DUPIN HYPERSURFACES WITH FOUR PRINCIPAL CURVATURES, II

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ABSTRACT. If M is an isoparametric hypersurface in a sphere S^n with four distinct principal curvatures, then the principal curvatures $\kappa_1, \ldots, \kappa_4$ can be ordered so that their multiplicities satisfy $m_1 = m_2$ and $m_3 = m_4$, and the cross-ratio r of the principal curvatures (the Lie curvature) equals -1. In this paper, we prove that if M is an irreducible connected proper Dupin hypersurface in $\mathbf{R}^n$ (or S^n) with four distinct principal curvatures with multiplicities $m_1 = m_2 \geq 1$ and $m_3 = m_4 = 1$, and constant Lie curvature $r = -1$, then M is equivalent by Lie sphere transformation to an isoparametric hypersurface in a sphere. This result remains true if the assumption of irreducibility is replaced by compactness and r is merely assumed to be constant.

1. INTRODUCTION

Let M be an immersed hypersurface in Euclidean space $\mathbf{R}^n$ or the unit sphere $S^n \subset \mathbf{R}^{n+1}$. A *curvature surface* of M is a smooth connected submanifold S such that for each point $x \in S$, the tangent space $T_x S$ is equal to a principal space of the shape operator A of M at x. This generalizes the classical notion of a line of curvature on a surface in $\mathbf{R}^3$. We use this definition in deference to what has become the standard definition in the Dupin hypersurface literature, even though a curvature surface can have any dimension from 1 through the dimension of M. The hypersurface M is said to be *Dupin* if it satisfies the condition

   (a) along each curvature surface, the corresponding principal curvature is constant.

The hypersurface M is called *proper Dupin* if, in addition to condition (a), it also satisfies the condition

   (b) the number g of distinct principal curvatures is constant on M.

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Pinkall [17] proved that both of these conditions are invariant under the group of Lie sphere transformations of S^n, which contains the group of Möbius (conformal) transformations of S^n as a subgroup. Thus, by stereographic projection, the theory of Dupin hypersurfaces in $\mathbf{R}^n$ or S^n is essentially the same.

Thorbergsson [23] showed that the number g of distinct principal curvatures of a compact proper Dupin hypersurface M immersed in S^n must be $1, 2, 3, 4$ or 6, the same as Münzner's [15, 16] restriction on the number of distinct principal curvatures of an isoparametric (constant principal curvatures) hypersurface in S^n. In the cases $g = 1, 2, 3$, compact proper Dupin hypersurfaces in S^n have been completely classified. For the totally umbilic case $g = 1$, M must be a great or small sphere. For $g = 2$, Cecil and Ryan [7] proved that M must be Möbius equivalent to a standard product of spheres (which is isoparametric)

$$S^k(r) \times S^{n-k-1}(s) \subset S^n, \quad r^2 + s^2 = 1.$$

In the case $g = 3$, Miyaoka [11] proved that M must be Lie equivalent to an isoparametric hypersurface in S^n, which by the work of Cartan [1] must be a tube of constant radius over a standard embedding of a projective plane $\mathbf{F}P^2$ into S^{3m+1}, where $\mathbf{F}$ is the division algebra $\mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$ (quaternions), $\mathbf{O}$ (Cayley numbers) for $m = 1, 2, 4, 8$, respectively.

The cases of compact proper Dupin hypersurfaces with $g = 4$ or 6 principal curvatures have not yet been classified, although Stolz [21] in the case $g = 4$ and Grove and Halperin [10] in the case $g = 6$ have shown that the multiplicities of the principal curvatures of a compact proper Dupin hypersurface must be the same as for an isoparametric hypersurface. In particular, in the case $g = 4$, the multiplicities must come in pairs, and the principal curvatures can be ordered in such a way that $m_1 = m_2$ and $m_3 = m_4$. In the case $g = 6$, all the principal curvatures must have the same multiplicity $m = 1$ or 2.

Miyaoka [12] introduced an important set of Lie invariants, the *Lie curvatures* of a Dupin hypersurface M, which are the cross-ratios of the principal curvatures taken four at a time. Obviously, for an isoparametric hypersurface, the Lie curvatures are constant, and a necessary condition for a Dupin hypersurface to be Lie equivalent to an isoparametric hypersurface is that it have constant Lie curvatures. At one time it was thought that perhaps every compact proper Dupin is Lie equivalent to an isoparametric hypersurface. However, by two separate constructions, Pinkall and Thorbergsson [19] ($g = 4$) and Miyaoka and Ozawa [14] ($g = 4$ or 6) produced compact proper Dupin hypersurfaces which do not have constant Lie curvatures and therefore cannot be Lie equivalent to an isoparametric hypersurface.

Miyaoka [12, 13] showed that a compact proper Dupin hypersurface immersed in S^n with $g = 4$ or 6 principal curvatures is Lie equivalent to an isoparametric hypersurface if it has constant Lie curvatures and it satisfies certain additional global conditions regarding the intersections of leaves of its various principal foliations. The goal of current research is to prove that the condition of constant Lie curvatures already suffices for the conclusion without assuming these additional conditions, and we have succeeded in doing this in the case $g = 4$ when one pair of the multiplicities is equal to one, as described below.

In contrast to the situation for compact proper Dupin hypersurfaces, there is a local method, due to Pinkall [17], for producing a Dupin hypersurface with any given number g of principal curvatures with any prescribed multiplicities $m_1, \ldots, m_g$. His method uses the basic constructions of building tubes, cylinders, cones and surfaces of revolution over a Dupin hypersurface W^{n-1} in $\mathbf{R}^n$ with g principal curvatures to get a Dupin hypersurface M^{n-1+k} in $\mathbf{R}^{n+k}$ with $g + 1$ principal curvatures. These constructions introduce a new principal curvature of multiplicity k which is easily seen to be constant along its curvature surfaces. The other principal curvatures are determined by the principal curvatures of W^{n-1}, and the Dupin property is preserved for these principal curvatures. These constructions are local in nature and only yield a compact proper Dupin hypersurface if the original manifold W^{n-1} is itself a sphere [2, Theorem 46]. Otherwise, the number of distinct principal curvatures is not constant on a compact manifold M^{n-1+k} obtained in this way, so it is not proper Dupin.

A Dupin hypersurface which is locally equivalent by a Lie sphere transformation to a hypersurface M^n obtained by one of these constructions is said to be *reducible*. Otherwise, the Dupin hypersurface is called *irreducible*. A Dupin hypersurface is called *locally irreducible* [6] if it does not contain any reducible open subset. Clearly, local irreducibility implies irreducibility.

In [4], we prove that any C^∞ proper Dupin hypersurface must be analytic. Using this result, we prove that if a connected proper Dupin hypersurface M has a reducible open subset, then M itself is reducible. That is, irreducibility implies local irreducibility. Analyticity allows us to work locally to obtain global results.

The primary work in this paper is local in nature and is accomplished in the setting of Lie sphere geometry. We concentrate on the case $g = 4$ with multiplicities $m_1 = m_2$, $m_3 = m_4$ and Lie curvature $r = -1$. In order to state our results we shall recall here some of the basic ideas in Lie sphere geometry.

Let $\mathbf{R}_2^{n+3}$ be a real vector space of dimension $n+3$ endowed with the metric of signature $(n+1,2)$,

$$(1.1) \qquad \langle x, y \rangle = -x^0 y^0 + x^1 y^1 + \cdots + x^{n+1} y^{n+1} - x^{n+2} y^{n+2}.$$

Let $e_0, \ldots, e_{n+2}$ denote the standard orthonormal basis with respect to this metric, with e_0 and e_{n+2} timelike. Let $\mathbf{R}P^{n+2}$ be the real projective space of lines through the origin in $\mathbf{R}_2^{n+3}$, and let Q^{n+1} be the quadric hypersurface determined by the equation $\langle x, x \rangle = 0$. This hypersurface is called the *Lie quadric*. The sphere S^n can be identified with the unit sphere in the Euclidean space $\mathbf{R}^{n+1}$ spanned by the vectors $e_1, \ldots, e_{n+1}$.

The points in Q^{n+1} are in bijective correspondence with the set of all oriented hyperspheres and point spheres in S^n. The Lie quadric contains projective lines but no linear subspaces of $\mathbf{R}P^{n+2}$ of higher dimension. Let Λ^{2n-1} denote the set of all projective lines in Q^{n+1}. It is an analytic manifold of dimension $2n-1$. The line $[x, y]$ determined by two points $[x]$ and $[y]$ of Q^{n+1} lies on Q^{n+1} if and only if $\langle x, y \rangle = 0$. This happens precisely when the hyperspheres in S^n corresponding to the points $[x]$ and $[y]$ are in oriented contact.

A *Lie sphere transformation* is a projective transformation of $\mathbf{R}P^{n+2}$ that maps Q^{n+1} to itself. A Lie sphere transformation preserves oriented contact of hyperspheres in S^n, since it takes lines on Q^{n+1} to lines on Q^{n+1}. The group G of Lie sphere transformations is isomorphic to $O(n+1,2)/\{\pm I\}$, where $O(n+1,2)$ is the orthogonal group for the metric (1.1). The group G acts transitively on Λ^{2n-1}.

The manifold Λ^{2n-1} of projective lines on Q^{n+1} has a *contact structure*, i.e., a globally defined 1-form ω such that $\omega \wedge d\omega^{n-1}$ never vanishes on Λ^{2n-1}. The condition $\omega = 0$ defines a codimension one distribution D on Λ^{2n-1} which has integral submanifolds of dimension $n-1$ but none of higher dimension. A *Legendre submanifold* is one of these integral submanifolds of maximal dimension, i.e., an immersion $\lambda : M^{n-1} \to \Lambda^{2n-1}$ such that $\lambda^* \omega = 0$.

An immersion $f : M^{n-1} \to S^n$ with field of unit normals $\xi : M^{n-1} \to S^n$ naturally induces a Legendre submanifold $\lambda = [Y_0, Y_1]$, where $[Y_0, Y_1]$ denotes the line in Q determined by the point sphere $Y_0 = (1, f, 0)$ and the tangent great sphere $Y_1 = (0, \xi, 1)$. In a similar way, an immersed submanifold $\phi : V \to S^n$ of codimension greater than one also induces a Legendre submanifold whose domain is the bundle B^{n-1} of unit normal vectors to $\phi(V)$ (see, for example, [3, p.79]).

In Section 2 we review the basic set-up of the method of moving frames in Lie sphere geometry developed in our previous paper [6].

In Section 3 we specialize the Lie frame for the case under study. Theorem 7 establishes sufficient conditions for when a proper Dupin

hypersurface with four principal curvatures having multiplicities $m_1 = m_2$, $m_3 = m_4$ and Lie curvature $r = -1$ is Lie equivalent to an isoparametric hypersurface.

In Section 4, we relate the Lie sphere definition of reducibility to the Pinkall constructions in Euclidean space. Theorems 15 and 16 establish sufficient conditions for reducibility in terms of quantities that arise naturally in the setting of moving Lie frames.

In Section 5, we prove one of our main results:

Theorem 23. *Suppose the connected proper Dupin hypersurface $\lambda : M^{n-1} \to \Lambda^{2n-1}$ has four distinct curvature spheres with multiplicities $m_1 = m_2 \geq 1$, $m_3 = m_4 = 1$, and Lie curvature $r = -1$. If λ is irreducible, then it is Lie equivalent to an isoparametric hypersurface.*

Isoparametric hypersurfaces with multiplicities $m_1 = m_2 \geq 1$ and $m_3 = m_4 = 1$ have been completely classified by Takagi [22]. They are all homogeneous spaces and are obtained by the construction of Ferus, Karcher, and Münzner [9] based on representations of Clifford algebras. Thus the Legendre submanifold induced by any Dupin hypersurface satisfying the conditions of this theorem is Lie equivalent to an open subset of the Legendre submanifold induced by one of Takagi's isoparametric hypersurfaces.

In [6, p.3], it was conjectured that if M is an irreducible proper Dupin hypersurface in S^n with four principal curvatures having respective multiplicities m_1, m_2, m_3, m_4, and M has constant Lie curvature, then the principal curvatures can be ordered so that $m_1 = m_2$, $m_3 = m_4$, and M is Lie equivalent to an isoparametric hypersurface in S^n. We still believe this conjecture to be true, although we have not yet been able to verify it in more generality than Theorem 23.

In Section 6, we prove in Theorem 25 that a compact proper Dupin hypersurface with $g > 2$ principal curvatures is irreducible. As a consequence of this, Theorem 23, and a result of Miyaoka [12] that a compact proper Dupin hypersurface with $g = 4$ and constant Lie curvature r must have $r = -1$, we obtain our second main result:

Theorem 28. *Let M be a compact connected proper Dupin hypersurface immersed in R^n with four distinct principal curvatures having multiplicities $m_1 = m_2 \geq 1$, $m_3 = m_4 = 1$, and constant Lie curvature. Then M is Lie equivalent to an isoparametric hypersurface.*

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2. Dupin hypersurfaces in Lie sphere geometry

In this section, we briefly recall how Dupin hypersurfaces can be studied in the context of Lie sphere geometry. In particular, we will summarize the basic set-up and main definitions of [6] that will be needed in the remainder of the paper. We will not, however, reproduce all the formulas from that paper, so the reader will need to consult that paper at times. Throughout this paper, equation references of the sort GD(3.36) will be to equation (3.36) of [6]. We will use the Einstein summation convention in this section.

We continue the review of Lie sphere geometry begun in the Introduction. Suppose that $\lambda = [Y_0, Y_1] : M^{n-1} \to \Lambda^{2n-1}$ is a Legendre submanifold. Let $p \in M^{n-1}$ and let r and s be real numbers at least one of which is non-zero. The sphere in S^n corresponding to the point $[K] = [rY_0(p) + sY_1(p)]$ in Q^{n+1} is called a *curvature sphere* of λ at p, if there exists a non-zero tangent vector $X \in T_pM$ such that $rdY_0(X) + sdY_1(X) \in \mathrm{Span}\{Y_0(p), Y_1(p)\}$. The vector X is called a *principal vector* corresponding to the curvature sphere $[K]$. The principal vectors corresponding to a given curvature sphere form a subspace of T_pM, and T_pM is the direct sum of these *principal spaces*.

To see the relationship between curvature spheres and principal curvatures, suppose now that $\lambda = [Y_0, Y_1]$ with $Y_0 = (1, f, 0), Y_1 = (0, \xi, 1)$. At a given $p \in M$, one can write the distinct curvature spheres in the form $[K_i] = [\kappa_i Y_0 + Y_1], 1 \le i \le g$. In the case where the map f is an immersion, these κ_i are the usual principal curvatures of the hypersurface f at p. The principal curvatures are not invariant under Lie sphere transformations. However, the cross-ratio of any four distinct principal curvatures is Lie invariant. These cross ratios are called *Lie curvatures* of λ.

As in Euclidean submanifold theory, a *curvature surface* is a smooth connected submanifold S of M such that for each point $p \in S$, the tangent space T_pS is equal to a principal space. A Legendre submanifold is called *Dupin* if along each curvature surface, the corresponding curvature sphere is constant. A Dupin submanifold is said to be *proper Dupin* if the number g of distinct curvature spheres is constant on M. These definitions agree with the usual definitions in the case where the Legendre submanifold is induced from an immersed hypersurface in S^n. Pinkall [17] showed that both of these properties are invariant under the group of Lie sphere transformations. At times, we will refer to Dupin submanifolds as "Dupin hypersurfaces," because of their close relationship with Dupin hypersurfaces in S^n.

We now begin to recall the notation and results from [6] in detail. We study Dupin hypersurfaces in Lie sphere geometry using the method of moving frames. Instead of using an orthonormal frame for the metric in (1.1), we consider a *Lie frame*, that is, an ordered set of vectors $Y_0, \ldots, Y_{n+2}$ in $\mathbf{R}_2^{n+3}$ satisfying $\langle Y_a, Y_b \rangle = k_{ab}$, for $0 \le a, b \le n+2$, with

$$(2.1) \qquad k = (k_{ab}) = \begin{pmatrix} 0 & 0 & -J \\ 0 & I_{n-1} & 0 \\ -J & 0 & 0 \end{pmatrix}, \quad \text{where} \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The space of all Lie frames can be identified with the orthogonal group $G = O(n+1, 2)$. In this space, one introduces the Maurer-Cartan forms,

$$(2.2) \qquad dY_a = \omega_a^b Y_b, 0 \le a, b \le n+2,$$

which satisfy the Maurer-Cartan structure equations of G,

$$(2.3) \qquad d\omega_b^a = -\omega_c^a \wedge \omega_b^c, \quad \text{for } 0 \le a, b, c \le n+2$$

Knowing from [4] that any proper Dupin hypersurface is real analytic, we assume from now on that all maps are real analtyic. A *Lie frame field* along a Legendre submanifold $\lambda : M^{n-1} \to \Lambda^{2n-1}$ is a real analytic map $Y : U \to G$ defined on an open subset U of M^{n-1} such that $\lambda(p) = [Y_0(p), Y_1(p)]$ for each $p \in U$. Here Y_a denotes the ath column of Y and $Y \in G$ means $\langle Y_a, Y_b \rangle = k_{ab}$, for all $a, b = 0, 1, \ldots, n+2$.

