A NOTE ON THE PAPER "ISOPARAMETRIC HYPERSURFACES WITH FOUR PRINCIPAL CURVATURES"

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In memory of Hongyou Wu

Abstract. In [6], employing commutative algebra, we showed that if the number of principal curvatures is 4 and if the multiplicities m_1 and m_2 of the principal curvatures satisfy $m_2 \geq 2m_1 - 1$, then the isoparametric hypersurface is of the type constructed by Ozeki-Takeuchi and Ferus-Karcher-Münzner [18], [11]. This leaves only four multiplicity pairs $(m_1, m_2) = (3,4), (4,5), (6,9)$ and $(7,8)$ unsettled. The proof eventually comes down to an algebro-geometric estimate [6] on the dimensions of certain singular varieties defined by the second fundamental form of the focal manifold of the smaller codimension, resorting at one point to a nontrivial topological result in [17].

In this note, we present a simple way for the same dimension estimate, which employs essentially no more than the implicit function theorem in calculus.

1. Introduction

An isoparametric hypersurface in a space form is one whose principal curvatures and their multiplicities are fixed constants. The long history of the study of isoparametric hypersurfaces dates back to 1918 when isoparametric surfaces in Euclidean 3-space arose in the study of geometric optics [13], [20], [19]; in contrast, their latest application to integrable systems came in as late as in 1995 [10], to the author's knowledge. The classification of such hypersurfaces started with Segre and Levi-Civita's papers [19], [14] for Euclidean space. Cartan soon afterwards settled the hyperbolic case [2] and found the spherical case deeply intriguing. He classified the spherical cases when g, the number of principal curvatures, is ≤ 3 [3], [4]; in particular, the case $g = 3$ furnished a very geometric description of the Cayley projective plane he had classified as a rank-one symmetric space. He then found two

1991 *Mathematics Subject Classification.* Primary 53C40.

Key words and phrases. isoparametric hypersurfaces.

The author was partially supported by NSF Grant No. DMS-0103838.

homogeneous examples of such hypersurfaces with $g = 4$ in S^5 and S^9 [5]. The interest was rekindled in the 1970's to study isoparametric hypersurfaces in the sphere, with remarkable advancement made [17]. One knows $g = 1, 2, 3, 4, 6$ and there are at most two different multiplicities $(m_1, m_2), m_1 \leq m_2$, where geometrically $m_1 + 1$ and $m_2 + 1$ are the codimensions of the two focal manifolds, M_+ and M_-, respectively, in the ambient sphere that the 1-parameter family of isoparametric hypersurfaces, associated with the given one, degenerates to. It is known that $m_1 = m_2 = 1$ or 2 when $g = 6$ [1]. The case $m_1 = m_2 = 1$ in S^7 was classified by Dorfmeister and Neher [9] (see also [15]) and the other case in S^{13} has recently been classified by Miyaoka [16]; they turn out to be homogeneous. The case when $g = 4$ seems to have gained the most scrutiny, due to the existence of families of inhomogeneous examples constructed by Ozeki-Takeuchi and Ferus-Karcher-Münzner, referred to collectively in this paper as of OT-FKM type [18], [11]. Using homotopy theory, it is shown in [21] that for $g = 4$, the multiplicity pairs (m_1, m_2) are exactly $(2, 2), (4, 5)$ or those of OT-FKM type.

In [6], employing commutative algebra, we showed that if $g = 4$ and $m_2 \geq 2m_1 - 1$, then the isoparametric hypersurface is of OT-FKM type. This leaves only four multiplicity pairs $(m_1, m_2) = (3, 4), (4, 5), (6, 9)$ and $(7, 8)$ unsettled. The proof eventually comes down to an algebro-geometric estimate [6, pp 68-73] on the dimensions of certain singular varieties defined by the second fundamental form of the focal manifold M_+, resorting at one point to a nontrivial topological result in [17] that states that $m_1 + m_2$ is an odd number when $2 \leq m_1 < m_2$.

In this note, we present a simple way for the same dimension estimate, which employs essentially no more than the implicit function theorem in calculus.

2. The estimate

We will follow closely the notation in [6]. Recall that we fix a normal basis $n_0, \cdots, n_{m_1}$ of M_+ and decompose the normal space of M_+ as the direct sum of eigenspaces E_+, E_- and E_0 of the shape operator S_{n_0} with eigenvalues $1, -1$ and 0, respectively. E_+, E_- and E_0, of dimensions m_2, m_2 and m_1, are parametrized by the vectors $x = (x_1, \cdots, x_{m_2}), y = (y_1, \cdots, y_{m_2})$ and $z = (z_1, \cdots, z_{m_1})$, respectively.

