

Strong Analytic Controllability for Hydrogen Control Systems

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1 Introduction

From a theoretical point of view, the essential problem of the manipulation of quantum dynamical phenomena is the problem of control. The introduction of control theory has put this manipulation on a firm theoretical footing. Classical concepts of geometric control provide a basis for many quantum results. For instance, the field of nuclear magnetic resonance is largely concerned with the geometric control of collections of interacting nuclear spins [1, 2, 3, 4]. The methods used to derive these results are those of geometric control applied to quantum systems: these group-theoretic methods allow easy mathematical treatment of Hamiltonian systems.

Although quantum computation has traditionally concerned itself with the manipulation of discrete systems with finite-dimensional state spaces, many quantum variables, such as position, momentum, or the amplitudes of electromagnetic fields, are continuous and act on infinite-dimensional state spaces. Because of developments in quantum error correction [5, 6, 7] and quantum teleportation [8, 9] of continuous variables, it is worthwhile addressing the question of quantum computation over continuous variables. In order to overcome the difficulties caused by infinite-dimensionality and the unboundedness of such continuous operators Huang, Tarn, and Clark (HTC) [10, 11] applied the analytic domain of Nelson [12] and proposed a theorem of strong analytic controllability of quantum systems acting on infinite-dimensional Hilbert spaces. This theorem is readily applicable to quantum control systems whose operators are continuous quantum variables.

In the microscopic world ruled by quantum mechanics, most interesting phenomena involve change. Time-dependent quantum systems have received

more and more attention with the development of modern computers, because we can now make detailed investigations of many “nonintegrable” time-dependent problems. With the advent of femtosecond lasers, nanostructure fabrication, and other laboratory advances, new research in time-dependent quantum systems has been developed in various fields, including atomic and optical physics, physical chemistry, solid state physics, nanoscience, and nanotechnology [13]. All real world quantum systems are influenced by interaction with their environment to a greater or lesser extent. Their environment changes as time passes, so the Hamiltonians used to describe these open quantum systems are time-dependent ones, as is the case in [14, 15]. Implementing controls on these time-dependent quantum systems LTCC(Lan, Tarn, Chi, and Clark)[16] proposed a theorem to account for time-dependent operators. This theorem can be applied to time-independent and time-dependent quantum systems, and can be applied to quantum systems with discrete and continuous operators acting on finite-dimensional or infinite-dimensional state spaces.

The application of group theory to strongly interacting particles has a long history [17]. Not only do certain systems possess obvious symmetries which allow a classification of their spectra, but it has also been suggested that one might look for certain transformations which allow passing from one level state to another, and thus get new insight into the structure of the system [18]. It is well known that the bound state subspace of a hydrogen atom for a fixed energy E_n spans a representation space of dimension n^2 of the symmetry group $SO(4)$ [19]. The symmetry group is $SO(3,1)$ for scattering states. When the atom interacts with an external electromagnetic field, it can make transitions to other energy sub-

spaces, because energy is transferred to or from the atom. Thus we can connect a representation space of $SO(4)$ or $SO(3,1)$ of some energy with representation spaces of other energies. It turns out that the totality of all states of the atom spans a representation space of $SO(4,2)$, and thus one can say that this dynamic group encompasses all the atomic states together with their transformations, including the transformations between states of different energy [20]. To study $SO(4,2)$, we examine its Lie algebra $so(4,2)$, which not only expresses the geometric structure property of the Lie group $SO(4,2)$, but also possesses a simple linear structure.

In this paper, we begin in section 2 with summarizing the theorem of strong analytic controllability of time-dependent quantum systems. In section 3, we first give the realization of $so(4,2)$ for negative energy and positive energy by solving differential equations, then show the representation states of $so(4,2)$, which is the representation states of $SO(4,2)$ as well. Using operators from the realization of $so(4,2)$ as interacting Hamiltonians, we construct a hydrogen atom control system in section 4. Based on the above and the strong analytic controllability theorem, we conclude that this hydrogen atom control system is strongly analytically controllable. The construction of this hydrogen atom control system provides theoretical direction on how to construct a new control system by adding controls and interacting Hamiltonians to the original Schrödinger equation so that this new control system is strongly analytically controllable. Finally we conclude this paper with section 5.