The notion of a curvature sphere of a Legendre submanifold $\lambda : M^{n-1} \to \Lambda^{2n-1}$ can be formulated in terms of Lie frames as follows. If Y is any Lie frame field along λ defined on a neighborhood of a point $p \in M$, then $[rY_0 + sY_1]$ is a curvature sphere of λ at p precisely when the following equation is satisfied at p,

$$(2.4) \qquad (r\omega_0^2 + s\omega_1^2) \wedge \ldots \wedge (r\omega_0^n + s\omega_1^n) = 0$$

This condition is equivalent to saying that the tangent sphere map

$$(2.5) \qquad [rY_0 + sY_1] : U \to Q \subset \mathbf{R}P^{n+2}$$

is singular at p in the sense that there exists a non-zero vector $X \in T_p M$ such that

$$(2.6) \qquad d(rY_0 + sY_1)_{(p)}(X) \in \text{span}\,\{Y_0(p), Y_1(p)\}$$

We now restrict our attention to the case where the Legendre submanifold $\lambda : M^{n-1} \to \Lambda^{2n-1}$ has $g = 4$ distinct curvature spheres of

multiplicities m_1, m_2, m_3 and m_4, respectively. We define sets

(2.7)
$$\begin{aligned}
\mathcal{I}_1 &= \{2, \ldots, m_1 + 1\} \\
\mathcal{I}_2 &= \{m_1 + 2, \ldots, m_1 + m_2 + 1\} \\
\mathcal{I}_3 &= \{m_1 + m_2 + 2, \ldots, m_1 + m_2 + m_3 + 1\} \\
\mathcal{I}_4 &= \{m_1 + m_2 + m_3 + 2, \ldots, m_1 + m_2 + m_3 + m_4 + 1\}.
\end{aligned}$$

and adopt the index conventions

(2.8)
$$\begin{aligned}
2 &\leq i, j, k, l \leq n \\
a, b, c, d &\in \mathcal{I}_1 \\
p, q, r, s &\in \mathcal{I}_2 \\
\alpha, \beta, \gamma, \delta &\in \mathcal{I}_3 \\
\mu, m, \sigma, \tau &\in \mathcal{I}_4
\end{aligned}$$

We next recall the following definition from [6].

Definition 1. Suppose that $\lambda : M \to \Lambda$ is a real analytic Legendre submanifold with $g = 4$ distinct curvature spheres at each point. A *first order frame field along* λ is an analytic Lie frame field $Y : U \subset M \to G$ such that

(2.9)
$$[Y_0], \quad [Y_1], \quad [Y_0 + Y_1], \quad [rY_0 + Y_1]$$

are the curvature spheres of λ at each point of U, and

(2.10)
$$\omega_0^a = 0, \quad \omega_1^p = 0, \quad \omega_0^\alpha + \omega_1^\alpha = 0, \quad r\omega_0^\mu + \omega_1^\mu = 0$$

for all a, p, α, μ.

Here $r : M \to \mathbf{R}$ is an analytic function never taking the values 0 or 1. Since we are free to put the four curvature spheres in any order, we can assume

(2.11)
$$-\infty < r < 0$$

Note that r is the cross-ratio of the curvature spheres in the appropriate order, and thus r is the Lie curvature of λ.

One can show that there exists a first order Lie frame field defined on some neighborhood of any point of any analytic Legendre submanifold with $g = 4$ distinct curvature spheres at each point. If Y is a first order frame field on an open set $U \subset M$, then its *associated coframe field* in U is the set of analytic 1-forms

(2.12)
$$\theta^a = \omega_1^a, \quad \theta^p = \omega_0^p, \quad \theta^\alpha = \omega_0^\alpha, \quad \theta^\mu = \omega_0^\mu$$

We now assume that the Legendre submanifold $\lambda : M^{n-1} \to \Lambda^{2n-1}$ is connected and proper Dupin with constant Lie curvature r.

Definition 2. A *second-order Lie frame field along* λ is a first order frame field $Y : U \to G$ such that

$$(2.13) \qquad \omega_0^1 = 0, \quad \omega_1^0 = 0, \quad \omega_1^1 = \omega_0^0$$

In [6], we show that it follows from the Dupin condition, that for any point $p \in M$, there exists a neighborhood U of p on which there is defined a second-order frame field along λ. If $Y : U \to G$ is a second order Lie frame field on an open set $U \subset M$, then any other second order frame field on U is given by

$$(2.14) \qquad \tilde{Y} = Y\, a(tI_2, B, 0, sL)$$

where $t, s : U \to \mathbf{R}$ are real analytic functions, t never zero,

$$(2.15) \qquad a(tI_2, B, 0, sL) = \begin{pmatrix} tI_2 & 0 & sL \\ 0 & B & 0 \\ 0 & 0 & t^{-1}I_2 \end{pmatrix}$$

$$(2.16) \qquad L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$(2.17) \qquad B = \begin{pmatrix} B_1 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & B_3 & 0 \\ 0 & 0 & 0 & B_4 \end{pmatrix}$$

where $B_i : U \to O(m_i)$ are real analytic maps.

With a second order frame, we have the following basic expressions for certain Maurer-Cartan forms in terms of the associated coframe. These equations define the analytic tensors F_{pa}^α, F_{pa}^μ, $F_{\alpha a}^\mu$, and $F_{\alpha p}^\mu$, which are crucial to our study.

$$(2.18) \qquad \begin{aligned} \omega_a^p &= F_{pa}^\alpha \theta^\alpha + r F_{pa}^\mu \theta^\mu \\ \omega_a^\alpha &= F_{pa}^\alpha \theta^p + (r-1) F_{\alpha a}^\mu \theta^\mu \\ \omega_p^\alpha &= F_{pa}^\alpha \theta^a + (r-1) F_{\alpha p}^\mu \theta^\mu \\ \omega_a^\mu &= r F_{pa}^\mu \theta^p + (r-1) F_{\alpha a}^\mu \theta^\alpha \\ \omega_p^\mu &= F_{pa}^\mu \theta^a + \frac{r-1}{r} F_{\alpha p}^\mu \theta^\alpha \\ \omega_\alpha^\mu &= F_{\alpha a}^\mu \theta^a + F_{\alpha p}^\mu \theta^p \end{aligned}$$

We also have the following formulas for the Maurer-Cartan forms ω_i^0 and ω_i^1,

$$(2.19) \qquad \omega_i^0 = D_{ij}\theta^j \text{ and } \omega_i^1 = E_{ij}\theta^j$$

The conditions defining a second order frame together with the structure equations impose many conditions on the analytic functions D_{ij} and E_{ij}, which are listed in equations GD(3.25) and GD(3.26) of [6]. In summary, what emerges are eight symmetric matrices

$$
(2.20) \quad
\begin{aligned}
D_1 &= (D_{ab}) & D_2 &= (D_{pq}) & D_3 &= (D_{\alpha\beta}) & D_4 &= (D_{\mu\nu}) \\
E_1 &= (E_{ab}) & E_2 &= (E_{pq}) & E_3 &= (E_{\alpha\beta}) & E_4 &= (E_{\mu\nu})
\end{aligned}
$$

and six matrices of analytic functions

$$
(2.21) \qquad D_{a\alpha}, \quad D_{pa}, \quad D_{p\alpha}, \quad D_{\mu a}, \quad D_{\mu\alpha}, \quad E_{p\mu}
$$

so that equations (2.19) become

$$
(2.22) \quad
\begin{aligned}
\omega_a^0 &= D_{ab}\theta^b + D_{a\alpha}\theta^\alpha - rD_{\mu a}\theta^\mu \\
\omega_p^0 &= D_{pa}\theta^a + D_{pq}\theta^q + D_{p\alpha}\theta^\alpha + rE_{p\mu}\theta^\mu \\
\omega_\alpha^0 &= -D_{a\alpha}\theta^a + D_{\alpha\beta}\theta^\beta + rD_{\mu\alpha}\theta^\mu \\
\omega_\mu^0 &= D_{\mu a}\theta^a + D_{\mu\alpha}\theta^\alpha + D_{\mu\nu}\theta^\nu
\end{aligned}
$$

and

$$
(2.23) \quad
\begin{aligned}
\omega_a^1 &= E_{ab}\theta^b - D_{pa}\theta^p + D_{a\alpha}\theta^\alpha - D_{\mu a}\theta^\mu \\
\omega_p^1 &= E_{pq}\theta^q + D_{p\alpha}\theta^\alpha + E_{p\mu}\theta^\mu \\
\omega_\alpha^1 &= D_{p\alpha}\theta^p + E_{\alpha\beta}\theta^\beta + D_{\mu\alpha}\theta^\mu \\
\omega_\mu^1 &= E_{p\mu}\theta^p + D_{\mu\alpha}\theta^\alpha + E_{\mu\nu}\theta^\nu
\end{aligned}
$$

The tensors in (2.20) satisfy the set of four linear equations GD(3.42), which relate these functions to the four multiplicities. The functions in (2.21) also arise in the following important expression for the exterior derivative of the form ω_0^0,

$$
(2.24) \quad
\begin{aligned}
d\omega_0^0 = {}&- D_{pa}\theta^a \wedge \theta^p + D_{a\alpha}\theta^a \wedge \theta^\alpha - D_{\mu a}\theta^a \wedge \theta^\mu \\
&+ D_{p\alpha}\theta^p \wedge \theta^\alpha + rE_{p\mu}\theta^p \wedge \theta^\mu + (r-1)D_{\mu\alpha}\theta^\alpha \wedge \theta^\mu
\end{aligned}
$$

In this set-up, we define the covariant derivatives of the F's, as the analytic functions on the right side of the equations

$$
(2.25) \quad
\begin{aligned}
dF_{pa}^\alpha + F_{pa}^\alpha \omega_0^0 + F_{pa}^\beta \omega_\beta^\alpha - F_{pb}^\alpha \omega_a^b - F_{qa}^\alpha \omega_p^q &= F_{paj}^\alpha \theta^j \\
dF_{pa}^\mu + F_{pa}^\mu \omega_0^0 + F_{pa}^\nu \omega_\nu^\mu - F_{pb}^\mu \omega_a^b - F_{qa}^\mu \omega_p^q &= F_{paj}^\mu \theta^j \\
dF_{\alpha a}^\mu + F_{\alpha a}^\mu \omega_0^0 + F_{\alpha a}^\nu \omega_\nu^\mu - F_{\alpha b}^\mu \omega_a^b - F_{\beta a}^\mu \omega_\alpha^\beta &= F_{\alpha aj}^\mu \theta^j \\
dF_{\alpha p}^\mu + F_{\alpha p}^\mu \omega_0^0 + F_{\alpha p}^\nu \omega_\nu^\mu - F_{\alpha q}^\mu \omega_p^q - F_{\beta p}^\mu \omega_\alpha^\beta &= F_{\alpha pj}^\mu \theta^j
\end{aligned}
$$

The F's satisfy the six algebraic equations GD(3.36), while their covariant derivatives satisfy the equations GD(3.37) through GD(3.41).

The covariant derivatives of the functions in (2.20) are defined in a way analogous to those of the F's in (2.25), except that the coefficient of ω_0^0 must be multiplied by two in every case. For example,

$$(2.26) \qquad dD_{ab} + 2D_{ab}\omega_0^0 - D_{cb}\omega_a^c - D_{ac}\omega_b^c = D_{abi}\theta^i$$

defines the covariant derivatives D_{abi} of D_{ab}. The covariant derivatives of the other tensors in (2.20) are defined similarly. The formula

$$(2.27) \qquad dD_{a\alpha} + 2D_{a\alpha}\omega_0^0 - D_{b\alpha}\omega_a^b - D_{a\beta}\omega_\alpha^\beta = D_{a\alpha i}\theta^i$$

defines the covariant derivatives $D_{a\alpha i}$ of $D_{a\alpha}$. The covariant derivatives of the other tensors in (3.28) are defined similarly. One last set of functions, R_i, are defined by

$$(2.28) \qquad \omega_{n+1}^0 = R_i\theta^i$$

All these covariant derivatives and the R_i are related in the set of equations GD(3.46) through GD(3.59) in [6, pages 25-30]. They are all real analytic functions.

3. A SUFFICIENT CONDITION TO BE ISOPARAMETRIC

Consider a Legendre map $\lambda : M^{n-1} \to \Lambda^{2n-1}$ that is proper Dupin with four distinct curvature spheres, constant Lie curvature r, and M is a connected real analytic manifold. For the rest of the paper we do not use the Einstein summation convention.

By the work of Münzner [15], [16], in order for λ to be Lie equivalent to an isoparametric hypersurface with four principal curvatures, it is necessary that the multiplicities of the four curvature spheres satisfy $m_1 = m_2$ and $m_3 = m_4$ and that the Lie curvature $r = -1$. In this section, we assume these necessary conditions and then find sufficient conditions in Theorem 7 for Lie equivalence to an isoparametric hypersurface.

Lemma 3. *If the Lie curvature $r = -1$, and the multiplicities satisfy $m_1 = m_2$ and $m_3 = m_4$, and if $Y : U \to G$ is any second order Lie frame field, then on U the symmetric matrices of (2.20) satisfy*

$$(3.1) \qquad \begin{aligned} D_1 &= d_1 I_{m_1} \\ E_2 &= e_2 I_{m_1} \\ D_3 + E_3 &= (d_3 + e_3) I_{m_3} \\ D_4 - E_4 &= (d_4 - e_4) I_{m_3} \end{aligned}$$

where $d_1, \ldots, e_4 : U \to \mathbf{R}$ are the real analytic functions

$$(3.2) \qquad d_1 = \frac{1}{m_1} \operatorname{trace} D_1, \ldots, e_4 = \frac{1}{m_3} \operatorname{trace} E_4$$

Proof. This follows from GD(3.42).   □

For a second order Lie frame field $Y : U \to G$, let

$$v_{a\alpha} = (F_{pa}^{\alpha}, F_{\alpha a}^{\mu}), \quad |v_{a\alpha}|^2 = 2\sum_p (F_{pa}^{\alpha})^2 + 4\sum_\mu (F_{\alpha a}^{\mu})^2$$

$$v_{p\alpha} = (F_{pa}^{\alpha}, F_{\alpha p}^{\mu}), \quad |v_{p\alpha}|^2 = 2\sum_a (F_{pa}^{\alpha})^2 + 4\sum_\mu (F_{\alpha p}^{\mu})^2$$

(3.3)

$$v_{a\mu} = (F_{pa}^{\mu}, F_{\alpha a}^{\mu}), \quad |v_{a\mu}|^2 = 2\sum_p (F_{pa}^{\mu})^2 + 4\sum_\alpha (F_{\alpha a}^{\mu})^2$$

$$v_{p\mu} = (F_{pa}^{\mu}, F_{\alpha p}^{\mu}), \quad |v_{p\mu}|^2 = 2\sum_a (F_{pa}^{\mu})^2 + 4\sum_\alpha (F_{\alpha p}^{\mu})^2$$

If the Lie curvature $r = -1$, and the multiplicities satisfy $m_1 = m_2$ and $m_3 = m_4$, then the middle four equations in GD(3.36) become, when the nonsummed indices of each range are set equal,

(3.4)

$$|v_{a\alpha}|^2 = d_1 - E_{aa} + E_{\alpha\alpha}$$

$$|v_{p\alpha}|^2 = e_2 - D_{pp} - D_{\alpha\alpha}$$

$$|v_{a\mu}|^2 = -d_1 - E_{aa} - E_{\mu\mu}$$

$$|v_{p\mu}|^2 = -e_2 - D_{pp} - D_{\mu\mu}$$

In addition, if all eight of the matrices $D_1, \ldots, E_4$ are scalar at every point of U, then equations (3.4) become

(3.5)

$$|v_{a\alpha}|^2 = d_1 - e_1 + e_3$$

$$|v_{p\alpha}|^2 = e_2 - d_2 - d_3$$

$$|v_{a\mu}|^2 = -e_4 - d_1 - e_1$$

$$|v_{p\mu}|^2 = -e_2 - d_2 - d_4$$

which shows that the functions on the left hand side do not depend on a, p, α, or μ in this case.

Remark 4. If D_1 is scalar at every point of U, then a frame change (2.14) of the form

(3.6) $\tilde{Y} = Ya(I_2, I, 0, sL)$

can be made so that $d_1 = 0$ at every point of U. This follows from GD(3.32), which shows that $\tilde{d}_1 = d_1 - s$.

Remark 5. If D_1 is scalar on U, then $D_{ab} = d_1\delta_{ab}$, for all a and b. If we define the covariant derivative of d_1 to be

(3.7) $dd_1 + 2d_1\omega_0^0 = \sum_i d_{1i}\theta^i$

then by GD(3.43)

$$\sum_j D_{abj}\theta^j = dD_{ab} + 2D_{ab}\omega_0^0 - \sum_c D_{cb}\omega_a^c - \sum_c D_{ac}\omega_b^c$$

(3.8)
$$= \delta_{ab}(dd_1 + 2d_1\omega_0^0) - d_1(\omega_a^b + \omega_b^a)$$

$$= \delta_{ab}\sum_i d_{1i}\theta^i$$

In particular, $d_{1i} = D_{aai}$, for all a and i. This same principle applies to all eight of the functions $d_1, \ldots, e_4$, when all eight of these matrices are scalar.

Lemma 6. *Suppose the Lie curvature $r = -1$, and the multiplicities satisfy $m_1 = m_2$ and $m_3 = m_4$. If $Y : U \to G$ is a second order Lie frame field for which*

(3.9)
$$|v_{a\alpha}| = |v_{p\mu}|, \quad and \quad |v_{a\mu}| = |v_{p\alpha}|$$

for all a, p, α, and μ, then the eight matrices $D_1, \ldots, E_4$ are scalar on U and Y can be adjusted by a change (3.6) on U so that

(3.10)
$$d_1 = 0$$

on U, and then

(3.11)
$$d_2 = e_1, \ d_4 = d_3, \ e_2 = 0, \ e_3 = -d_3, \ e_4 = d_3$$

(3.12)
$$|v_{a\alpha}| = |v_{p\mu}| = |v_{p\alpha}| = |v_{a\mu}|$$

on U, for all a, p, α, and μ, and

(3.13)
$$\omega_{n+1}^0 = 0$$

on U. That is, by (2.28), $R_i = 0$ on U, for all i.