We are interested in the components of the second fundamental form of M_+ restricted to $E_+ \oplus E_-$, denoted by $p_0, p_1, \cdots, p_{m_1}$, which are explicitly given in equation (10.2) on p 52 of [6]. The important point is that $p_1, \cdots, p_{m_1}$ are bihomogeneous polynomials of bidegree $(1, 1)$ in x and y, i.e., they are linear in x and y, respectively.

As on p 72 of [6], for $n \leq m_1$, consider the map

$$f_n : (x, y) \in \mathbb{C}^{m_2} \times \mathbb{C}^{m_2} \mapsto (p_1(x, y), \cdots, p_n(x, y)).$$

Let Z_n be the singular set of f_n, where the rank of df_n is $< n$, and set

$$J_n := Z_n \cap f_n^{-1}(0).$$

The key point is to estimate the dimension of J_n to obtain [6, p 73]

$$\dim(J_n) \leq m_1 + m_2 - 2$$

to warrant the primeness of the ideal $(p_1, \cdots, p_n)$ for $n \leq m_1 - 1$, and the reducedness of the ideal $(p_1, \cdots, p_{m_1})$ when $m_2 \geq 2m_1 - 1$. More precisely, we aim to assert

$$\dim(J_n) \leq \dim(f_n^{-1}(0)) - 2 \tag{1}$$

for $n \leq m_1 - 1$, and

$$\dim(J_{m_1}) \leq \dim(f_{m_1}^{-1}(0)) - 1. \tag{2}$$

As on p 72 of [6], $\mathbb{C}^{m_2} \times \mathbb{C}^{m_2}$ is stratified into locally closed sets (i.e., Zariski open sets in their respective closures) $X_{-1}, X_0, X_1, \cdots, X_{n-1}$ such that df_n has rank

$$J := n - j - 1$$

on X_j. We have

$$Z_n = \cup_{j \geq 0} X_j.$$

Definition 1. *We say a point $(x, y) \in Z_n$ is generic in Z_n if there is a $p_\lambda = c_1 p_1 + \cdots + c_n p_n$, where $\lambda := [c_1 : \cdots : c_n] \in \mathbb{C}P^{n-1}$, such that $dp_\lambda(x, y) = 0$ and the rank of $p_\lambda = 2(m_2 - r)$ for some $r \leq m_1$ (respectively, for some $r \leq m_1 - 1$) if $m_2 \geq 2m_1$ (respectively, if $m_2 = 2m_1 - 1$). We say (x, y) is nongeneric in Z_n otherwise.*

Lemma 1. *For a generic choice of λ, we always have that p_λ is of rank $= 2(m_2 - r)$ for some $r \leq m_1$ (respectively, for some $r \leq m_1 - 1$) if $m_2 \geq 2m_1$ (respectively, if $m_2 = 2m_1 - 1$).*

Proof. We may assume $m_1 \geq 2$ as it is straightforward to verify that the isoparametric hypersurface is of OT-FKM type when $m_1 = 1$ [6, p 61]. Observe that the lemma is immediate in the case $m_2 \geq 2m_1$, because automatically the rank of $p_\lambda \geq 2(m_2 - m_1)$ for a generic choice of λ since the same is true for all $p_1, \cdots, p_n$ [6, Lemma 49, p 64].

Hence, we may assume now that $m_2 = 2m_1 - 1$. Suppose the lemma is not true. Then p_λ will have rank $= 2(m_2 - m_1)$ for generic λ; in particular, this is true for $p_1, \cdots, p_n$.

Recall [18, p 536, I] the second fundamental matrices $S_1, \cdots, S_{m_1}$ of M_+ are related to $A_1, \cdots, A_{m_1}$ by

$$S_a = \begin{pmatrix} 0 & A_a & B_a \\ A_a^{tr} & 0 & C_a \\ B_a^{tr} & C_a^{tr} & 0 \end{pmatrix}. \tag{3}$$

Observe that Lemma 49 of [6] implies

$$B_a^{tr} B_a = I/2 \tag{4}$$

for all $a = 1, \cdots, n$ by the fact that all p_a now have the same rank $2(m_2 - m_1)$, or equivalently, that m_1 is both the kernel dimension of A_a and the rank of B_a for each a. This means that the column vectors of $\sqrt{2}B_a$ are orthonormal vectors for each a. Since the column space of each B_a, being of dimension m_1, lives in $\mathbb{R}^{m_2} = \mathbb{R}^{2m_1-1}$, the column spaces of B_a and B_c, for $a \neq c$, must intersect nontrivially for dimension reason. We may thus arrange so that some nonzero v in the intersection is the first column vector of both $\sqrt{2}B_a$ and $\sqrt{2}B_c$. However, polarizing (4), we obatin

$$B_a^{tr} B_c = -B_c^{tr} B_a$$

for all $a \neq c$. It follows that

$$< v, v > = - < v, v >$$

with $< \cdot, \cdot >$ the Euclidean inner product. That is, we have derived $v = 0$, which is absurd. $\square$