2 A Strong Analytic Controllability Theorem for Time-dependent Quantum Systems

Quantum control systems as described by the Schrödinger equation are bilinear systems with re-

spect to controls and states

$$\begin{aligned} \frac{\partial}{\partial t} \psi(t, x) &= (H_0(t, x) + \sum_{l=1}^r u_l(t) H_l(t, x)) \psi(t, x), \\ \psi(t_0, x) &= \psi_0, \end{aligned} \tag{1}$$

where $H_0(t, x), H_l(t, x)$, $l = 1, 2, \dots, r$, are skew-Hermitian operators on $S_{\mathcal{H}}$, $\psi(t, x) \in S_{\mathcal{H}}$, and $u_l(t)$, $l = 1, \dots, r$ are piece-wise constant functions. In fact, the above skew-Hermitian operator $H(t, x)$ can be written as $iH'(t, x)$, which H' is an observable, an Hermitian operator.

Operators involved in system (1) are generally unbounded and act on an infinite-dimensional Hilbert space which consists of quantum states ψ . In order to find a domain on which the exponentiations of these operators converge, we introduce a so-called analytic domain D_{ω} proposed by Nelson [12], a dense domain invariant under the action of operators in system (1). In addition, the solution of the Schrödinger equation can be expressed globally in exponential form on this domain, which is also invariant under the action of the exponentiations of these operators.

For system (1), a theorem [10, 11] tells us that the transitivity of states on $S_{\mathcal{H}}$ requires infinite control manipulations based on the piece-wise constant control set $\{u_l(t)\}$. Obviously, switching control manipulations infinitely is meaningless in practice. So practical interest directs us to consider controllability on a finite-dimensional submanifold M of the unit sphere Hilbert space $S_{\mathcal{H}}$. Moreover, a finite-dimensional tangent space generated by $H_0(t, x)\psi(t, x), \dots, H_r(t, x)\psi(t, x)$ is defined on $M \cap D_{\omega}$ and $\dim M \cap D_{\omega} = m$ when assuming $\dim M = m$. That is, the finite-dimensional tangent space generated by $H_0(t, x)\psi(t, x), \dots, H_r(t, x)\psi(t, x)$ is densely defined on M . Further the manifold M can be given by the closure of the set $\{e^{t_0 H_{p_0}} e^{t_1 H_{p_1}} \dots e^{t_r H_{p_r}} \psi_0\}$, with (p_0, p_1, \dots, p_r) any permutation of $(0, 1, \dots, r)$ and $t_i \in \mathbb{R}^1, i = 0, 1, \dots, r$.

On analytic domain, the controllability is modified to analytic controllability.

Definition 2.1 For system (1), if D_{ω} exists for \mathcal{L} , and for any $\psi_0, \psi_d \in D_{\omega} \cap M$, there exist

$u_1(t), \dots, u_r(t)$, and T ($\forall T$) s.t. the solution of control system (1) satisfies $\psi(t_0, x) = \psi_0$, $\psi(T, x) = \psi_d$, then the system is called *analytically controllable* (strongly analytically controllable) on M , and we say the corresponding unitary Lie group is *analytically transitive* on M .

Let $R_t(\psi_0)$ denote the reachable set of system (1) from starting state ψ_0 at time t , and let $R(\psi_0) = \bigcup_{t>t_0} R_t(\psi_0)$ denote the reachable set starting from ψ_0 at time greater than t_0 .

Definition 2.2 A time-dependent control system is said to be *completely controllable* if $R(\psi_0) = M$ holds for all $\psi_0 \in M$. If $R_t(\psi_0) = M$ for all time $t > t_0$ and for all $\psi_0 \in M$, then this system is said to be *strongly completely controllable* on M . If $R_t(\psi_0) = M \cap D_\omega$ for all time $t > t_0$ and for all $\psi_0 \in M \cap D_\omega$, then this system is said to be *strongly analytically controllable* on M .

With the help of Kunita' method [21], we derive the following strong analytic controllability theorem in [16, 22].

Theorem 2.1 Considering quantum control system (1), let

$$\begin{aligned} \mathcal{B}(t, x) &= \mathcal{L}(H_1(t, x), \dots, H_r(t, x)) \\ B_1 &= [H_0, B_0] + \frac{\partial}{\partial t} B \\ &\vdots \\ B_n &= [H_0, B_{n-1}] + \frac{\partial}{\partial t} B_{n-1} \\ &\vdots \\ \mathcal{C} &= \mathcal{L}\{B, B_1, \dots, B_n, \dots\}. \end{aligned}$$

Suppose $\dim \mathcal{C}(t, x)\psi(t, x) = m$ holds for all $\psi \in M \cap D_\omega$, and $[\mathcal{B}, \mathcal{C}](t, x) \subset \mathcal{B}(t, x)$ holds for all (t, x) . Then the time-dependent quantum control system (1) is *strongly analytically controllable*.