Proof. As described in Remark 4, a frame change (3.6) will give (3.10), and then (3.11) follows from GD(3.42) by linear algebra. Putting (3.11) into (3.5), we obtain (3.12). Finally, to prove (3.13), use GD(3.46i), for any c and any $a = b$, to get

$$d_{1c} = D_{aac} = -R_c - \frac{2}{m_1}\sum_{p,\alpha} D_{p\alpha}(F_{pc}^\alpha + 2\delta_{ac}F_{pa}^\alpha)$$

(3.14)
$$- \frac{2}{m_1}\sum_{p,\mu} E_{p\mu}(F_{pc}^\mu + 2\delta_{ac}F_{pa}^\mu)$$

$$+ \frac{8}{m_1}\sum_{\alpha,p,\mu} (F_{pa}^\mu F_{pa}^\alpha F_{\alpha c}^\mu + F_{pa}^\mu F_{pc}^\alpha F_{\alpha a}^\mu + F_{pc}^\mu F_{pa}^\alpha F_{\alpha a}^\mu)$$

and GD(3.62) with $a = b$ and any c, to get

$$\sum_{p,\alpha} D_{p\alpha}(F_{pc}^{\alpha} + 2\delta_{ac}F_{pa}^{\alpha}) + \sum_{p,\mu} E_{p\mu}(F_{pc}^{\mu} + 2\delta_{ac}F_{pa}^{\mu})$$

(3.15)

$$= 4\sum_{\alpha,p,\mu}(F_{pa}^{\mu}F_{pa}^{\alpha}F_{\alpha c}^{\mu} + F_{pa}^{\mu}F_{pc}^{\alpha}F_{\alpha a}^{\mu} + F_{pc}^{\mu}F_{pa}^{\alpha}F_{\alpha a}^{\mu})$$

Substitute (3.15) into (3.14) to get

(3.16) $$d_{1c} = -R_c$$

Since $d_1 = 0$ on U, we have $d_{1j} = 0$ on U, for every j, by (3.7). Therefore,

(3.17) $$R_c = 0$$

on U, for all c.

For any $p = q$ and for all c in GD(3.51i)

(3.18) $$e_{2c} = E_{ppc} = R_c + 2\sum_{\mu} E_{p\mu}F_{pc}^{\mu} + 2\sum_{\alpha} D_{p\alpha}F_{pc}^{\alpha}$$

By GD(3.51ii), we have for any $p = q$ and any s

$$e_{2s} = E_{pps} = R_s + \frac{2}{m_1}\sum_{a,\alpha} D_{a\alpha}(F_{sa}^{\alpha} + 2\delta_{ps}F_{pa}^{\alpha})$$

(3.19)

$$+ \frac{2}{m_1}\sum_{a,\mu} D_{\mu a}(F_{sa}^{\mu} + 2\delta_{ps}F_{pa}^{\mu})$$

$$- \frac{8}{m_1}\sum_{a,\alpha,\mu}(F_{\alpha s}^{\mu}F_{pa}^{\mu}F_{pa}^{\alpha} + F_{\alpha p}^{\mu}F_{sa}^{\mu}F_{pa}^{\alpha} + F_{\alpha p}^{\mu}F_{pa}^{\mu}F_{sa}^{\alpha})$$

By GD(3.63), for any $p = q$ and any s,

$$\sum_{a,\alpha} D_{a\alpha}(F_{sa}^{\alpha} + 2\delta_{ps}F_{pa}^{\alpha}) + \sum_{a,\mu} D_{\mu a}(F_{sa}^{\mu} + 2\delta_{ps}F_{pa}^{\mu})$$

(3.20)

$$= 4\sum_{a,\alpha,\mu}(F_{\alpha p}^{\mu}F_{pa}^{\alpha}F_{sa}^{\mu} + F_{\alpha p}^{\mu}F_{sa}^{\alpha}F_{pa}^{\mu} + F_{\alpha s}^{\mu}F_{pa}^{\alpha}F_{pa}^{\mu})$$

Substitute (3.20) into (3.19) to get

(3.21) $$e_{2s} = R_s$$

for all s. Since $e_2 = 0$ on U, we have $e_{2j} = 0$ on U, for all j, and therefore

(3.22) $$R_s = 0$$

for all s. By GD(3.48iii), for any $\alpha = \beta$ and for any γ

$$d_{3\gamma} = D_{\alpha\alpha\gamma} = R_\gamma + \frac{2}{m_1}\sum_{a,p} D_{pa}(F^\gamma_{pa} + 2\delta_{\alpha\gamma}F^\alpha_{pa})$$

(3.23)
$$+ \frac{4}{m_1}\sum_{p,\mu} E_{p\mu}(F^\mu_{\gamma p} + 2\delta_{\alpha\gamma}F^\mu_{\alpha p})$$

$$+ \frac{8}{m_1}\sum_{a,p,\mu}(F^\mu_{\gamma a}F^\alpha_{pa}F^\mu_{\alpha p} + F^\mu_{\gamma p}F^\alpha_{pa}F^\mu_{\alpha a} + F^\gamma_{pa}F^\mu_{\alpha a}F^\mu_{\alpha p})$$

By GD(3.64) for all $\alpha = \beta$ and for all γ

(3.24)
$$-\sum_{a,\mu} D_{\mu a}(F^\mu_{\gamma a} + 2\delta_{\alpha\gamma}F^\mu_{\alpha a}) - \sum_{p,\mu} E_{p\mu}(F^\mu_{\gamma p} + 2\delta_{\alpha\gamma}F^\mu_{\alpha p})$$

$$= 4\sum_{a,p,\mu}(F^\mu_{\gamma a}F^\alpha_{pa}F^\mu_{\alpha p} + F^\mu_{\gamma p}F^\alpha_{pa}F^\mu_{\alpha a} + F^\gamma_{pa}F^\mu_{\alpha a}F^\mu_{\alpha p})$$

By GD(3.52iii), for any $\alpha = \beta$ and for all γ

$$e_{3\gamma} = E_{\alpha\alpha\gamma} = R_\gamma - \frac{2}{m_1}\sum_{a,p} D_{pa}(F^\gamma_{pa} + 2\delta_{\alpha\gamma}F^\alpha_{pa})$$

(3.25)
$$+ \frac{4}{m_1}\sum_{a,\mu} D_{\mu a}(F^\mu_{\gamma a} + 2\delta_{\alpha\gamma}F^\mu_{\alpha a})$$

$$+ \frac{8}{m_1}\sum_{a,p,\mu}(F^\mu_{\gamma a}F^\alpha_{pa}F^\mu_{\alpha p} + F^\mu_{\gamma p}F^\alpha_{pa}F^\mu_{\alpha a} + F^\gamma_{pa}F^\mu_{\alpha a}F^\mu_{\alpha p})$$

Now $e_3 + d_3 = 0$ on U, so $e_{3\gamma} + d_{3\gamma} = 0$ on U, for all γ, so (3.23) and (3.25) added together give on U, for all γ,

$$0 = 2R_\gamma$$

(3.26)
$$+ \frac{4}{m_1}\left(\sum_{p,\mu} E_{p\mu}(F^\mu_{\gamma p} + 2\delta_{\alpha\gamma}F^\mu_{\alpha p}) + \sum_{a,\mu} D_{\mu a}(F^\mu_{\gamma a} + 2\delta_{\alpha\gamma}F^\mu_{\alpha a})\right)$$

$$+ \frac{16}{m_1}\sum_{a,p,\mu}(F^\mu_{\gamma a}F^\alpha_{pa}F^\mu_{\alpha p} + F^\mu_{\gamma p}F^\alpha_{pa}F^\mu_{\alpha a} + F^\gamma_{pa}F^\mu_{\alpha a}F^\mu_{\alpha p})$$

and this with (3.24) implies

(3.27)
$$R_\gamma = 0$$

on U for all γ. In the same way, by GD(3.48iv)

(3.28)
$$d_{3\mu} = D_{\alpha\alpha\mu} = R_\mu - D_{\mu\alpha\alpha} + 6\sum_a D_{a\alpha}F^\mu_{\alpha a} + 2\sum_p D_{p\alpha}F^\mu_{\alpha p}$$

By GD(3.52iv),

$$(3.29) \qquad e_{3\mu} = E_{\alpha\alpha\mu} = R_\mu + D_{\mu\alpha\alpha} + 2\sum_a D_{a\alpha} F^\mu_{\alpha a} + 6\sum_p D_{p\alpha} F^\mu_{\alpha p}$$

Adding these equations together and using (3.27) and the fact that $d_{3\gamma} + e_{3\gamma} = 0$ on U, we get on U, for every α and μ,

$$(3.30) \qquad \sum_a D_{a\alpha} F^\mu_{\alpha a} + \sum_p D_{p\alpha} F^\mu_{\alpha p} = 0$$

Finally, by GD(3.49iv), for any $\mu = \nu$ and for any σ,

$$d_{4\sigma} = D_{\mu\mu\sigma} = -R_\sigma + \frac{2}{m_1}\sum_{a,p} D_{pa}(F^\sigma_{pa} + 2\delta_{\mu\sigma} F^\mu_{pa})$$

$$(3.31) \qquad + \frac{4}{m_1}\sum_{p,\alpha} D_{p\alpha}(F^\sigma_{\alpha p} + 2\delta_{\mu\sigma} F^\mu_{\alpha p})$$

$$+ \frac{8}{m_1}\sum_{a,p,\alpha}(F^\sigma_{\alpha a} F^\mu_{pa} F^\mu_{\alpha p} + F^\sigma_{\alpha p} F^\mu_{pa} F^\mu_{\alpha a} + F^\sigma_{pa} F^\mu_{\alpha a} F^\mu_{\alpha p})$$

By GD(3.65) with $\mu = \nu$ and for any σ,

$$(3.32) \qquad \begin{aligned} &\sum_{a,\alpha} D_{a\alpha}(F^\sigma_{\alpha a} + 2\delta_{\mu\sigma} F^\mu_{\alpha a}) + \sum_{p,\alpha} D_{p\alpha}(F^\sigma_{\alpha p} + 2\delta_{\mu\sigma} F^\mu_{\alpha p}) \\ &= -4\sum_{a,p,\alpha}(F^\sigma_{\alpha a} F^\mu_{pa} F^\mu_{\alpha p} + F^\sigma_{\alpha p} F^\mu_{pa} F^\mu_{\alpha a} + F^\sigma_{pa} F^\mu_{\alpha a} F^\mu_{\alpha p}) \end{aligned}$$

By GD(3.53iv), for all $\mu = \nu$ and for any σ,

$$e_{4\sigma} = E_{\mu\mu\sigma} = R_\sigma + \frac{2}{m_1}\sum_{a,p} D_{pa}(F^\sigma_{pa} + 2\delta_{\mu\sigma} F^\mu_{pa})$$

$$(3.33) \qquad - \frac{4}{m_1}\sum_{a,\alpha} D_{a\alpha}(F^\sigma_{\alpha a} + 2\delta_{\mu\sigma} F^\mu_{\alpha a})$$

$$- \frac{8}{m_1}\sum_{a,p,\alpha}(F^\sigma_{\alpha a} F^\mu_{pa} F^\mu_{\alpha p} + F^\sigma_{\alpha p} F^\mu_{pa} F^\mu_{\alpha a} + F^\sigma_{pa} F^\mu_{\alpha a} F^\mu_{\alpha p})$$

Now $e_4 = d_4$ on U implies that $e_{4\sigma} - d_{4\sigma} = 0$ on U, so by (3.31) and (3.33) we get

$$0 = e_{4\sigma} - d_{4\sigma} = 2R_\sigma$$

(3.34)
$$- \frac{4}{m_1} \left(\sum_{a,\alpha} D_{a\alpha}(F^\sigma_{\alpha a} + 2\delta_{\mu\sigma} F^\mu_{\alpha a}) + \sum_{p,\alpha} D_{p\alpha}(F^\sigma_{\alpha p} + \delta_{\mu\sigma} F^\mu_{\alpha p}) \right)$$

$$- \frac{16}{m_1} \sum_{a,p,\alpha} (F^\sigma_{\alpha a} F^\mu_{pa} F^\mu_{\alpha p} + F^\sigma_{\alpha p} F^\mu_{pa} F^\mu_{\alpha a} + F^\sigma_{pa} F^\mu_{\alpha a} F^\mu_{\alpha p})$$

$$= 2R_\sigma$$

by (3.32). Therefore, on U,

(3.35) $$R_\sigma = 0$$

for every σ. Therefore, (3.13) holds by (3.17), (3.22), (3.27), and (3.35). □

Theorem 7. *Suppose the multiplicities satisfy $m_1 = m_2$, $m_3 = m_4$, and the Lie curvature is $r = -1$. Suppose that for any point in M there exists a second order frame field $Y : U \to G$ along λ on an open set $U \subset M$ about the point, such that equations (3.9) hold on U, for all a, p, α, μ. If for some a, α*

(3.36) $$|v_{a\alpha}| > 0$$

on an open dense subset of U; and if

(3.37) $$d\omega_0^0 = 0$$

on U, then $\lambda : M \to \Lambda$ is Lie equivalent to an isoparametric hypersurface.

Proof. Given any point of M, let $Y : U \to G$ be a second order frame field about the point satisfying the hypotheses. By Lemma 6, we may assume Y satisfies (3.10), (3.11), and (3.12) on U. Thus, (3.36) implies that all the functions in (3.12) are positive on U. These properties are preserved by any frame change of the form

(3.38) $$\tilde{Y} = Y a(tI_2, I, 0, 0)$$

where t is any nowhere zero real analytic function on U, in which case

(3.39) $$\tilde{\omega}_0^0 = \omega_0^0 + d\log|t|$$

on U. We may assume that U is contractible. Then (3.37) implies that

(3.40) $$\omega_0^0 = df$$

for some real analytic function f on U. Making the frame change (3.38) with $t = e^{-f}$, we have

$$(3.41) \qquad\qquad\qquad \tilde{\omega}_0^0 = 0$$

on U. We now continue with this frame and drop the tildes. By (2.24), our hypothesis $d\omega_0^0 = 0$ on U implies that $D_{a\alpha}$ and its covariant derivatives $D_{a\alpha j}$ are identically zero on U. Then GD(3.54) with (3.10) and (3.11) implies that

$$(3.42) \qquad\qquad \begin{aligned} 0 &= (d_3 - d_2) F_{pa}^\alpha \\ 0 &= (d_4 + e_4 - 2e_1 - e_3 - d_3) F_{\alpha a}^\mu = 2(d_3 - d_2) F_{\alpha a}^\mu \end{aligned}$$

on U, for any a, p, α, and μ. Thus,

$$(3.43) \qquad\qquad\qquad 0 = (d_3 - d_2)^2 |v_{a\alpha}|^2$$

at every point of U, for all a, α, and so (3.36) implies

$$(3.44) \qquad\qquad\qquad d_3 - d_2 = 0$$

on an open dense subset of U, hence on all of U, by continuity. So, (3.11) becomes

$$(3.45) \qquad d_1 = e_2 = 0, \quad e_1 = d_2 = d_3 = d_4 = e_4 = -e_3$$

on U. Since $d\omega_0^0 = 0$ and $\omega_{n+1}^0 = 0$ on U, we get from GD(3.47) that

$$(3.46) \qquad\qquad\qquad d_{2a} = d_{2\alpha} = d_{2\mu} = 0$$

on U, for all a, α, μ, and from GD(3.48) that

$$(3.47) \qquad\qquad\qquad d_{3p} = 0$$

on U, for all p. Since $d_2 = d_3$ on U, it follows that d_2 is covariant constant on U, and therefore d_2 must be constant on U, since $\omega_0^0 = 0$ in (3.7) and U is connected. Putting (3.45) into (3.5) and using (3.36), we have

$$(3.48) \qquad\qquad\qquad -2d_2 = |v_{a\alpha}|^2 > 0$$

on an open dense subset of U. Therefore, d_2 is a negative constant. Making a frame change (3.38) with the constant

$$(3.49) \qquad\qquad\qquad t = \sqrt{-d_2}$$

we have, by GD(3.32), that

$$(3.50) \qquad\qquad\qquad \tilde{d}_2 = \frac{1}{t^2} d_2 = -1$$

at every point of U. Hence, we may assume that

$$(3.51) \qquad\qquad\qquad d_2 = -1$$

on U. We have thus proved that for any point of M, there exists a second order frame field $Y : U \to G$ on a neighborhood of that point for which $\omega_0^0 = 0$, $\omega_{n+1}^0 = 0$, and

$$(3.52) \qquad d_1 = e_2 = 0, \quad e_1 = d_2 = d_3 = d_4 = e_4 = -e_3 = -1$$

on U. The following equations then follow from the structure equations

$$dY_A = \sum_{B=0}^{n+2} \omega_A^B Y_B$$

and the properties of our frame.

$$(3.53)$$
$$dY_0 = \sum_p \theta^p Y_p + \sum_\alpha \theta^\alpha Y_\alpha + \sum_\mu \theta^\mu Y_\mu$$

$$dY_1 = \sum_a \theta^a Y_a - \sum_\alpha \theta^\alpha Y_\alpha + \sum_\mu \theta^\mu Y_\mu$$

$$dY_{n+1} = -\sum_a \theta^a Y_a + \sum_\alpha \theta^\alpha Y_\alpha - \sum_\mu \theta^\mu Y_\mu$$

$$dY_{n+2} = -\sum_p \theta^p Y_p - \sum_\alpha \theta^\alpha Y_\alpha - \sum_\mu \theta^\mu Y_\mu$$

If we let

$$(3.54) \qquad W_1 = Y_0 + Y_{n+2}, \quad W_2 = Y_1 + Y_{n+1}$$

then equations (3.53) show that

$$(3.55) \qquad dW_1 = 0, \quad dW_2 = 0$$

on U, so W_1 and W_2 are constant vectors (assuming U connected). In addition, they span a time-like line in $\mathbf{R}_2^{n+3}$, since

$$(3.56)$$
$$\langle W_1, W_1 \rangle = -2$$
$$\langle W_2, W_2 \rangle = -2$$
$$\langle W_1, W_2 \rangle = 0$$

Then W_1, W_2, $W_1 - W_2$, and $W_1 + W_2$ are four points on this time-like line such that

$$(3.57)$$
$$\langle Y_0, W_1 \rangle = 0$$
$$\langle Y_1, W_2 \rangle = 0$$
$$\langle Y_0 + Y_1, W_1 - W_2 \rangle = 0$$
$$\langle -Y_0 + Y_1, W_1 + W_2 \rangle = 0$$

on U. If $\tilde{Y} : \tilde{U} \to G$ is another Lie frame field satisfying (3.52), then on the intersection $\tilde{U} \cap U$ (supposed nonempty) they must be related by

$$\tilde{Y} = Ya(I_2, B, 0, 0)$$

where $B : U \cap \tilde{U} \to O(n-1)$ is an analytic map. In particular,

$$\tilde{Y}_0 = Y_0, \ \tilde{Y}_1 = Y_1, \ \tilde{Y}_{n+1} = Y_{n+1}, \ \tilde{Y}_{n+2} = Y_{n+2}$$

and therefore (3.57) holds for $\tilde{Y}$, for the same vectors W_i, for $i = 1, 2, 3, 4$, and thus (3.57) holds on all of M for the four curvature spheres. By Cecil's Theorem 5.6 ([3, pp 102-103]), $\lambda : M \to \Lambda$ is Lie equivalent to the Legendre submanifold induced by an isoparametric hypersurface. $\square$

4. REDUCIBILITY

Before we return to the case of a Dupin hypersurface with four principal curvatures, we prove some general results about reducible Dupin hypersurfaces.

