric hypersurface with four principal curvatures, and they cannot be completed to be a compact proper Dupin hypersurface with four principal curvatures. Given these facts, Cecil and Jensen

[15, pp.3-4] proposed the following conjecture. In the original formulation of the conjecture, it was assumed that M is locally irreducible, but as we pointed out earlier, irreduciblity impiles local irreducibility.

Conjecture 3.1. *If M is an irreducible connected, proper Dupin hypersurface in S^n with four principal curvatures having respective multiplicities m_1, m_2, m_3, m_4 and constant Lie curvature, then $m_1 = m_2, m_3 = m_4$, and M is Lie equivalent to an isoparametric hypersurface.*

Note that in order for the multiplicities of an isoparametric hypersurface to satisfy $m_1 = m_2, m_3 = m_4$, the principal curvatures must be ordered in such a way that the Lie curvature r is equal to -1, that is the four principal curvatures are in *harmonic division* (see [44, p.59]) in the sense of projective geometry.

This conjecture is still open in its full generality, although we have made considerable progress which is summarized in the following theorem [13].

Theorem 3.2. *Let M be an irreducible connected proper Dupin hypersurface in S^n with four principal curvatures having multiplicities $m_1 = m_2 \geq 1$, $m_3 = m_4 = 1$, and constant Lie curvature $r = -1$. Then M is Lie equivalent to an isoparametric hypersurface.*

In the case where all the multiplicities are equal to one, the hypotheses of the theorem can be weakened to merely constant Lie curvature, without needing to assume that $r = -1$ [15], in order to obtain Lie equivalence to an isoparametric hypersurface (see also Niebergall [37], who proved this result under some additional assumptions). To our knowledge, not much progress has been made thus far in proving that irreducibility and constant Lie curvature implies that $m_1 = m_2, m_3 = m_4$, although it is possible that a proof of this part of Conjecture 3.1 may follow the lines of the proof [14] that irreducibility implies that all the multiplicities must be equal in the case $g = 3$.

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