Lemma 2. *If $(x, y) \in X_j$ with $J \leq n - 2$, then (x, y) is generic in Z_n.*

Proof. Since

$$df_n : \mathbb{C}^{2m_2} \to \mathbb{C}^n$$

is of rank J at (x, y), the kernel $\mathcal{K}$ of

$$df_n^{tr} : \mathbb{C}^n \to \mathbb{C}^{2m_2}$$

is of dimension $n - J$ at (x, y). This says precisely that, for any $\lambda = (c_1, \cdots, c_n)$, the linear combination $p_\lambda = c_1 p_1 + \cdots + c_n p_n$ satisfies $dp_\lambda = 0$ at (x, y) if and only if $\lambda \in \mathcal{K}$. Therefore, if $J \leq n - 2$, or $\dim(\mathcal{K}) = n - J \geq 2$, then up to scaling there is at least a one-parameter family of $\lambda \in \mathcal{K}$ for which $dp_\lambda = 0$ at (x, y); the generic choice of such p_λ will respect Lemma 1. This is because by the realness of $p_1, \cdots, p_n$, the above condition $dp_\lambda = c_1 dp_1 + \cdots + c_n dp_n = 0$ for $\lambda \in \mathcal{K}$ splits into

$$dp_\lambda^{re} = dp_\lambda^{im} = 0$$

at (x, y), where re and im denote the real and imaginary parts of a complex number. Since

$$d(\cos(\theta)p_\lambda^{re} - \sin(\theta)p_\lambda^{im}) = 0,$$

and

$$d(\sin(\theta)p_\lambda^{re} + \cos(\theta)p_\lambda^{im}) = 0,$$

for all θ at (x, y) by multiplying p_λ by $e^{\sqrt{-1}\theta}$, we see p_λ^{re} and p_λ^{im} play the role of p_a and p_c in Lemma 1 for a generic choice of $\lambda \in \mathcal{K}$. $\square$

Fix $(c_0, d_0) \in X_j \subset \mathbb{C}^{m_2} \times \mathbb{C}^{m_2}$, $j \geq 0$, where the rank of df_n is J. For the sake of simplicity in Taylor expansions, we assume $p_1 = \cdots = p_n = 0$ at (c_0, d_0) since we will be most interested in J_n later on. Without loss of generality assume $x_1, \cdots, x_s$ and $y_1, \cdots, y_t$, where $s + t = J$, are, respectively, the coordinates of x and y such that

$$\partial(p_1, \cdots, p_J)/\partial(x_1, \cdots, x_s, y_1, \cdots, y_t) \neq 0$$

at (c_0, d_0). The implicit function theorem in calculus says that we can replace these coordinates by $p_1, \cdots, p_J$ so that in a neighborhood of $(c_0, d_0) \in \mathbb{C}^{m_2} \times \mathbb{C}^{m_2}$ the function f_n is of the form

$$\begin{aligned} f_n : &(p_1, \cdots, p_J, x_{s+1}, \cdots, x_{m_2}, y_{t+1}, \cdots, y_{m_2}) \\ &\mapsto (p_1, \cdots, p_J, \cdots, p_n). \end{aligned} \tag{5}$$

It follows that the rank of f_n is J in the neighborhood precisely when

$$\begin{aligned} \partial p_a(p_1, \cdots, p_J, x_{s+1}, \cdots, x_{m_2}, y_{t+1}, \cdots, y_{m_2})/\partial x_\alpha = 0, \\ \partial p_a(p_1, \cdots, p_J, x_{s+1}, \cdots, x_{m_2}, y_{t+1}, \cdots, y_{m_2})/\partial y_\mu = 0, \end{aligned} \tag{6}$$

where $a > J, \alpha > s$ and $\mu > t$. Note that p_n satisfies (6).