Pinkall [17] introduced the basic constructions of building tubes, cylinders, cones, and surfaces of revolution over a Dupin hypersurface M^{n-1} in $\mathbf{R}^n$ with g principal curvatures to get a Dupin hypersurface W^{n-1-k} in $\mathbf{R}^{n+k}$ with $g + 1$ principal curvatures. In general, these constructions introduce a new principal curvature of multiplicity k, which is easily seen to be constant along its curvature surfaces. The other principal curvatures are determined by the principal curvatures of M^{n-1}, and the Dupin property is preserved for these principal curvatures. A Dupin hypersurface that is locally Lie equivalent to a hypersurface obtained by one of these constructions is said to be reducible. In Theorem 4 of his paper, Pinkall gave a formulation of reducibility in terms of Lie sphere geometry. As in the paper [5], we use this formulation as our definition of reducibility on an open subset of a Dupin submanifold as follows.

Definition 8. We define the Dupin submanifold $\lambda : M \to \Lambda$ to be *reducible on an open subset* $O \subset M$ if some curvature sphere maps O into some fixed linear subspace of $\mathbf{R}P^{n+2}$ of codimension at least two. We say that λ is *reducible* if it is reducible on M. Define λ to be *locally irreducible* if it is not reducible on any open subset of M. Define λ to be *irreducible* if it is not reducible.

Proposition 9. *If a connected, proper Dupin submanifold is reducible on an open subset U of M, then it is reducible. Thus, a connected,*

proper Dupin submanifold is locally irreducible if and only if it is irreducible.

Proof. Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be a connected, proper Dupin submanifold. Suppose that there exists an open subset $U \subset M$ such that the restriction of λ to U is reducible. By the definition of reducibility, there exists a curvature sphere $[K]$ of λ and two linearly independent vectors $v_i \in \mathbf{R}_2^{n+3}$, $i = 1, 2$, such that

$$\langle K, v_i \rangle = 0$$

on the set U. Since λ is analytic, the curvature sphere map $[K]$ is analytic on M, and so the functions $\langle K, v_i \rangle$ are analytic on M. Since these functions equal zero on the open set U, they are equal to zero on all of the connected manifold M, and thus $\lambda : M \to \Lambda$ is reducible. $\square$

This result has ramifications for the case of proper Dupin hypersurfaces with $g = 3$ principal curvatures.

Corollary 10. *Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be an irreducible proper Dupin submanifold with $g = 3$ principal curvatures. Then λ is Lie equivalent to an isoparametric hypersurface in S^n.*

Proof. In the case where all the principal curvatures have multiplicity one, this was proven by Pinkall [18]. For the case of higher multiplicities, this was proven in [5, p. 175] under the assumption that λ is locally irreducible. By Proposition 9, we see that the hypothesis of irreducibility is sufficient. $\square$

We now prove a characterization of reducibility for proper Dupin submanifolds. Note that we need only use three of Pinkall's four constructions, since as Pinkall showed, the cone construction is locally Lie equivalent to the tube construction.

Proposition 11. *Let $\nu : W^{d-1} \to \Lambda^{2d-1}$ be a connected, reducible proper Dupin submanifold. Then ν is Lie equivalent to a proper Dupin submanifold μ which is obtained from a lower dimensional proper Dupin submanifold λ by one of Pinkall's three constructions (tube, cylinder, surface of revolution).*

Proof. It is possible that the curvature sphere $[K]$ of ν locally lies in a linear subspace of codimension even higher than two. For each $x \in W$, let m_x be the largest positive integer such that for some neighborhood U_x, the curvature sphere map $[K]$ restricted to U_x is contained in a linear space of codimension $m_x + 1$ in $\mathbf{R}P^{d+2}$. By hypothesis, we know that $m_x \geq 1$ for all $x \in W$. Choose x_0 to be a point where m_x attains

its maximum value m. Then there exist linearly independent vectors $v_1, \ldots, v_{m+1}$ in $\mathbf{R}_2^{d+3}$ such that on an open set U_{x_0} about x_0,

$$(4.1) \qquad\qquad \langle K, v_i \rangle = 0$$

for $1 \leq i \leq m + 1$. Since ν is analytic, the curvature sphere map $[K] : W \to Q$ is analytic, and since the analytic functions $\langle K, v_i \rangle$ equal zero on the open set U_{x_0}, they must equal zero on the whole connected manifold W. Thus, (4.1) holds on all of W, and the function $m_x = m$ for all $x \in W$.

The rest of the proof is essentially the same as the proof of Theorem 2.8 in [3, pp. 145-147]. Specifically, let E be the linear subspace in $\mathbf{R}_2^{d+3}$ of codimension $m + 1$ whose orthogonal complement $E^\perp$ is spanned by the vectors $v_1, \ldots, v_{m+1}$. The signature of $\langle\, , \rangle$ on $E^\perp$ must be $(m + 1, 0)$, $(m, 1)$, or $(m, 0)$. Then, as in the proof of Theorem 2.8, one can show that there is a Lie transformation A such that $\mu = A\nu$ is obtained from a proper Dupin submanifold $\lambda : M^{n-1} \to \Lambda^{2n-1}$, where $n = d - m$, by the surface of revolution, tube, or cylinder construction, depending on whether the signature of the inner product on $E^\perp$ is $(m + 1, 0)$, $(m, 1)$, or $(m, 0)$, respectively. The proof of Theorem 2.8 deals specifically with the case where $[K]$ has multiplicity m, and so μ has one more curvature sphere than λ. In the case where the multiplicity of $[K]$ is greater than m, one must make some slight adjustments in the exposition of the proof. In that case, the curvature sphere $A[K]$ is equal to one of the curvature spheres of μ induced from a curvature sphere $[k]$ of λ, and the multiplicity of $[K]$ is $m + q$, where q is the multiplicity of $[k]$ as a curvature sphere of λ. In that case, μ and λ have the same number of distinct curvature spheres. $\qquad\square$

We return to the case where $\lambda : M^{n-1} \to \Lambda^{2n-1}$ is a proper Dupin submanifold with four curvature spheres of multiplicities $m_1 = m_2$, $m_3 = m_4$, and with constant Lie curvature $r = -1$.

Definition 12. For a second order frame field $Y : U \to G$, let $\mathcal{I}_1'$ be the set of all $a \in \mathcal{I}_1$ such that

$$(4.2) \qquad\qquad F_{pa}^\alpha = 0, \; F_{pa}^\mu = 0, \; F_{\alpha a}^\mu = 0$$

on U, for all p, α, μ.

Proposition 13. *If $m_1 = m_2$, $m_3 = m_4$, and $r = -1$, and if $\mathcal{I}_1'$ is nonempty for the second order frame field $Y : U \to G$, then the symmetric matrices D_1, E_2, D_3, D_4, E_3, and E_4 are scalar matrices,*

$$(4.3) \qquad\qquad E_1 = \begin{pmatrix} (d_1 + e_3)I_m & 0 \\ 0 & * \end{pmatrix}$$

where m is the cardinality of $\mathcal{I}_1'$, and

$$(4.4) \qquad\qquad d_1 + e_2 = 0$$

at every point of U. Here $d_1, \ldots, e_4$ are defined in (3.2).

Proof. By Lemma 3, just the assumptions on the multiplicities and on r imply that D_1, E_2, $D_3 + E_3$, and $D_4 - E_4$ are scalar matrices on U. They are given by (3.1). For each $a \in \mathcal{I}_1'$ and $e \in \mathcal{I}_1$, the left side of GD(3.36ii) is zero, so that

$$(4.5) \qquad 0 = 2 \sum_p F_{pa}^\alpha F_{pa}^\beta + 4 \sum_\mu F_{\alpha a}^\mu F_{\beta a}^\mu = (d_1 - E_{aa})\delta_{\alpha\beta} + E_{\alpha\beta}$$

on U for all α, β. Therefore, $E_3 = e_3 I_{m_3}$ is a scalar matrix, where

$$(4.6) \qquad\qquad e_3 = E_{aa} - d_1$$

for every $a \in \mathcal{I}_1'$. Since $D_3 + E_3$ is scalar, it follows that

$$(4.7) \qquad\qquad D_3 = d_3 I_{m_3}$$

is scalar also. Then for any $a \in \mathcal{I}_1'$, $e \in \mathcal{I}_1$, and $\alpha = \beta$ in GD(3.36ii), we have

$$(4.8) \qquad 0 = 2 \sum_p F_{pa}^\alpha F_{pe}^\alpha + 4 \sum_\mu F_{\alpha a}^\mu F_{\alpha e}^\mu = (d_1 - E_{ae}) + e_3 \delta_{ae}$$

on U. Therefore,

$$(4.9) \qquad\qquad E_{ae} = d_1 + e_3 \delta_{ae}$$

from which (4.3) follows. In the same way, for all $a = b \in \mathcal{I}_1'$, the left side of GD(3.36iv) is zero, and so we have

$$(4.10) \qquad 0 = -2 \sum_p F_{pa}^\mu F_{pa}^\nu - 4 \sum_\alpha F_{\alpha a}^\nu F_{\alpha a}^\nu = E_{\mu\nu} + (2d_1 + e_3)\delta_{\mu\nu}$$

on U, for all μ, ν. Therefore, $E_4 = e_4 I_{m_3}$ is a scalar matrix, where

$$(4.11) \qquad\qquad e_4 = -(2d_1 + e_3)$$

Since $D_4 - E_4$ is a scalar matrix, it follows that

$$(4.12) \qquad\qquad D_4 = d_4 I_{m_3}$$

is scalar also. Finally, if $a \in \mathcal{I}_1'$, then the left side of GD(3.36i) is zero, so (4.4) holds. $\qquad\square$

Corollary 14. *Under the hypotheses of Proposition 13, the second order frame $Y : U \to G$ can be chosen so that*

$$(4.13) \qquad d_1 = 0,\ e_2 = 0,\ e_3 = -e_4,\ d_3 = d_4,\ e_a = e_3,\ \forall a \in \mathcal{I}_1'$$

Proof. If we make a change of frame of the form (2.14)

$$(4.14) \qquad \tilde{Y} = Ya(tI_2, B, 0, sL)$$

with $t = 1$, $B = I$ and $s = d_1$, then by GD(3.32) and (4.4)

$$(4.15) \qquad \tilde{d}_1 = 0 = \tilde{e}_2$$

Dropping the tildes, we see that two of the remaining equations in (4.13) then follow from (4.6) and (4.11). It remains to prove that $d_3 = d_4$ in this frame. For this we use GD(3.42). In fact, using the already established equations in (4.13) and GD(3.42ii), we have

$$(4.16) \qquad (m_1 + \frac{m_3}{2})(d_3 + e_3) = m_1(e_1 - d_2) - \frac{m_3}{2}(d_4 + e_3)$$

From GD(3.42iv), we have

$$(4.17) \qquad (m_1 + \frac{m_3}{2})(-e_3 - d_4) = m_1(-e_1 + d_2) + \frac{m_3}{2}(e_3 + d_3)$$

Adding (4.16) and (4.17), we get

$$(4.18) \qquad (m_1 + \frac{m_3}{2})(d_3 - d_4) = \frac{m_3}{2}(d_3 - d_4)$$

from which we conclude that $d_3 = d_4$. $\qquad\qquad\square$

We shall call a second order frame $Y : U \to G$ *normalized* if it satisfies (4.13).

For a second order frame $Y : U \to G$, let

$$(4.19) \qquad \begin{aligned} f &= 2\sum_{a,p,\alpha}(F_{pa}^{\alpha})^2, \qquad g = 2\sum_{a,p,\mu}(F_{pa}^{\mu})^2 \\ h &= 2\sum_{a,p,\mu}(F_{\alpha a}^{\mu})^2, \qquad k = 2\sum_{p,\alpha,\mu}(F_{\alpha p}^{\mu})^2 \end{aligned}$$

By GD(3.10), a change of second order frame field (2.14) multiplies these functions by a nowhere zero function. In particular, the zero sets of these functions are globally well defined.

Theorem 15. *Suppose* $\lambda : M^{n-1} \to \Lambda^{2n-1}$ *is proper Dupin with multiplicities* $m_1 = m_2$, $m_3 = m_4$, *and Lie curvature* $r = -1$. *If at least three of* f, g, h, *and* k *are zero on* M, *then* $\lambda : M \to \Lambda^{2n-1}$ *is reducible.*

Proof. Suppose $f = g = h = 0$ at every point of M. Then

$$(4.20) \qquad F_{pa}^{\alpha} = F_{pa}^{\mu} = F_{\alpha a}^{\mu} = 0$$

for all a, p, α, μ at each point of U for any second order frame field $Y : U \to G$. Note that (4.20) is equivalent to

$$(4.21) \qquad \omega_a^p = \omega_a^{\alpha} = \omega_a^{\mu} = 0$$

on U, for all a, p, α, μ. Because $\mathcal{I}_1' = \mathcal{I}_1$ in this case, we may assume that Y is normalized, and then (4.13) implies that $E_1 = e_1 I_m$ is scalar with $e_1 = e_3$. By GD(3.37), GD(3.38), and GD(3.39) together with (4.20), we have that

$$(4.22) \qquad D_{p\alpha} = D_{a\alpha} = D_{pa} = E_{p\mu} = D_{\mu a} = D_{\mu\alpha} = 0$$

at every point of U. Therefore,

$$(4.23) \qquad\qquad d\omega_0^0 = 0, \ \omega_a^0 = 0 = \omega_p^1$$

at every point of U. Moreover,

$$(4.24) \qquad\qquad\qquad \omega_{n+1}^0 = 0$$

on U, because GD(3.46i), GD(3.46ii), and (4.20) imply that $R_a = R_p = 0$ on U, and GD(3.46iii), GD(3.46iv), and (4.20) imply that $R_\alpha = D_{a\alpha a} = 0$ and $R_\mu = D_{\mu aa} = 0$ on U. It then follows from GD(3.51) that e_3 is covariant constant; that is,

$$(4.25) \qquad\qquad\qquad de_3 + 2e_3\omega_0^0 = 0$$

on U.