3 The realization and the representation of $so(4, 2)$

Kleinert [23] pointed out that the internal structure of the quantum mechanical system of the hydrogen

atom can be described completely in terms of simple group operations in the representation space of the non-compact group $SO(4, 2)$. $SO(4, 2)$ is the group of rotation in six-dimensional pseudo-Euclidean space with the metric

$$g = \text{diag}(1, 1, 1, 1, -1, -1). \quad (2)$$

It is a 15-parameter group and is generated by 15 generators $L_{\alpha\beta} = -L_{\beta\alpha}$, $\alpha, \beta = 1, \dots, 6$ which form the Lie algebra $so(4, 2)$ with the commutation rules

$$[L_{\alpha\beta}, L_{\alpha\gamma}] = ig_{\alpha\alpha} L_{\beta\gamma}. \quad (3)$$

All information contained in the Schrödinger theory of the hydrogen atom can be expressed in a completely algebraic language; that is, representation states of $so(4, 2)$ can be brought in one-to-one correspondence with the states of the hydrogen atom.

The Lie algebra $so(4, 2)$ appears as the tangent space to its Lie group $SO(4, 2)$ at the identity with a bilinear product. The structure of the Lie group $SO(4, 2)$ is reflected in the structure of its Lie algebra $so(4, 2)$. Thus, it is much easier to deal with Lie algebra $so(4, 2)$ than with a Lie group $SO(4, 2)$ since the Lie algebra $so(4, 2)$ is a linear vector space.

It is known [24] that the Lie algebra $so(4, 2)$ satisfies the following commutation relations:

$$[\mathbf{L}, \mathbf{L}] = i\mathbf{L} \quad (4)$$

$$[\mathbf{L}, \mathbf{A}] = i\mathbf{A} \quad (5)$$

$$[\mathbf{L}, \mathbf{B}] = i\mathbf{B} \quad (6)$$

$$[\mathbf{L}, \mathbf{S}] = 0 \quad (7)$$

$$[\mathbf{A}, \mathbf{A}] = i\mathbf{L} \quad (8)$$

$$[\mathbf{S}, \mathbf{A}] = i\mathbf{B} \quad (9)$$

$$[\mathbf{A}, \mathbf{B}] = i\mathbf{S} \quad (10)$$

$$[\mathbf{B}, \mathbf{B}] = -i\mathbf{L} \quad (11)$$

$$[\mathbf{S}, \mathbf{B}] = i\mathbf{A} \quad (12)$$

$$[\mathbf{C}, \mathbf{A}] = i\mathbf{\Gamma} \quad (13)$$

$$[\mathbf{D}, \mathbf{A}] = 0 \quad (14)$$

$$[\mathbf{C}, \mathbf{B}] = 0 \quad (15)$$

$$[\mathbf{D}, \mathbf{B}] = -i\mathbf{\Gamma} \quad (16)$$

$$[\mathbf{C}, \mathbf{L}] = 0 \quad (17)$$

$$[\mathbf{D}, \mathbf{L}] = 0 \quad (18)$$

$$[\mathbf{L}, \mathbf{\Gamma}] = i\mathbf{\Gamma} \quad (19)$$

$$[C, S] = -iD \quad (20)$$

$$[S, D] = iC \quad (21)$$

$$[D, C] = iS \quad (22)$$

$$[\mathbf{\Gamma}, \mathbf{A}] = -iC \quad (23)$$

$$[\mathbf{\Gamma}, \mathbf{B}] = -iD \quad (24)$$

$$[\mathbf{\Gamma}, C] = -i\mathbf{A} \quad (25)$$

$$[\mathbf{\Gamma}, S] = 0 \quad (26)$$

$$[\mathbf{\Gamma}, D] = -i\mathbf{B} \quad (27)$$

$$[\mathbf{\Gamma}, \mathbf{\Gamma}] = -i\mathbf{L}, \quad (28)$$

where $[\mathbf{A}, \mathbf{B}] = iS$ means $[A_i, B_j] = i\delta_{ij}S$.