We first handle the case when (c_0, d_0) is generic in Z_n. We may assume $dp_n(c_0, d_0) = 0$ and p_n is of rank $2(m_2 - r)$ for some $r \leq m_1$ (respectively, $r \leq m_1 - 1$) when $m_2 \geq 2m_1$ (respectively, $m_2 = 2m_1 - 1$); the rank is exactly that of the Hessian $H(p_n)$ of p_n at (c_0, d_0), which is in turn independent of the coordinate system chosen (this is the basis of Morse Theory). Thus, we can first calculate the rank of $H(p_n)$ with respect to the original coordinates $x_1, \cdots, x_{m_2}, y_1, \cdots, y_{m_2}$, for which p_n is bihomogeneous of degree $(1, 1)$, to obtain

$$H(p_n) = \begin{pmatrix} 0 & A_n \\ A_n^{tr} & 0 \end{pmatrix}, \tag{7}$$

where A_n is the matrix such that $p_n(X, Y) = 2X^{tr} A_n Y$ with $X = (x_1, \cdots, x_{m_2})$ and $Y = (y_1, \cdots, y_{m_2})$. In particular, the rank of A_n is $m_2 - r$.

On the other hand, $H(p_n)$, calculated with respect to the coordinates $p_1, \cdots, p_J, x_{s+1}, \cdots, x_{m_2}$ and $y_{t+1}, \cdots, y_{m_2}$, is

$$H(p_n) = \begin{pmatrix} A & B^{tr} \\ B & C \end{pmatrix}, \tag{8}$$

where A is of size J-by-J, which is the matrix of the second derivatives of p_n with respect to $p_1, \cdots, p_J$, and C is of size $(2m_2-J)$-by-$(2m_2-J)$, which is the matrix of the second derivatives of p_n with respect to $x_{s+1}, \cdots, x_{m_2}, y_{t+1}, \cdots, y_{m_2}$, etc.

Let R be the rank of C at (c_0, d_0), $k \leq J$ be the rank of B and $l \leq J$ be the rank of the matrix $\begin{pmatrix} A & B^{tr} \end{pmatrix}$. Since the rank of $H(p_n)$ is $2m_2 - 2r$, we see

$$R \geq 2m_2 - 2r - k - l. \tag{9}$$

We may assume the upper left R-by-R block of C is nonsingular without loss of generality. The implicit function theorem applied to the $2m_2 - J$ equations in (6) for p_n enables us to solve the first R variables of $x_{s+1}, \cdots, x_{m_2}, y_{t+1}, \cdots, y_{m_2}$ in terms of $p_1, \cdots, p_J$ and the remaining $2m_2 - J - R$ variables of $x_{s+1}, \cdots, x_{m_2}, y_{t+1}, \cdots, y_{m_2}$, to be denoted by $z_1, \cdots, z_{2m_2-J-R}$. Therefore, the irreducible component $\mathcal{V}$ of X_j containing (c_0, d_0), being locally contained in the space parametrized by the variables $p_1, \cdots, p_J$ and $z_1, \cdots, z_{2m_2-J-R}$, satisfies, by (9),

$$\dim(\mathcal{V}) \leq 2m_2 - R \leq 2r + k + l.$$

Setting $p_1 = \cdots = p_J = 0$, we see $\mathcal{V} \cap J_n$ satisfies

$$\dim(\mathcal{V} \cap J_n) \leq 2m_2 - R - J \leq 2r + k + l - J \leq 2r + J. \tag{10}$$

Case 1. $m_2 \geq 2m_1 + 1$.

With $r \leq m_1$ in the generic case and $J \leq m_1 - 1$ in general, we have

$$2r + J \leq 3m_1 - 1 \leq m_1 + m_2 - 2,$$

which is the desired dimension estimate.

Case 2. $m_2 = 2m_1 - 1$.

With $r \leq m_1 - 1$ in the generic case and $J \leq m_1 - 1$ in general, we obtain once more

$$2r + J \leq 3m_1 - 3 = m_1 + m_2 - 2.$$

Case 3. $m_2 = 2m_1$.

We may assume $n = m_1$, because for $n \leq m_1 - 1$ we would have $J \leq n - 1 \leq m_1 - 2$, and so with $r \leq m_1$ we obtain

$$\dim(\mathcal{V} \cap J_n) \leq 2r + J \leq 3m_1 - 2 = m_1 + m_2 - 2.$$

Now with $n = m_1$, we see

$$\dim(\mathcal{V} \cap J_{m_1}) \le 2r + J \le 3m_1 - 1;$$

on the other hand, the m_1 cuts $p_1 = \cdots = p_{m_1} = 0$ give the dimension estimate

$$\dim(f_{m_1}^{-1}(0)) \ge 2m_2 - m_1 = 3m_1.$$

In other words,

$$\dim(\mathcal{V} \cap J_n) \le \dim(f_n^{-1}(0)) - 2$$

if $n \le m_1 - 1$, and

$$\dim(\mathcal{V} \cap J_{m_1}) \le \dim(f_{m_1}^{-1}(0)) - 1,$$

which are (1) and (2) we are after. We are done with the case when (c_0, d_0) is generic in Z_n.