Let $V(u)$ be the subspace of $\mathbf{R}_2^{n+3}$ defined by the span of the vectors

$$(4.26) \qquad\qquad Y_0, Y_p, Y_\alpha, Y_\mu, Y_{n+2}, e_3Y_1 - Y_{n+1}$$

for all p, α, μ at the point $u \in U$. Then $V(u)$ does not depend on the choice of normalized second order frame field at u, since any other is given by (2.14) with $s = 0$. Let V be the span of $V(u)$ for all $u \in M$. We want to prove that V is a subspace of codimension $m + 1$, because the curvature sphere $[Y_0]$ on U takes all of its values in V then shows that λ is reducible on U, and therefore λ is reducible by Proposition 8. Since the codimension of $V(u)$ is $m + 1$ for any $u \in M$, we will obtain our result if we prove that $V(u)$ is constant on the domain U of any normalized second order frame field Y. This will be true if we show that the derivatives of the vectors spanning $V(u)$ are zero modulo $V(u)$, for every $u \in U$. This follows from (4.23), and (4.24) and (4.25),

and (4.21). In fact, if $\equiv$ denotes equality modulo $V(u)$, then

(4.27)
$$dY_0 \equiv 0$$
$$dY_p \equiv \omega_p^1 Y_1 + \omega_p^a Y_a + \omega_p^{n+1} Y_{n+1} = 0$$
$$dY_\alpha \equiv \omega_\alpha^1 Y_1 + \omega_\alpha^a Y_a + \omega_\alpha^{n+1} Y_{n+1}$$
$$= (e_3 Y_1 - Y_{n+1})\theta^\alpha \equiv 0$$
$$dY_\mu \equiv \omega_\mu^1 Y_1 + \omega_\mu^a Y_a + \omega_\mu^{n+1} Y_{n+1}$$
$$= (-e_3 Y_1 + Y_{n+1})\theta^\mu \equiv 0$$
$$dY_{n+2} \equiv \omega_{n+2}^1 Y_1 + \omega_{n+2}^a Y_a + \omega_{n+2}^{n+1} Y_{n+1} = 0$$
$$d(e_3 Y_1 - Y_{n+1}) \equiv de_3 Y_1 + e_3(\omega_1^1 Y_1 + \theta^a Y_a) - (\omega_{n+1}^a Y_a + \omega_{n+1}^{n+1} Y_{n+1})$$
$$= (de_3 + e_3\omega_0^0)Y_1 + \omega_0^0 Y_{n+1} = (-e_3 Y_1 + Y_{n+1})\omega_0^0 \equiv 0$$

$\square$

Theorem 16. *Let $\lambda : M \to \Lambda$ be a proper Dupin hypersurface for which $r = -1$, $m_1 = m_2$, and $m_3 = m_4$. If $Y : U \to G$ is a second order frame field such that*

(4.28)
$$d\omega_0^0 = 0$$

on U, and

(4.29)
$$\mathcal{I}_1' \neq \emptyset$$

(see Definition 12), then there exists a nonempty open subset W of U such that the curvature sphere $[Y_0] : W \to \mathbf{R}P^{n+2}$ takes values in a constant linear subspace of codimension at least 2. Thus, λ is reducible on W.

Proof. By Proposition 13 and Corollary 14, we may assume (4.13) and (4.3) hold. Choosing U to be contractible, we may assume

(4.30)
$$\omega_0^0 = 0$$

on U. In fact, if U is contractible, then $\omega_0^0 = df$, for some function f on U. A change of frame $\tilde{Y} = Ya(tI_2, I, 0, 0)$ doesn't affect the assumptions already made, and $\tilde{\omega}_0^0 = \omega_0^0 + dt/t$. Thus, take $t = e^{-f}$. There exists a dense open subset of U on which Y can be chosen to diagonalize E_1 and D_2. Let $\tilde{U}$ be an open connected component of this open dense subset. It follows from GD(3.46) and (4.28), that for all $a \in \mathcal{I}_1'$ and for all j,

(4.31)
$$0 = d_{aj} = D_{aaj} = -R_j$$

on $\tilde{U}$, and hence

(4.32) $$\omega_{n+1}^0 = 0$$

on $\tilde{U}$. From (4.28) and (2.24), we know that the six sets of invariants $D_{a\alpha}$, etc. and their covariant derivatives are identically zero on U. Using (4.13), we get from GD(3.54)

(4.33)
$$0 = D_{e\alpha p} = (-d_p + d_3)F_{pe}^\alpha$$
$$0 = D_{e\alpha\mu} = -2(e_e + e_3)F_{\alpha e}^\mu$$

on $\tilde{U}$, for all e, p, α, μ; from GD(3.55)

(4.34)
$$0 = D_{pe\alpha} = (e_e + e_3 + d_p - d_3)F_{pe}^\alpha$$
$$0 = D_{pe\mu} = (e_e + e_3 + d_p - d_3)F_{pe}^\mu$$

on $\tilde{U}$, for all e, p, α, μ; from GD(3.56)

(4.35)
$$0 = D_{p\alpha e} = (e_e + e_3)F_{pe}^\alpha$$
$$0 = D_{p\alpha\mu} = 2(d_p - d_3)F_{\alpha p}^\mu$$

on $\tilde{U}$, for all e, p, α, μ; from GD(3.57)

(4.36)
$$0 = D_{\mu ep} = (d_p - d_3)F_{pe}^\mu$$
$$0 = D_{\mu e\alpha} = -2(e_e + e_3)F_{\alpha e}^\mu$$

on $\tilde{U}$, for all e, p, α, μ; from GD(3.58)

(4.37)
$$0 = D_{\mu\alpha e} = -2(e_e + e_3)F_{\alpha e}^\mu$$
$$0 = D_{\mu\alpha p} = 2(d_p - d_3)F_{\alpha p}^\mu$$

on $\tilde{U}$, for all e, p, α, μ; and from GD(3.59)

(4.38)
$$0 = E_{p\mu e} = -(e_e + e_3)F_{pe}^\mu$$
$$0 = E_{p\mu\alpha} = 2(d_p - d_3)F_{\alpha p}^\mu$$

on $\tilde{U}$, for all e, p, α, μ. In summary, we have

(4.39) $\quad 0 = (e_e + e_3)F_{pe}^\alpha, \quad 0 = (e_e + e_3)F_{pe}^\mu, \quad 0 = (e_e + e_3)F_{\alpha e}^\mu$

on $\tilde{U}$ for all e, p, α, μ, and

(4.40) $\quad 0 = (d_p - d_3)F_{pe}^\mu, \quad 0 = (d_p - d_3)F_{\alpha p}^\mu, \quad 0 = (d_p - d_3)F_{pe}^\alpha$

on $\tilde{U}$, for all e, p, α, μ. In the present proof, equations (4.40) are not needed, but we record them here for use in the proof of Theorem 23 below. For each $e \in \mathcal{I}_1$, define the analytic function on U

(4.41)
$$A_e = \sum_\alpha |v_{e\alpha}|^2 + \sum_\mu |v_{e\mu}|^2 = 2\sum_{p,\alpha}(F_{pe}^\alpha)^2 + 2\sum_{p,\mu}(F_{pe}^\mu)^2 + 8\sum_{\alpha,\mu}(F_{\alpha e}^\mu)^2$$

Let

$$(4.42) \qquad n(A_e) = \{x \in U : A_e(x) > 0\}$$

an open subset of U. Let $\partial n(A_e)$ be the boundary of $n(A_e)$ in U. If

$$(4.43) \qquad \tilde{U} \cap n(A_e) = \emptyset$$

for all $e \in \mathcal{I}_1$, then Theorem 15 applies and we conclude that λ is reducible on $\tilde{U}$. The proof of Theorem 15 shows that e_3 is constant on $\tilde{U}$ in this case, by (4.25) and the fact that $\omega_0^0 = 0$ now. If

$$(4.44) \qquad \tilde{U} \cap n(A_c) \neq \emptyset$$

for some $c \in \mathcal{I}_1$, then there exists a point

$$(4.45) \qquad x \in \left(\tilde{U} \cap (\cup_e n(A_e)) \right) \setminus \cup_e \partial n(A_e)$$

and for such a point there exists a connected open neighborhood W of x such that $W \subset \tilde{U}$ and for every $e \in \mathcal{I}_1$, either A_e is identically zero on W or A_e is always positive on W. Let

$$(4.46) \qquad \widetilde{\mathcal{I}_1}' = \{a \in \mathcal{I}_1 : A_a = 0 \text{ on } W\}$$

and let $\mathcal{I}_1''$ be the complement of $\widetilde{\mathcal{I}_1}'$ in $\mathcal{I}_1$. Thus,

$$(4.47) \qquad \mathcal{I}_1'' = \{c \in \mathcal{I}_1 : A_c > 0 \text{ on } W\}$$

and $\mathcal{I}_1'' \neq \emptyset$. If $c \in \mathcal{I}_1''$, then for each $x \in W$, there exists p, α, or μ such that $F_{pc}^{\alpha}(x) \neq 0$ or $F_{pc}^{\mu}(x) \neq 0$ or $F_{\alpha c}^{\mu}(x) \neq 0$. Therefore,

$$(4.48) \qquad e_c = -e_3$$

on W, for any $c \in \mathcal{I}_1''$, by (4.39). We have

$$(4.49) \qquad e_3 > 0$$

on W, because if $c \in \mathcal{I}_1''$, then by (3.4),

$$(4.50) \qquad 0 < A_c = 2m_3(e_3 - e_c) = 4m_3 e_3$$

on W, by (4.48). We next prove that

$$(4.51) \qquad \omega_c^a = 0$$

on W, for all $a \in \widetilde{\mathcal{I}_1}'$ and $c \in \mathcal{I}_1''$. To do this, we observe that $E_{ac} = 0$ on W, for all $a \in \widetilde{\mathcal{I}_1}'$ and all $c \in \mathcal{I}_1''$, since E_1 is diagonalized on W. Therefore, using (4.48), we have

$$(4.52) \quad \begin{aligned} \sum_j E_{acj}\theta^j &= dE_{ac} + 2E_{ac}\omega_0^0 - \sum_e E_{ec}\omega_a^c - \sum_e E_{ae}\omega_c^e \\ &= (e_a - e_c)\omega_a^c = 2e_3\omega_a^c \end{aligned}$$

on W. But GD(3.50) with (4.28) and the definition of $\widetilde{\mathcal{I}}_1{}'$ imply that

$$(4.53) \qquad\qquad\qquad E_{acj} = 0$$

on W, for all $a \in \widetilde{\mathcal{I}}_1{}'$, $c \in \mathcal{I}_1''$, and all j. Then (4.51) follows from (4.49), (4.52), and (4.53). In addition, e_3 must be constant on W. In fact, from GD(3.52) it is seen that $e_{3e} = 0$, $e_{3p} = 0$, and $e_{3\mu} = 0$ on W. If $a \in \widetilde{\mathcal{I}}_1{}'$, then $e_3 = e_a$ on W, so $e_{3\alpha} = e_{a\alpha} = E_{aa\alpha} = 0$ by GD(3.50). Hence, $e_{3j} = 0$ on W for all j, and so

$$(4.54) \qquad\qquad de_3 = de_3 + 2e_3\omega_0^0 = \sum_j e_{3j}\theta^j = 0$$

on W.

The rest of the proof is now similar to the last part of the proof of Theorem 15. For each $x \in W$, let $V(x)$ be the subspace of $\mathbf{R}_2^{n+3}$ defined by the span of the vectors

$$(4.55) \qquad\qquad Y_0,\ Y_c,\ Y_p,\ Y_\alpha,\ Y_\mu,\ Y_{n+2},\ e_3 Y_1 - Y_{n+1}$$

at the point $x \in W$, for all $c \in \mathcal{I}_1''$, p, α, μ . Let V be the span of $V(x)$ for all $x \in W$. We want to prove that V is a subspace of codimension $m+1$, where $m \geq 1$ is the cardinality of $\widetilde{\mathcal{I}}_1{}'$. Because the codimension of $V(x)$ is $m+1$ for any $x \in W$, we will obtain our result if we prove that $V(x)$ is constant on W. This will be true if we show that the derivatives of the vectors spanning $V(x)$ are zero modulo $V(x)$, for every $x \in W$. This follows because

$$(4.56) \qquad\qquad \omega_p^a = 0, \quad \omega_\alpha^a = 0, \quad \omega_\mu^a = 0$$

on W, for all $a \in \widetilde{\mathcal{I}}_1{}'$, by (2.18) and the definition of $\widetilde{\mathcal{I}}_1{}'$. $\square$

Corollary 17. *Let $\lambda : M \to \Lambda$ be an irreducible proper Dupin hypersurface for which $r = -1$, $m_1 = m_2$, and $m_3 = m_4$. If $Y : U \to G$ is a second order frame field such that $d\omega_0^0 = 0$ on U, then*

$$(4.57) \qquad\qquad\qquad \mathcal{I}_1' = \emptyset$$

5. One pair of multiplicities is 1

Assume now that $\lambda : M^{n-1} \to \Lambda^{2n-1}$ is the Legendre lift of a Dupin hypersurface with four principal curvature, constant Lie curvature $r = -1$ and multiplicities $m_1 = m_2 \geq 2$ and $m_3 = m_4 = 1$. For these multiplicities, the index sets $\mathcal{I}_3$ and $\mathcal{I}_4$ consist of one element each

$$\mathcal{I}_3 = \{2m_1 + 2\}, \ \mathcal{I}_4 = \{2m_1 + 3\}$$

It is convenient to continue writing α and μ for these values, respectively. In addition, D_3, D_4, E_3, and E_4 are automatically scalar matrices in this case, for any second order frame field.

Proposition 18. *If $m_1 = m_2 \geq 2$, $m_3 = m_4 = 1$, and $r = -1$, then for any point in M there exists a second order frame field $Y : U \to G$ about the point, for which*

$$(5.1) \qquad\qquad\qquad d_1 = e_2$$

For any such frame field,

$$(5.2) \qquad\qquad F_{pa}^{\alpha} F_{pa}^{\mu} F_{\alpha a}^{\mu} = 0 = F_{pa}^{\alpha} F_{pa}^{\mu} F_{\alpha p}^{\mu}$$

on U, for all p, a; and

$$(5.3) \qquad\qquad d_{1a} = 0 = R_a, \quad d_{1p} = 0 = R_p$$

on U, for all a, p.

Proof. Let $Y : U \to G$ be a second order frame field. Then $D_1 = d_1 I_{m_1}$ and $E_2 = e_2 I_{m_1}$ are scalar matrices, by the first two equations in GD(3.42). A second order frame change (2.14) of the form $\tilde{Y} = Y a(I_2, I, 0, sL)$ has

$$(5.4) \qquad\qquad \tilde{e}_2 - \tilde{d}_1 = e_2 - d_1 + 2s$$

by GD(3.32). Taking $s = (d_1 - e_2)/2$, we obtain a second order frame field for which

$$(5.5) \qquad\qquad\qquad d_1 = e_2$$

We assume this done for our frame Y. Setting $a = b = c$ in GD(3.62i), we find that

$$(5.6) \qquad\qquad \sum_p D_{p\alpha} F_{pc}^{\alpha} + \sum_p E_{p\mu} F_{pc}^{\mu} = 4 \sum_p F_{pc}^{\alpha} F_{pc}^{\mu} F_{\alpha c}^{\mu}$$

By GD(3.51i), for each $p = q$ we have

$$(5.7) \qquad\qquad e_{2c} = E_{ppc} = R_c + 2E_{p\mu} F_{pc}^{\mu} + 2D_{p\alpha} F_{pc}^{\alpha}$$

which shows that

$$(5.8) \qquad\qquad E_{p\mu} F_{pc}^{\mu} + D_{p\alpha} F_{pc}^{\alpha}$$

is independent of $p \in \mathcal{I}_2$. Therefore, (5.6) becomes

$$(5.9) \qquad\qquad D_{p\alpha} F_{pc}^{\alpha} + E_{p\mu} F_{pc}^{\mu} = \frac{4}{m_1} \sum_q F_{qc}^{\alpha} F_{qc}^{\mu} F_{\alpha c}^{\mu}$$

for all $p \in \mathcal{I}_2$. Using GD(3.46i) with $a = b = c$, and using (5.6), we find

$$(5.10) \qquad\qquad d_{1c} = D_{ccc} = -R_c$$

By (5.5), $d_{1c} = e_{2c}$, so combining (5.7) and (5.10) and substituting the result into (5.6) gives

$$(5.11) \qquad R_c = \frac{4}{m_1} \sum_p F_{pc}^{\alpha} F_{pc}^{\mu} F_{\alpha c}^{\mu}$$

Multiply F_{pcc}^{α} by $2F_{pc}^{\alpha}$ and F_{pcc}^{μ} by $2F_{pc}^{\mu}$ in GD(3.37) and GD(3.38), respectively, subtract the latter from the former and use GD(3.36i), to get

$$(5.12) \qquad \begin{aligned} -d_{1c} &= ((F_{pc}^{\alpha})^2 - (F_{pc}^{\mu})^2)_c = 2F_{pc}^{\alpha}F_{pcc}^{\alpha} - 2F_{pc}^{\mu}F_{pcc}^{\mu} \\ &= 2(F_{pc}^{\alpha}D_{p\alpha} + F_{pc}^{\mu}E_{p\mu}) - 12F_{pc}^{\mu}F_{\alpha c}^{\mu}F_{pc}^{\alpha} \end{aligned}$$

Therefore, for each $c \in \mathcal{I}_1$,

$$(5.13) \qquad F_{pc}^{\mu} F_{\alpha c}^{\mu} F_{pc}^{\alpha}$$

is independent of p, by (5.8) and the fact that d_{1c} is independent of p. We show now that this implies (5.2). At any point of U, let $\mathcal{V}$ denote the vector subspace spanned by the Y_p. This subspace is invariant under a change of frame (2.14). On $\mathcal{V}$, for fixed c, define the bilinear form

$$S : \mathcal{V} \times \mathcal{V} \to \mathbf{R}$$

$$(5.14) \qquad S(\sum_p u^p Y_p, \sum_q v^q Y_q) = \sum_{p,q} F_{pc}^{\mu} F_{\alpha c}^{\mu} F_{qc}^{\alpha} u^p v^q$$

By GD(3.10), S does not depend on the choice of Y. Then (5.13) is the value of $S(Y_p, Y_p)$, so this value is independent of p. Let

$$(5.15) \qquad K = S(Y_p, Y_p)$$

for every $p \in \mathcal{I}_2$, an analytic function on U. If t and s are nonzero real numbers such that $t^2 + s^2 = 1$, and if $p \neq q$, then replacing Y_p and Y_q with $tY_p + sY_q$ and $-sY_p + tY_q$ gives another allowable frame Y, so S has the same value on each:

$$(5.16) \qquad S(tY_p + sY_q, tY_p + sY_q) = S(-sY_p + tY_q, -sY_p + tY_q)$$

so

$$(5.17) \qquad K + 2tsS(Y_p, Y_q) = K - 2stS(Y_p, Y_q)$$

Therefore,

$$(5.18) \qquad S(Y_p, Y_q) = 0$$

whenever $p \neq q$, so S is a multiple of the inner product on $\mathcal{V}$; that is,

$$(5.19) \qquad S(\sum_p u^p Y_p, \sum_p u^p Y_p) = K \sum_p (u^p)^2$$

for some function K on U. On the other hand,

$$(5.20) \qquad S(\sum_p u^p Y_p, \sum_p u^p Y_p) = F^\mu_{\alpha c}(\sum_p F^\mu_{pc} u^p)(\sum_p F^\alpha_{pc} u^p)$$

is the product of two linear polynomials in the u^p. Such a factorization of a sum of two or more squares is impossible over the reals unless K is identically zero on U. Therefore, the first equation (5.2) must hold at every point of U. The proof of the second equation (5.2) is done in the same way with the roles of a and p reversed. $\qquad\square$

Proposition 19. *If $m_1 = m_2 \geq 2$, $m_3 = m_4 = 1$, and $r = -1$, and if $Y : U \to G$ is a second order frame field for which $d_1 + e_2 = 0$ on U, then $d\omega^0_0 = 0$ on U.*

Proof. As seen in the proof of Proposition 18, we may assume Y chosen so that $d_1 = e_2$ on U, in which case the hypothesis implies that $d_1 = e_2 = 0$ on U. Then GD(3.36i) says

$$(5.21) \qquad\qquad (F^\alpha_{pa})^2 = (F^\mu_{pa})^2$$

on U, for all a, p. Then adding together GD(3.36ii) and GD(3.36iv), respectively, subtracting GD(3.36iii) from GD(3.36v), we find

$$(5.22) \qquad\qquad e_3 = -e_4, \quad d_3 = d_4$$

respectively, on U. Substituting (5.21) into (5.2) we find

$$(5.23) \qquad\qquad F^\mu_{pa} F^\mu_{\alpha a} = 0$$

on U, for all a, p, and

$$(5.24) \qquad\qquad F^\mu_{pa} F^\mu_{\alpha p} = 0$$

on U, for all a, p. These equations are true even for $a \neq b$ and $p \neq q$, as follows. In fact, take the covariant derivative of (5.21) to get

$$(5.25) \qquad\qquad F^\alpha_{pa} F^\alpha_{paj} = F^\mu_{pa} F^\mu_{paj}$$

for all a, p, j. Take $j = b \neq a$ and use GD(3.37) and GD(3.38) to get

$$(5.26) \qquad F^\alpha_{pa}(-F^\mu_{pa} F^\mu_{\alpha b} - 2F^\mu_{pb} F^\mu_{\alpha a}) = F^\mu_{pa}(F^\alpha_{pa} F^\mu_{\alpha b} + 2F^\alpha_{pb} F^\mu_{\alpha a})$$

on U. This, together with (5.2), yields

$$(5.27) \qquad F^\alpha_{pa} F^\mu_{pa} F^\mu_{\alpha b} + F^\alpha_{pa} F^\mu_{pb} F^\mu_{\alpha a} + F^\alpha_{pb} F^\mu_{pa} F^\mu_{\alpha a} = 0$$

on U, for all a, b, p. By (5.23) and (5.21), the middle term is 0 on U, while (5.23) implies that the third term is 0 on U. Thus (5.27) is equivalent to $F^\alpha_{pa} F^\mu_{pa} F^\mu_{\alpha b} = 0$ on U, for all a, b, p, which by (5.21), is equivalent to

$$(5.28) \qquad\qquad F^\mu_{pa} F^\mu_{\alpha b} = 0$$

on U, for all a, b, p. In the same way, if we take $j = q \neq p$ in (5.26), and use GD(3.37) and GD(3.38), we get

$$(5.29) \qquad F_{pa}^{\alpha}(F_{pa}^{\mu} F_{\alpha q}^{\mu} + 2 F_{\alpha p}^{\mu} F_{qa}^{\mu}) = F_{pa}^{\mu}(-F_{pa}^{\alpha} F_{\alpha q}^{\mu} - 2 F_{\alpha p}^{\mu} F_{qa}^{\alpha})$$

on U, which simplifies to

$$(5.30) \qquad F_{pa}^{\alpha} F_{pa}^{\mu} F_{\alpha q}^{\mu} + F_{pa}^{\alpha} F_{qa}^{\mu} F_{\alpha p}^{\mu} + F_{qa}^{\alpha} F_{pa}^{\mu} F_{\alpha p}^{\mu} = 0$$

on U, for all $a, p \neq q$. By (5.24) and (5.21), the middle term is 0 on U, while (5.24) implies that the third term is 0 on U. Thus (5.30) is equivalent to $F_{pa}^{\alpha} F_{pa}^{\mu} F_{\alpha q}^{\mu} = 0$ on U, for all $a, p \neq q$, which by (5.21), is equivalent to

$$(5.31) \qquad F_{pa}^{\mu} F_{\alpha q}^{\mu} = 0$$

on U, for all a, $p \neq q$. By (5.24), this holds also for $p = q$.

For a function $f : U \to \mathbf{R}$, let

$$(5.32) \qquad n(f) = \{x \in U : f(x) \neq 0\}$$

Let spt (f) denote the closure of $n(f)$ in U, namely, the support of f in U. Then (5.21) implies that

$$(5.33) \qquad n(F_{pa}^{\alpha}) = n(F_{pa}^{\mu})$$

for all a, p. Define open subsets U_2 and U_3 of U by

$$(5.34) \qquad U_2 = \cup_{a,p} n(F_{pa}^{\mu}), \quad U_3 = \cup_a n(F_{\alpha a}^{\mu})$$

Then (5.28) implies that

$$(5.35) \qquad U_2 \cap U_3 = \emptyset$$

and (5.31) implies that

$$(5.36) \qquad U_2 \cap (\cup_q n(F_{\alpha q}^{\mu})) = \emptyset$$

Let $U_1 \subset U$ be the open subset of U defined by

$$(5.37) \qquad U_1 = U \setminus ((\cup_{a,p} \mathrm{spt}\,(F_{pa}^{\mu})) \cup (\cup_a \mathrm{spt}\,(F_{\alpha a}^{\mu})))$$

The closure of $U_2 \cup U_3$ in U is clearly the complement of U_1 in U.

On U_1, the functions f, g, and h in (4.19) are identically zero, by (5.33) and the definition of U_1. Hence, by (4.23) in the proof of Theorem 15, we have $d\omega_0^0 = 0$ on U_1.

On U_2 we have

$$(5.38) \qquad F_{\alpha a}^{\mu} = 0, \quad F_{\alpha p}^{\mu} = 0$$

for all a, p, by (5.35) and (5.36), respectively. Then by (2.25), $F_{\alpha a j}^{\mu} = 0$ on U_2, for all j. By GD(3.39), we have

$$(5.39) \quad 0 = F_{\alpha a a}^{\mu} = -\frac{1}{2} D_{\mu\alpha} - \frac{1}{2} \sum_p F_{pa}^{\mu} F_{pa}^{\alpha} + \frac{1}{2} \sum_p F_{pa}^{\mu} F_{pa}^{\alpha} = -\frac{1}{2} D_{\mu\alpha}$$

on U_2. By GD(3.39) and (5.38) we have

$$(5.40) \qquad 0 = F^{\mu}_{\alpha a \alpha} = D_{\mu a} + 3 \sum_p F^{\mu}_{\alpha p} F^{\alpha}_{pa} = D_{\mu a}$$

on U_2, for all a. By GD(3.39) and (5.38), we have

$$(5.41) \qquad 0 = F^{\mu}_{\alpha a \mu} = D_{a \alpha} + 3 \sum_p F^{\mu}_{\alpha p} F^{\mu}_{pa} = D_{a \alpha}$$

on U_2, for all a. Similarly, by the second equation in (5.38), we have $F^{\mu}_{\alpha p j} = 0$ on U_2, for all p, j. From GD(3.40) and (5.38) we get

$$(5.42) \qquad 0 = F^{\mu}_{\alpha p \alpha} = -E_{p \mu} - 3 \sum_a F^{\alpha}_{pa} F^{\mu}_{\alpha a} = -E_{p \mu}$$

on U_2, for all p. Similarly,

$$(5.43) \qquad 0 = F^{\mu}_{\alpha p \mu} = -D_{p \alpha} - 3 \sum_a F^{\mu}_{\alpha a} F^{\mu}_{pa} = -D_{p \alpha}$$

on U_2, for all p. Finally, we want to show that $D_{pa} = 0$ on U_2, for all a, p. By GD(3.37) and the fact that $F^{\mu}_{\alpha a} = 0$ and $F^{\mu}_{\alpha p} = 0$ at every point of U_2, for all a, p, we have

$$(5.44) \qquad F^{\alpha}_{pa\alpha} = -D_{pa}$$

on U_2, for all a, p. In the same way using GD(3.38), we have

$$(5.45) \qquad F^{\mu}_{pa\mu} = -D_{pa}$$

on U_2, for all a, p. Taking $j = \alpha$ in (5.25), we find

$$(5.46) \qquad F^{\alpha}_{pa} F^{\alpha}_{pa\alpha} = F^{\mu}_{pa} F^{\mu}_{pa\alpha}$$

on U_2, for all a, p. Substitute (5.44) into this to get

$$(5.47) \qquad -F^{\alpha}_{pa} D_{pa} = F^{\mu}_{pa} F^{\mu}_{pa\alpha}$$

on U_2, for all a, p. Similarly, taking $j = \mu$ in (5.25), we get

$$(5.48) \qquad F^{\alpha}_{pa} F^{\alpha}_{pa\mu} = F^{\mu}_{pa} F^{\mu}_{pa\mu}$$

on U_2, for all a, p. Substitute (5.45) into this to get

$$(5.49) \qquad F^{\alpha}_{pa} F^{\alpha}_{pa\mu} = -F^{\mu}_{pa} D_{pa}$$

on U_2, for all a, p. By GD(3.41) we have $F^{\alpha}_{pa\mu} = -F^{\mu}_{pa\alpha}$, so substitute this into (5.49) to get

$$(5.50) \qquad F^{\alpha}_{pa} F^{\mu}_{pa\alpha} = F^{\mu}_{pa} D_{pa}$$

on U_2, for all a, p. Now multiply (5.47) by F^{μ}_{pa} to get

$$(5.51) \qquad -F^{\alpha}_{pa} F^{\mu}_{pa} D_{pa} = (F^{\mu}_{pa})^2 F^{\mu}_{pa\alpha}$$

on U_2, for all a, p. Multiply (5.50) by F_{pa}^α to get

$$(5.52) \qquad\qquad F_{pa}^\alpha F_{pa}^\mu D_{pa} = (F_{pa}^\alpha)^2 F_{pa\alpha}^\mu$$

on U_2, for all a, p. By (5.21), we see that the right sides of (5.51) and (5.52) are the same. Therefore the left sides must be equal at every point of U_2, but they are negatives of each other, so we get, again using (5.21)

$$(5.53) \qquad\qquad F_{pa}^\mu D_{pa} = 0, \quad F_{pa}^\alpha D_{pa} = 0$$

on U_2, for all a, p.

The next step is to show

$$(5.54) \qquad\qquad \omega_{n+1}^0 = 0$$

on U_2. By (2.28), this is equivalent to showing $R_j = 0$ on U_2, for all j. Now $d_1 = 0$ on U_2 implies that all of its covariant derivatives $d_{1j} = 0$ on U_2, since $dd_1 + 2d_1\omega_0^0 = \sum_j d_{1j}\theta^j$ defines its covariant derivatives. Using the fact that $D_{\mu\alpha}$, $D_{\mu a}$, $D_{a\alpha}$, $E_{p\mu}$, $D_{p\alpha}$, and all their covariant derivatives are zero on U_2, we find the consequences of $d_{1j} = 0$ on U_2 to be as follows. By GD(3.46i) and (5.38),

$$(5.55) \qquad\qquad 0 = d_{1a} = D_{bba} = -R_a$$

on U_2, for all a. By GD(3.46ii),

$$(5.56) \qquad\qquad 0 = d_{1p} = D_{aap} = -R_p$$

on U_2, for all p. By GD(3.46iii) and (5.53),

$$(5.57) \qquad\qquad 0 = d_{1\alpha} = D_{aa\alpha} = -R_\alpha$$

on U_2. By GD(3.46iv) and (5.53),

$$(5.58) \qquad\qquad 0 = d_{1\mu} = D_{aa\mu} = -R_\mu$$

on U_2. This completes the proof of (5.54).

The next step is to show that the covariant derivatives e_{3j} of e_3 satisfy

$$(5.59) \qquad\qquad e_{3j} = 0$$

on U_2, where by (2.26) these are defined by

$$(5.60) \qquad\qquad de_3 + 2e_3\omega_0^0 = \sum_j e_{3j}\theta^j$$

To prove (5.59), we use the equations GD(3.52) to get

$$(5.61) \qquad\qquad e_{3a} = E_{\alpha\alpha a} = R_a = 0$$

on U_2, for all a,

$$(5.62) \qquad\qquad e_{3p} = E_{\alpha\alpha p} = R_p = 0$$

on U_2, for all p,

$$(5.63) \quad e_{3\alpha} = E_{\alpha\alpha\alpha} = R_\alpha - \frac{6}{m_1} \sum_{a,p} D_{pa} F_{pa}^\alpha + \frac{24}{m_1} \sum_a F_{\alpha a}^\mu F_{pa}^\alpha F_{\alpha p}^\mu = 0$$

on U_2, by (5.53) and (5.38), and

$$(5.64) \qquad\qquad e_{3\mu} = E_{\alpha\alpha\mu} = R_\mu + D_{\mu\alpha\alpha} = 0$$

on U_2, since (5.39) implies that $D_{\mu\alpha\alpha} = 0$ on U_2. This completes the proof of (5.59), and therefore (5.60) becomes

$$(5.65) \qquad\qquad de_3 + 2e_3\omega_0^0 = 0$$

on U_2. This implies that on the subset of U_2 where $e_3 \neq 0$, the 1-form

$$(5.66) \qquad\qquad \omega_0^0 = -\frac{1}{2}\frac{1}{e_3}de_3$$

and therefore ω_0^0 is closed on this subset of U_2. Thus, if e_3 is never zero on U_2, then ω_0^0 is closed on all of U_2. We now prove that

$$(5.67) \qquad\qquad e_3(x) \neq 0$$

for every $x \in U_2$. Let $x \in U_2$. We may assume that our second order frame Y on U diagonalizes E_1 at x. It is easily checked that if the vectors Y_a are changed by an orthogonal matrix, then the set U_2 does not change, and e_3 is unchanged. Let

$$(5.68) \qquad\qquad \mathcal{I}_1'(x) = \{a \in \mathcal{I}_1 : F_{pa}^\alpha(x) = 0, \;\; \forall p\}$$

and let

$$(5.69) \qquad\qquad \mathcal{I}_1''(x) = \mathcal{I}_1 \setminus \mathcal{I}_1'(x)$$

We want to show that if the set $\mathcal{I}_1$ is arranged with the indices in $\mathcal{I}_1'(x)$ first followed by the indices in $\mathcal{I}_1''(x)$, then

$$(5.70) \qquad\qquad E_1(x) = e_3(x) \begin{pmatrix} I_m & 0 \\ 0 & -I_{m_1-m} \end{pmatrix}$$

where m is the cardinality of $\mathcal{I}_1'(x)$. To prove this, suppose that $a \in \mathcal{I}_1'(x)$. Then $F_{pa}^\alpha(x) = 0$ for all p, and of course $F_{\alpha a}^\mu = 0$ on all of U_2, so GD(3.36ii) implies that (since $d_1 = 0$)

$$(5.71) \qquad\qquad 0 = -e_a(x) + e_3(x)$$

as claimed. Next suppose that $a \in \mathcal{I}_1''(x)$, so that $F_{pa}^\mu(x) = \pm F_{pa}^\alpha(x) \neq 0$, for some p. Using GD(3.59i) and the fact that $E_{p\mu} = 0$ on U_2, we have

$$(5.72) \qquad\qquad 0 = E_{p\mu a}(x) = (e_4(x) - e_a(x))F_{pa}^\mu(x)$$

which implies that $e_a(x) = e_4(x) = -e_3(x)$, by (5.22). This completes the proof of (5.70) and allows us to prove (5.67) as follows. For our given $x \in U_2$, GD(3.36ii) implies that

$$(5.73) \qquad \sum_p (F^\alpha_{pa}(x))^2 = -e_a(x) + e_3(x)$$

for all a. But (5.70) implies that $e_a(x) = \pm e_3(x)$, so if $e_3(x) = 0$, then $e_a(x) = 0$ as well, and we must conclude from (5.73) that $F^\alpha_{pa}(x) = 0$, for all a, p, contradicting the fact that $x \in U_2$. We conclude that e_3 is never zero on U_2, and therefore ω^0_0 is closed on U_2, so in particular,

$$(5.74) \qquad D_{pa} = 0$$

on U_2, for all a, p.