To handle the case when (c_0, d_0) is nongeneric in Z_n, note that now $(c_0, d_0) \in X_0$ by Lemma 2. Let $\mathcal{X}$ be the irreducible component of X_0 containing (c_0, d_0). Since the boundary points p of $\mathcal{X}$ belong to X_j for some $j > 0$, Lemma 2 implies that p are generic in Z_n. Therefore, by (9), the rank R of the matrix C defined in (8) at such a p satisfies, with $r \le m_1$ if $m_2 \ge 2m_1$ and $r \le m_1 - 1$ if $m_2 = 2m_1 - 1$, that

$$R \ge 2m_1 - 2J, \tag{11}$$

which continues to hold at all points $q \in \mathcal{X} \subset X_0$ close to p. Denote the rank of the matrix C in (8) at q by R_0. Since the size of the matrix A in (8) at such a q is $(n-1)$-by-$(n-1)$, we see that R_0 satisfy, by (11),

$$\begin{aligned} R_0 &\ge R - 2(n-1-J) \ge 2(m_1 - n + 1) \\ &\ge 4 \qquad \text{if } n \le m_1 - 1, \\ &\ge 2 \qquad \text{if } n = m_1. \end{aligned} \tag{12}$$

Since at each step of the cutting from $\mathcal{X}$ down to $\mathcal{X} \cap J_n$, we can introduce a generic p_λ to cut, it follows that the resulting variety at each step is of pure dimension, even though some of whose components may not intersect X_j for all $j > 0$. Hence. for the purpose of dimension estimate, we may look at an irreducible component of $\mathcal{X} \cap J_n$ with a boundary point in X_j for some $j > 0$. Equation (12) implies, by the first inequality in (10), which is generally true, that

$$\begin{aligned} \dim(\mathcal{X} \cap J_n) &\le 2m_2 - R_0 - (n-1) \\ &\le 2m_2 - n - 2, \qquad \text{if } n \le m_1 - 1, \\ &\le 2m_2 - n - 1, \qquad \text{if } n = m_1. \end{aligned}$$

That is, the dimension estimates (1) and (2) hold in the nongeneric case as well.

Remark 1. *In the third line above the last displayed formula on p 63 of* [6]*, the symbol p refers to any point of the variety; in particular, it applies to the generically chosen point* $z = (h, k)$*. Meanwhile, the formulae in the last line on the same page are slightly misleading. They should read, instead, with the Einstein summation convention prevailing,*

$$\overline{F}^{\mu}_{\alpha a} k_\mu = f_{ab}(h,k) F^{\mu}_{\alpha b} k_\mu, \qquad \overline{F}^{\mu}_{\alpha a} h_\alpha = f_{ab}(h,k) F^{\mu}_{\alpha b} h_\alpha,$$

where $f_{ab}(z) := r_{ab}(z)/q_a(z)$*. In other words, the equations say* $\overline{p}_a(\cdot, k) = f_{ab}(h,k) p_b(\cdot, k)$ *and* $\overline{p}_a(h, \cdot) = f_{ab}(h,k) p_b(h, \cdot)$*.*

Since the image of the projection $W \to \mathbb{R}^{m_2}$ *sending* z *to* h *is dense as* z *varies generically in* W*, we see that*

$$\overline{p}_a(h, \cdot) = f_{ab}(h) p_b(h, \cdot) \tag{13}$$

holds in a neighborhood around a generic point h_0 *with* $f_{ab}(h)$ *the collection of the terms in* h *of the analytic* $f_{ab}(h,k)$*. Since* $\overline{p}_a$ *and* p_b *are linear in* h*, taking a second order partial derivative of* (13) *against* h *at* h_0*, which we denote by* $f''_{ab}(h_0)$ *, gives*

$$f''_{ab}(h_0) p_b(h_0, k) = 0.$$

This implies $f''_{ab}(h_0) = 0$ *for all* a, b *since the map*

$$k :\mapsto (p_1(h_0, k), \cdots, p_{m_1}(h_0, k))$$

is surjective (see pp 60-62 of [6] *for the n-spanning property) for a generic* h_0*. Similarly, all partial derivatives of* f_{ab} *of order* ≥ 2 *are zero at* h_0*. We conclude that* $f_{ab}(h)$ *are linear functions in* h*. As a consequence,* $\overline{p}_a = c_{ab} p_b$ *for some constants* c_{ab}*.*

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