Finally, it remains to prove that $d\omega^0_0 = 0$ on U_3. To do this, we first observe that $F^\alpha_{pa} = 0 = F^\mu_{pa}$ on U_3, for every a, p. It follows that their covariant derivatives $F^\alpha_{paj} = 0 = F^\mu_{paj}$ on U_3, for all a, p, j. By GD(3.37i), then

$$(5.75) \qquad 0 = F^\alpha_{paa} = D_{p\alpha}$$

on U_3, for all a, p. By GD(3.37ii),

$$(5.76) \qquad 0 = F^\alpha_{pap} = -D_{a\alpha}$$

on U_3, for all a, p. By GD(3.37iii)

$$(5.77) \qquad 0 = F^\alpha_{pa\alpha} = -D_{pa} + 2F^\mu_{\alpha p}F^\mu_{\alpha a} - 2F^\mu_{\alpha p}F^\mu_{\alpha a} = -D_{pa}$$

on U_3, for all a, p. By GD(3.38i)

$$(5.78) \qquad 0 = F^\mu_{paa} = -E_{p\mu}$$

on U_3, for all a, p. By GD(3.38ii)

$$(5.79) \qquad 0 = F^\mu_{pap} = D_{\mu a}$$

on U_3, for all a, p. It remains to prove that $D_{\mu\alpha} = 0$ on U_3. To do this, we first observe that there is a dense open subset V of U_3 on which we may make an analytic change of frame field

$$(5.80) \qquad \tilde{Y} = Ya(I_2, B, 0, 0)$$

where $B = B_1 \oplus I_{m_1} \oplus 1 \oplus 1$, with $B_1 : V \to O(m_1)$ an analytic map, such that $\tilde{E}_1$ is diagonal at each point (see, for example, [20]). For such a change of frame, $\tilde{\omega}^0_0 = \omega^0_0$ and the sets $\tilde{U}_2 = U_2$ and $\tilde{U}_3 = U_3$. So, without loss of generality, we may assume $\tilde{Y} = Y$ on V and all of the above properties of Y on U_3 remain true on V. For a point $x \in V$, let

$$(5.81) \qquad \mathcal{I}'_1(x) = \{a \in \mathcal{I}_1 : F^\mu_{\alpha a}(x) = 0\}$$

and let its complement be

(5.82) $$\mathcal{I}_1''(x) = \mathcal{I}_1 \setminus \mathcal{I}_1'(x)$$

If $a \in \mathcal{I}_1'(x)$, then GD(3.36ii) implies that

(5.83) $$0 = -e_a(x) + e_3(x)$$

If $a \in \mathcal{I}_1''(x)$, then by (5.76) and GD(3.54iv) and using (5.22), we have

(5.84) $$0 = D_{a\alpha\mu} = (-2e_a - e_3 - d_3 + d_4 + e_4)F_{\alpha a}^\mu = -2(e_a + e_3)F_{\alpha a}^\mu$$

at x, and so $F_{\alpha a}^\mu(x) \neq 0$ implies that

(5.85) $$e_a(x) = -e_3(x)$$

We next show that $e_3(x) \neq 0$. Since $x \in V \subset U_3$, the set $\mathcal{I}_1''(x)$ is non-empty. For $a \in \mathcal{I}_1''(x)$, GD(3.36ii) and (5.85) give

$$4(F_{\alpha a}^\mu)^2 = -e_a(x) + e_3(x) = 2e_3(x)$$

The left side of this equation is non-zero for $a \in \mathcal{I}_1''(x)$, and thus $e_3(x) \neq 0$. Therefore, at every point of V, with this frame Y the matrix E_1 has the form (5.70), where $m = m(x)$ is the cardinality of $\mathcal{I}_1'(x)$. Since $e_3(x) \neq 0$, this shows that on each connected component of V, $\mathcal{I}_1'$ and $\mathcal{I}_1''$ must be constant and equations (5.83) and (5.85) hold at every point of this component. Therefore, we may use GD(3.50i) to find that the covariant derivative of e_a, for any $a \in \mathcal{I}_1''$, is

(5.86) $$e_{aa} = E_{aaa} = 6D_{\mu\alpha}F_{\alpha a}^\mu$$

on V, while by (5.85) and GD(3.52i)

(5.87) $$-e_{aa} = e_{3a} = E_{\alpha\alpha a} = R_a + 2D_{\mu\alpha}F_{\alpha a}^\mu = 2D_{\mu\alpha}F_{\alpha a}^\mu$$

on V, by (5.2) and (5.11). Adding (5.87) and (5.86), we get

(5.88) $$0 = 8D_{\mu\alpha}F_{\alpha a}^\mu$$

on this connected component, for any $a \in \mathcal{I}_1''$. Therefore,

(5.89) $$D_{\mu\alpha} = 0$$

on each connected component of V, therefore on all of V. It follows from (2.24) that $d\omega_0^0 = 0$ on V, and therefore $d\omega_0^0 = 0$ on the closure of V in U_3, which is all of U_3. Therefore, this is true for our originally chosen frame field Y on U, and $d\omega_0^0 = 0$ on U_1, U_2, and U_3, so it is also zero on the closure of the union of these three sets in U, which is all of U. $\qquad\square$

Theorem 20. *Suppose $m_1 = m_2 \geq 2$, $m_3 = m_4 = 1$, and $r = -1$. For any point of M, there exists a second order frame field $Y : U \to G$ about the point, for which $d\omega_0^0 = 0$ on U.*

Proof. Given a point of M, we know by Proposition 18 that there exists a second order frame field $Y : U \to G$ about the point for which $d_1 = e_2$ on U and (5.2) holds on U, for all a, p. We will show that $d\omega_0^0 = 0$ on U, for this frame. To do this, we decompose U into a disjoint union of subsets U_1 and U_2, where

$$(5.90) \qquad\qquad U_1 = \{x \in U : d_1(x) = 0\}$$

a closed subset of U, and

$$(5.91) \qquad\qquad U_2 = \{x \in U : d_1(x) \neq 0\}$$

an open subset of U. Let $\overset{\circ}{U}_1$ denote the interior of U_1. Then $d_1 = 0$ on $\overset{\circ}{U}_1$, implies $d\omega_0^0 = 0$ on $\overset{\circ}{U}_1$, by Proposition 19.

Next consider the open set U_2, where d_1 is never zero. On each connected component of U_2, d_1 is either always positive or always negative. To be specific, suppose that

$$(5.92) \qquad\qquad e_2 = d_1 < 0$$

on the connected component $\tilde{U}$ of U_2. Then GD(3.36i) becomes

$$(5.93) \qquad 2(F_{pa}^\alpha)^2 - 2(F_{pa}^\mu)^2 = -(d_1 + e_2) = -2d_1 > 0$$

on $\tilde{U}$ for all a, p, which implies that

$$(5.94) \qquad\qquad F_{pa}^\alpha \neq 0$$

at every point of $\tilde{U}$, for all a, p. This combined with (5.2) implies that

$$(5.95) \qquad\qquad F_{pa}^\mu F_{\alpha a}^\mu = 0, \quad F_{pa}^\mu F_{\alpha p}^\mu = 0$$

on $\tilde{U}$, for all a, p. This holds for an arbitrary orthogonal change of the frame vectors Y_a or of the frame vectors Y_p. By polarization, it follows that

$$(5.96) \qquad F_{pa}^\mu F_{\alpha b}^\mu + F_{pb}^\mu F_{\alpha a}^\mu = 0, \quad F_{pa}^\mu F_{\alpha q}^\mu + F_{qa}^\mu F_{\alpha p}^\mu = 0$$

on $\tilde{U}$, for all a, b, p, q. Polarization means the following. For any p consider the bilinear form

$$(5.97) \qquad\qquad \Phi_p = \sum_{a,b} F_{pa}^\mu F_{\alpha b}^\mu \theta^a \otimes \theta^b$$

Then (5.95) says that $\Phi_p(Y_a, Y_a) = 0$ for any a, and this is true for any orthogonal change of the frame vectors Y_a, so it must be true that $\Phi_p(v, v) = 0$ for any unit vector v. Hence, Φ_p is an alternating form and must satisfy

$$(5.98) \qquad\qquad \Phi_p(u, v) + \Phi_p(v, u) = 0$$

for all vectors u, v. This implies the first equation in (5.96). The second equation follows in the same way.

The next step is to prove that

$$(5.99) \qquad\qquad F^{\mu}_{pa} = 0$$

on $\tilde{U}$, for all a, p. We prove this by assuming the contrary and then deducing a contradiction to our assumption that $d_1 \neq 0$ on $\tilde{U}$. Suppose, then, there exists a and p and $x \in \tilde{U}$ such that

$$(5.100) \qquad\qquad F^{\mu}_{pa}(x) \neq 0$$

Then there is an open set O of $\tilde{U}$, containing x, on which F^{μ}_{pa} is never zero, for this a, p. Hence $F^{\mu}_{\alpha a} = 0$ on O, for this a, by (5.95), and then

$$(5.101) \qquad\qquad F^{\mu}_{\alpha b} = 0$$

on O, for all b, by the first equation in (5.96). In the same way we see that

$$(5.102) \qquad\qquad F^{\mu}_{\alpha q} = 0$$

on O, for all q. These two equations are the same as (5.38), and in the same way as we derived (5.39) through (5.43), we get

$$(5.103) \qquad D_{\mu\alpha} = D_{\mu b} = D_{b\alpha} = E_{q\mu} = D_{q\alpha} = 0$$

on O, for all b, q. There exists an open dense subset $\tilde{O}$ of O on which we may assume D_2 and E_1 have been diagonalized at each point. Then

$$(5.104) \qquad \begin{aligned} 0 &= D_{q\alpha b} = (e_3 - e_2 + e_b)F^{\alpha}_{qb} \\ 0 &= E_{p\mu a} = (e_4 - e_2 - e_a)F^{\mu}_{pa} \end{aligned}$$

on $\tilde{O}$, by GD(3.56) and GD(3.59). By (5.94) we can conclude that

$$(5.105) \qquad\qquad e_b = e_2 - e_3$$

on $\tilde{O}$, for all b, so by continuity, $E_1 = (e_2 - e_3)I_{m_1}$ is scalar on all of O. But, by (5.100) and the second equation of (5.104), we have

$$(5.106) \qquad\qquad e_a = e_4 - e_2 = -e_3 - e_2$$

on O, for the particular value of a, therefore for all a, since E_1 is scalar on O. Subtracting (5.106) from (5.105), we conclude that $e_2 = 0$ on O, which contradicts (5.92).

Therefore, (5.99) holds on $\tilde{U}$, for all a, p. Then all covariant derivatives $F^{\mu}_{paj} = 0$ on $\tilde{U}$, for all a, p, j, so GD(3.38) implies that

$$(5.107) \qquad\qquad 0 = F^{\mu}_{pa\mu} = -D_{pa}$$

on $\tilde{U}$, for all a, p, and

(5.108)
$$0 = F_{pap}^{\mu} = D_{\mu a} - 3F_{pa}^{\alpha} F_{\alpha p}^{\mu}$$
$$0 = F_{paa}^{\mu} = -E_{p\mu} + 3F_{pa}^{\alpha} F_{\alpha a}^{\mu}$$

on $\tilde{U}$, for all a, p. These two formulas hold for any orthogonal change of the vectors Y_a or Y_p on $\tilde{U}$. The 1-forms on $\tilde{U}$

(5.109)
$$\alpha_a = \sum_p F_{pa}^{\alpha} \theta^p, \quad \beta = 3 \sum_p F_{\alpha p}^{\mu} \theta^p$$

are invariant under such changes of frame, and the bilinear form

(5.110)
$$\psi_a = \alpha_a \otimes \beta$$

has the property that $\psi_a(Y_p, Y_p) = 3F_{pa}^{\alpha} F_{\alpha p}^{\mu} = D_{\mu a}$, for every p. By the argument containing (5.13)-(5.20), this implies that

(5.111)
$$D_{\mu a} = 0$$

on $\tilde{U}$, for every a. In the same way, using the second equation in (5.108), we conclude that

(5.112)
$$E_{p\mu} = 0$$

on $\tilde{U}$, for all p. Then (5.108) implies that

(5.113)
$$F_{\alpha p}^{\mu} = 0 = F_{\alpha a}^{\mu}$$

on $\tilde{U}$, for all a, p, since F_{pa}^{α} is never zero on $\tilde{U}$, for all a, p. We remark here for the proof of Corollary 21 below, that (5.99) and (5.113) imply

(5.114)
$$g = h = k = 0$$

on $\tilde{U}$, and therefore $\lambda : \tilde{U} \to \Lambda$ is reducible, by Theorem 15. For the proof at hand, we use (5.99) and (5.113) in GD(3.39) to get

(5.115)
$$0 = F_{\alpha aa}^{\mu} = -\frac{1}{2} D_{\mu\alpha}$$
$$0 = F_{\alpha a\mu}^{\mu} = D_{a\alpha}$$

on $\tilde{U}$, for all a, p, and in GD(3.40) to get

(5.116)
$$0 = F_{\alpha p\mu}^{\mu} = -D_{p\alpha}$$

on $\tilde{U}$, for all p. Then (5.107), (5.111), (5.112), (5.115), and (5.116) imply that $d\omega_0^0 = 0$ on $\tilde{U}$.

In the same way we prove that if $d_1 > 0$ on a connected component $\tilde{U}$ of U_2, then $d\omega_0^0 = 0$ on $\tilde{U}$. Therefore, $d\omega_0^0 = 0$ on all of U_2. If U_1 has no interior, then the closure of U_2 in U is all of U, and thus $d\omega_0^0 = 0$ on all of U, by continuity. If U_1 has nonempty interior, then we have

proven that $d\omega_0^0 = 0$ on this interior, and therefore $d\omega_0^0 = 0$ on all of U, which is the closure in U of the union of the interior of U_1 with U_2. $\square$

Corollary 21. *Suppose $m_1 = m_2 \geq 2$, $m_3 = m_4 = 1$, and $r = -1$. If the Dupin hypersurface is irreducible, then for any second order frame field $Y : U \rightarrow G$ along it, we have*

$$(5.117) \qquad\qquad d_1 + e_2 = 0$$

on U.

Proof. Let $Y : U \rightarrow G$ be any second order frame field along the Dupin hypersurface. We know that D_1 and E_2 are scalar matrices at every point of U. Any change of second order frame field is given by (2.14), and thus GD(3.32) shows that under such a change the function $d_1 + e_2$ is multiplied by an everywhere positive function on U. It follows that if (5.117) holds for some second order frame field in U, then it holds for any second order frame field in U.

By Proposition 18, about any point $x \in U$ there exists a second order frame field for which $d_1 = e_2$. Seeking a contradiction, suppose that $d_1(x) \neq 0$, for some x in the domain of this frame field. Shrinking the domain, if necessary, we may assume that $d_1 \neq 0$ on the whole domain of the frame field, and then the proof of Theorem 20, as remarked after (5.114), shows that the Dupin hypersurface is reducible on some open subset of x, and thus it is reducible by Proposition 9. This contradicts our assumption that the Dupin hypersurface is irreducible. Hence, d_1 must be zero at every point of its domain. $\square$

Corollary 22. *Suppose $m_1 = m_2 \geq 1$, $m_3 = m_4 = 1$, and $r = -1$. If $Y : U \rightarrow G$ is a second order frame field along the Dupin hypersurface such that one of $\mathcal{I}_1'$, $\mathcal{I}_2'$, $\mathcal{I}_3'$, or $\mathcal{I}_4'$ is nonempty (see Definition 12), then the hypersurface is reducible.*

Proof. If $\mathcal{I}_1'$ or $\mathcal{I}_2'$ is nonempty, then there is no loss in generality in assuming that $\mathcal{I}_1'$ is nonempty, in which case the result follows from Theorem 16 and Theorem 20. If $\mathcal{I}_3'$ or $\mathcal{I}_4'$ is nonempty, there is no loss in generality in assuming that $\mathcal{I}_3'$ is nonempty. Since $m_3 = 1$, this means that the functions f, h, and k, defined in (4.19), are identically zero on U. Therefore, the result follows from Theorem 15. $\square$

Theorem 23. *Suppose the connected proper Dupin hypersurface $\lambda : M^{n-1} \rightarrow \Lambda^{2n-1}$ has four distinct curvature spheres with multiplicities $m_1 = m_2 \geq 1$, $m_3 = m_4 = 1$, and Lie curvature $r = -1$. If λ is irreducible, then it is Lie equivalent to an isoparametric hypersurface.*

Proof. The case $m_1 = m_2 = 1$ was handled in [5, Theorem 4.1, p. 33].

Assume now that $m_1 = m_2 \geq 2$. In order to apply Theorem 7, we must find a second order frame field, defined on a neighborhood of any given point, that satisfies the conditions of Theorem 7. Let $x \in M$ and let $Y : U \to G$ be a second order frame field, where $x \in U$. Then $d_1 + e_2 = 0$, by Corollary 21, and $d\omega_0^0 = 0$ on U, by Proposition 19. By Proposition 18, we may adjust Y so that $d_1 = 0 = e_2$ on U. As observed in (5.22) of the proof of Proposition 19, we also have $e_3 = -e_4$ and $d_3 = d_4$.

Consider the functions A_a, for any $a \in \mathcal{I}_1$, defined in (4.41) in the proof of Theorem 16. Irreducibility implies that for each $a \in \mathcal{I}_1$, A_a must be positive on a dense open subset of U, by Theorem 16. In the same way, for each p, the analytic function

$$(5.118) \qquad A_p = |v_{p\alpha}|^2 + |v_{p\mu}|^2 = 2 \sum_a ((F_{pa}^\alpha)^2 + (F_{pa}^\mu)^2) + 8(F_{\alpha p}^\mu)^2$$

must be positive on a dense open subset of U. There is also a dense open subset of U on which E_1 and D_2 can be diagonalized by our frame field. The intersection of these three dense open subsets is a dense open subset of U. Let W be a connected component of it. Then (4.39) holds on W, and A_a is positive on W for all $a \in \mathcal{I}_1$, so we have

$$(5.119) \qquad\qquad\qquad e_a = -e_3$$

on W, for all $a \in \mathcal{I}_1$. Similarly, A_p is positive on W for all p, so by (4.40) we have

$$(5.120) \qquad\qquad\qquad d_p = d_3$$

on W, for all $p \in \mathcal{I}_2$. Therefore, on each connected component of this open dense subset we have $E_1 = -E_3$ and $D_2 = D_3$ are scalar matrices for some choice of second order frame field, hence for all choices of second order frame fields. By continuity, it follows that this is true on all of U for Y, that is,

$$(5.121) \qquad\qquad\qquad e_1 = -e_3, \quad d_2 = d_3$$

on U. Now GD(3.42iii) implies that

$$(5.122) \qquad (m_1 + \frac{1}{2})(d_3 + e_3) = -(m_1 + \frac{1}{2})(d_3 + e_3)$$

on U, which implies that

$$(5.123) \qquad\qquad\qquad d_3 = -e_3$$

on U. Plugging our known values of $d_1, \ldots, e_4$ into (3.5), we find that (3.9) holds on U. Finally, it follows now from (3.4) that

$$(5.124) \qquad\qquad\qquad |v_{a\alpha}|^2 = \frac{1}{2} A_a$$

which is positive on a dense open subset of U. Therefore, all the conditions of Theorem 7 are satisfied and we may conclude that λ is Lie equivalent to an isoparametric hypersurface. $\square$

Remark 24. There exist reducible proper Dupin hypersurfaces with four curvature spheres satisfying the conditions $m_1 = m_2$, $m_3 = m_4$, and $r = -1$ that are not Lie equivalent to an isoparametric hypersurface (see [3, pp. 107-108]).

6. Compact proper Dupin hypersurfaces

Our main goal in this section is to prove the following result.

Theorem 25. *Let W^{d-1} be a compact, connected, proper Dupin hypersurface immersed in S^d (or $\mathbf{R}^d$) with $g > 2$ distinct principal curvatures. Then W is irreducible; that is, the Legendre submanifold induced by the hypersurface W is irreducible.*

Remark 26. Thorbergsson [23] proved that a compact proper Dupin hypersurface W^{d-1} immersed in S^d must be taut, and a taut immersion must be an embedding (see Cecil-Ryan [8, p. 121]), so W must be embedded in S^d. In this same paper he also proved that the number g of distinct principal curvatures of a compact proper Dupin hypersurface must be 1, 2, 3, 4, or 6, the same as for an isoparametric hypersurface in a sphere.

Proof. The proof is by contradiction. We assume that $W^{d-1} \subset S^d$ is reducible and obtain a contradiction. The proof is essentially the same as the proof of Theorem 2 of [2] (see also Theorem 2.11 of [3, p. 148]). In that theorem, however, we assumed that the Dupin hypersurface $M^{n-1} \subset S^n$ to which W is reducible is immersed in S^n. In the present proof we do not need to make such an assumption because we can handle the situation as follows.

Assume that there exists a reducible, compact, connected, proper Dupin hypersurface W^{d-1} immersed in S^d with $g > 2$ principal curvatures. Let $\nu : W \to \Lambda^{2d-1}$ be the Legendre submanifold induced by this immersion. By Proposition 11, ν is Lie equivalent to a proper Dupin submanifold $\mu : W \to \Lambda^{2d-1}$ that is obtained from a lower dimensional proper Dupin submanifold $\lambda : M^{n-1} \to \Lambda^{2n-1}$ by one of the three standard constructions of Pinkall [17]. As shown in Section 4.2 of [3], all three of the constructions begin with a proper Dupin submanifold $\lambda : M^{n-1} \to \Lambda^{2n-1}$ and produce a Dupin submanifold $\mu : M \times S^m \to \Lambda^{2(n+m)-1}$. Thus, W is diffeomorphic to $M \times S^m$, and M must be compact since W is compact. We are also assuming that ν (and thus μ) has $g > 2$ distinct curvature spheres at each point. As

shown in Propositions 2.1, 2.3 and 2.5 of Section 4.5 of [3], however, for μ obtained by the tube and cylinder constructions, there always exist points on $M \times S^m$ at which the number of distinct curvature spheres is two. Thus, μ cannot be obtained by the tube or cylinder construction.

Thus, if $g > 2$, then μ must be obtained from λ by the surface of revolution construction. Proposition 2.7 of [3, p. 144] shows that for the surface of revolution construction, the number k of distinct curvature spheres on M must be $g-1$ or g. We also have the following relationship between the sum β of the $\mathbf{Z}_2$-Betti numbers of W and M,

$$\beta(W) = \beta(M \times S^m) = 2\beta(M)$$

On the other hand, Thorbergsson [23] showed that for a compact proper Dupin hypersurface embedded in S^d, β is equal to twice the number of distinct curvature spheres. Thus, we have $\beta(W) = 2g$.

If the point sphere map of $\lambda : M \to \Lambda^{2n-1}$ is an immersion, or even if there is a Lie sphere transformation A such that the point sphere map of $A\lambda$ is an immersion, then we have an immersed proper Dupin hypersurface $f : M \to S^n$ to which Thorbergsson's theorem applies, and $\beta(M) = 2k$, where k is the number of distinct curvature spheres of M. Thus, we have

$$\beta(W) = 2g, \quad \beta(M) = 2k$$

where $k = g - 1$ or $k = g$, and

$$\beta(W) = 2\beta(M)$$

Combining these equations, we get

$$2g = 2(2k) = 4k$$

for $k = g - 1$ or $k = g$. Clearly, $k = g$ is impossible, and $k = g - 1$ yields

$$2g = 4(g - 1) = 4g - 4$$

and thus $g = 2$, contradicting the assumption that $g > 2$.

It remains to show that in the case of the surface of revolution construction, there is a Lie sphere transformation A such that the point sphere map of $A\lambda$ is an immersion. That follows from the following lemma.

Lemma 27. *Let $\mu : M^{n-1} \times S^m \to \Lambda^{2(n+m)-1}$ be a Legendre submanifold which is obtained from a proper Dupin submanifold $\lambda : M \to \Lambda^{2n-1}$ by the surface of revolution construction. If there exists a Lie sphere transformation B such that the point sphere map of $B\mu$ is an immersion, then there exists a Lie sphere transformation A such that the point sphere map of $A\lambda$ is an immersion.*

We shall first apply this lemma to complete the proof of Theorem 25 and then give the proof of the lemma. Since the point sphere map of $\nu : W \to \Lambda^{2d-1}$ is given to be an immersion , and ν is Lie equivalent to $\mu : W \to \Lambda^{2d-1}$, we know that the Lie sphere transformation B in the lemma exists. Therefore, a Lie sphere transformation A exists such that the point sphere map of $A\lambda$ is an immersion, which is what we need to complete the proof of Theorem 25. $\qquad\square$

Proof of Lemma 27. We begin by reviewing the surface of revolution construction (see [3, pp. 141-144]). Let $e_0, \ldots, e_{n+m+2}$ be an orthonormal basis of $\mathbf{R}_2^{n+m+3}$ with e_0 and e_{n+m+2} timelike and the rest spacelike. Let $\mathbf{R}P^{n+m+2}$ be the projective space determined by $\mathbf{R}_2^{n+m+3}$ with corresponding Lie quadric Q^{n+m+1}. Let

$$(6.1) \qquad \mathbf{R}_2^{n+3} = \operatorname{span}\{e_0, e_1, \ldots, e_{n+1}, e_{n+m+2}\} \subset \mathbf{R}_2^{n+m+3}$$

and let $\mathbf{R}P^{n+2}$ and Q^{n+1} be the corresponding projective space and Lie quadric. Let Λ^{2n-1} and $\Lambda^{2(n+m)-1}$ be the space of projective lines on Q^{n+1} and Q^{n+m+1}, respectively. Finally, let $u_k = e_{k+1}$ for $k = 1, \ldots, n+m$, and let

$$(6.2) \qquad \mathbf{R}^n = \operatorname{span}\{e_2, \ldots, e_{n+1}\} = \operatorname{span}\{u_1, \ldots, u_n\}$$

$$(6.3) \qquad \mathbf{R}^{n+m} = \operatorname{span}\{e_2, \ldots, e_{n+m+1}\} = \operatorname{span}\{u_1, \ldots, u_{n+m}\}$$

Consider a proper Dupin submanifold $\lambda : M^{n-1} \to \Lambda^{2n-1}$ with g distinct curvature spheres. We can parametrize λ by using the Euclidean projection $f : M \to \mathbf{R}^n$ and Euclidean field of unit normals $\xi : M \to \mathbf{R}^n$ as follows (see [3, p. 82]),

$$(6.4) \qquad\qquad\qquad \lambda = [k_1, k_2]$$

where

$$(6.5) \quad k_1 = (1 + f \cdot f, 1 - f \cdot f, 2f, 0)/2, \quad k_2 = (f \cdot \xi, -f \cdot \xi, \xi, 1)$$

The map $[k_1] : M \to Q^{n+1}$ is the point sphere map of λ, and the map $[k_2] : M \to Q^{n+1}$ is the tangent hyperplane map of λ.

We want to construct a Legendre submanifold μ by "revolving" f around an "axis" $\mathbf{R}^{n-1} \subset \mathbf{R}^n \subset \mathbf{R}^{n+m}$, for $\mathbf{R}^{n+m}$ as in (6.3). The domain of μ will be $M \times S^m$. Note that we do not assume that f is an immersion, nor that the range of f is disjoint from the axis $\mathbf{R}^{n-1}$.

For simplicity, we assume that the axis $\mathbf{R}^{n-1}$ contains the origin of $\mathbf{R}^n$ and that the orthonormal basis vectors have been chosen so that

$$(6.6) \qquad\qquad \mathbf{R}^{n-1} = \operatorname{span}\{u_1, \ldots, u_{n-1}\} \subset \mathbf{R}^n \subset \mathbf{R}^{n+m}$$

for $\mathbf{R}^n$ and $\mathbf{R}^{n+m}$ as in (6.2) and (6.3), respectively. We write the sphere S^m in the form

(6.7) $S^m = \{y = y_0 u_n + \cdots + y_m u_{n+m} : y_0^2 + \cdots + y_m^2 = 1\}$

We now define a new Legendre submanifold $\mu : M \times S^m \to \Lambda^{2(n+m)-1}$ by its Euclidean projection $F : M \times S^m \to \mathbf{R}^{n+m}$ and its Euclidean field of unit normals $\eta : M \times S^m \to \mathbf{R}^{n+m}$, so that $\mu = [K_1, K_2]$, where

(6.8) $K_1 = (1 + F \cdot F, 1 - F \cdot F, 2F, 0)/2, \quad K_2 = (F \cdot \eta, -F \cdot \eta, \eta, 1)$

and the maps F and η are defined as follows. First we decompose the maps f and ξ into components along $\mathbf{R}^{n-1}$ and orthogonal to $\mathbf{R}^{n-1}$ in $\mathbf{R}^n$ and write,

(6.9) $f(x) = \hat{f}(x) + f_n(x)u_n, \quad \hat{f}(x) \in \mathbf{R}^{n-1}$

(6.10) $\xi(x) = \hat{\xi}(x) + \xi_n(x)u_n, \quad \hat{\xi}(x) \in \mathbf{R}^{n-1}$

Then for $x \in M$, $y \in S^m$, we define the maps F and η in (6.8) by

(6.11) $F(x, y) = \hat{f}(x) + f_n(x)y$

(6.12) $\eta(x, y) = \hat{\xi}(x) + \xi_n(x)y$

In [3, pp. 141-144], we show that the curvature spheres of μ are of two types. The first type is

(6.13) $[K(x, y)] = [\xi_n(x)K_1(x, y) - f_n(x)K_2(x, y)]$

This is the new curvature sphere introduced by the surface of revolution construction. The second type is

(6.14) $[K(x, y)] = [rK_1(x, y) + sK_2(x, y)]$

where r, s are real numbers such that

(6.15) $[k(x)] = [rk_1(x) + sk_2(x)]$

is a curvature sphere of λ at $x \in M$.

 Now we turn to the question of whether there exists a Lie sphere transformation A such that the point sphere map of the Legendre submanifold $A\lambda : M \to \Lambda^{2n-1}$ is an immersion. The point sphere map $Z(x)$ of the Legendre submanifold $A\lambda$ is determined by the condition

(6.16) $\langle Z(x), e_{n+m+2} \rangle = 0$

The map $Z(x)$ is an immersion if and only if $Z(x)$ is not a curvature sphere of $A\lambda$, for any $x \in M$. Since the curvature spheres of $A\lambda$ are of the form $A[k]$, where $[k]$ is a curvature sphere of λ, we see that Z is an immersion if and only if

(6.17) $\langle Ak(x), e_{n+m+2} \rangle \neq 0$

for all curvature spheres $[k(x)]$ of λ at all points $x \in M$. If we apply the Lie sphere transformation $A^{-1} \in O(n+1,2)$ to (6.17), we get the condition

$$(6.18) \qquad \langle k(x), A^{-1}e_{n+m+2} \rangle \neq 0$$

for all curvature spheres $[k(x)]$ of λ at all points $x \in M$. Note that $v = A^{-1}e_{n+m+2}$ is a unit timelike vector in $\mathbf{R}_2^{n+3}$. Thus, there exists a Lie sphere transformation A such that the point sphere map Z of $A\lambda$ is an immersion if and only if there exists a unit timelike vector $v \in \mathbf{R}_2^{n+3}$ such that

$$(6.19) \qquad \langle k, v \rangle \neq 0$$

for all curvature sphere maps k of λ.

It is a hypothesis of Lemma 27 that there exists a Lie sphere transformation $B \in O(n+m+1,2)$ such that the point sphere map W of the Legendre submanifold $B\mu$ is an immersion. Thus, there exists a unit timelike vector $q \in \mathbf{R}_2^{n+m+3}$ such that

$$(6.20) \qquad \langle K, q \rangle \neq 0$$

for all curvature sphere maps K of μ. We can write q in coordinates as

$$(6.21) \qquad q = (q_0, q_1, \hat{q}, w, q_{n+m+2})$$

where $\hat{q} = (q_2, \ldots, q_n) \in \mathbf{R}^{n-1}$ and $w = (q_{n+1}, \ldots, q_{n+m+1})$. For a curvature sphere $K(x,y)$ as in (6.14), one can compute that

$$
\begin{aligned}
(6.22) \qquad \langle K(x,y), q \rangle &= -q_0(r(\frac{1+f \cdot f}{2}) + sf \cdot \xi) \\
&+ q_1(r(\frac{1-f \cdot f}{2}) - sf \cdot \xi) + (r\hat{f}(x) + s\hat{\xi}(x)) \cdot \hat{q} \\
&+ (rf_n(x) + s\xi_n(x))(y \cdot w) - sq_{n+m+2}
\end{aligned}
$$

since $F \cdot F = f \cdot f$, $\eta \cdot \eta = \xi \cdot \xi$, and $F \cdot \eta = f \cdot \xi$. Note that all terms depend only on $x \in M$ except the term

$$(6.23) \qquad (rf_n(x) + s\xi_n(x))(y \cdot w)$$

If we take $y = u_n$, then

$$(6.24) \qquad \langle K(x,y), q \rangle = \langle k(x), v \rangle$$

for $[k(x)]$ as in (6.15) and $v = \pi(q)$, where π is orthogonal projection of $\mathbf{R}_2^{n+m+2}$ onto its subspace $\mathbf{R}_2^{n+3}$ in (6.1) given by

$$(6.25) \qquad v = \pi(\sum_{i=0}^{n+m+2} q_i e_i) = \sum_{i=0}^{n+1} q_i e_i + q_{n+m+2}e_{n+m+2}$$

Note that v is timelike, since q is a unit timelike vector, and

$$(6.26) \qquad \langle v, v \rangle = \langle q, q \rangle - (q_{n+2}^2 + \cdots + q_{n+m+1}^2)$$

Thus, v is a timelike vector such that

$$(6.27) \qquad \langle k, v \rangle \neq 0$$

for all curvature sphere maps $[k]$ of λ. So there exists a Lie sphere transformation $A \in O(n+1, 2)$ such that the point sphere map of $A\lambda$ is an immersion. $\qquad \square$

Theorem 28. *Let M be a compact connected proper Dupin hypersurface immersed in R^n with four distinct principal curvatures having multiplicities $m_1 = m_2 \geq 1$, $m_3 = m_4 = 1$, and constant Lie curvature. Then M is Lie equivalent to an isoparametric hypersurface.*

Proof. As noted in Remark 26, M must, in fact, be embedded in $\mathbf{R}^n$. Miyaoka [12, p. 252] showed that if M is a compact connected proper Dupin hypersurface embedded in $\mathbf{R}^n$ with four distinct principal curvatures and constant Lie curvature r, then r must equal -1. (Miyaoka's therorem states that $r = 1/2$, but that case corresponds to $r = -1$ under the ordering of the principal curvatures used here). Then Theorem 25 implies that M is irreducible, and finally Theorem 23 implies that M is Lie equivalent to an isoparametric hypersurface. $\qquad \square$

Remark 29. Theorem 25 and Corollary 10 imply that a compact proper Dupin hypersurface immersed in $\mathbf{R}^n$ with $g = 3$ principal curvatures must be Lie equivalent to an isoparametric hypersurface. This was first proven by Miyaoka [11], who used different methods and did not focus on the notion of irreducibility.

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