A realization of the Lie algebra $so(4, 2)$ is a homomorphism which associates a concrete set of operators with abstract basis vectors of the Lie algebra [24]. It is more of a physical concept than a mathematical one. In quantum mechanics such operators are often differential operators expressed in terms of the position operator \mathbf{r} and momentum operator \mathbf{p} , and acting on a Hilbert space of quantum mechanical states. In the following we will give a realization of $so(4, 2)$ for negative energy and positive energy respectively by solving differential equations based on Bacry's method, with which Bacry derived the realization of $so(4, 1)$ for negative energy [25]. Dothan [26] proposed that these generators constitute of a finite-dimensional spectrum generating algebra. Adams [24] described a beautiful algebraic method to construct another type of 15 generators of $so(4, 2)$. These two sets of 15 generators are equivalent according to the $so(4, 2)$ commuting structure.

For the negative energy case, Greiner and Müller [27] describes a way to relate $so(4)$ to the hydrogen atom Hamiltonian which is expressed in the atomic units as follows:

$$H = \frac{p^2}{2} - \frac{1}{r}, \quad (29)$$

where p is the momentum and r is the distance between the electron and the proton in the hydrogen atom. That is, they showed that the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ and the Runge-Lenz vector $\mathbf{A} = \frac{1}{\sqrt{-2H}} \frac{1}{2}(\mathbf{L} \times \mathbf{p} - \mathbf{p} \times \mathbf{L}) + \frac{\mathbf{r}}{r}$ generate the Lie algebra $so(4)$, which is part of the Lie algebra of

$so(4, 2)$. Moreover, \mathbf{L} and \mathbf{A} commute with Hamiltonian H . Hence, these operators can only transform a quantum state to another with the same energy level. When quantum systems interact with environment, they may absorb or release energy, thus changing their energy levels. $so(4, 2)$ is more interesting because it permits to transform an energy level into another energy level [24, 26, 23, 28, 19]. Hence, $so(4, 2)$ is represented fully on the energy eigenstates — discrete and continuous, energy degenerate and non-degenerate.

In order to find the other operators $\mathbf{B}, \mathbf{\Gamma}, S, C,$ and D in terms of \mathbf{r} and \mathbf{p} , it is much simpler to use Poisson brackets instead of commutators because equations (4) –(28) lead to differential equations which can be integrated in a simple way under Poisson brackets [25]. By changing the coordinate space from \mathbf{r}, \mathbf{p} to r, H, l [25], where $l^2 = r^2 p^2 - (\mathbf{r} \cdot \mathbf{p})^2$, we solve the differential equations under the Poisson brackets form. Further, following the above computation under the classical Poisson bracket the operators may need to be symmetrized to be Hermitian to get the following negative-energy realization of $so(4, 2)$ [22]:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \quad (30)$$

$$\mathbf{A} = \frac{1}{\sqrt{-2H}} [(\mathbf{L} \times \mathbf{p} - \mathbf{p} \times \mathbf{L})/2 - \frac{\mathbf{r}}{r}] \quad (31)$$

$$\begin{aligned} \mathbf{B} = & \frac{\mathbf{pr} + r\mathbf{p}}{2} \cos \zeta - \frac{1}{\sqrt{-2H}} \frac{\mathbf{p}(\mathbf{r} \cdot \mathbf{p}) + (\mathbf{p} \cdot \mathbf{r})\mathbf{p}}{2} \sin \zeta \\ & + \frac{1}{\sqrt{-2H}} \frac{\mathbf{r}}{r} \sin \zeta \end{aligned} \quad (32)$$

$$\begin{aligned} \mathbf{\Gamma} = & -\frac{\mathbf{pr} + r\mathbf{p}}{2} \sin \zeta - \frac{1}{\sqrt{-2H}} \frac{\mathbf{p}(\mathbf{r} \cdot \mathbf{p}) + (\mathbf{p} \cdot \mathbf{r})\mathbf{p}}{2} \cos \zeta \\ & + \frac{1}{\sqrt{-2H}} \frac{\mathbf{r}}{r} \cos \zeta \end{aligned} \quad (33)$$

$$S = -\frac{\mathbf{r} \cdot \mathbf{p} + \mathbf{p} \cdot \mathbf{r}}{2} \sin \zeta - \frac{1}{\sqrt{-2H}} (1 + 2Hr) \cos \zeta \quad (34)$$

$$C = -\frac{\mathbf{r} \cdot \mathbf{p} + \mathbf{p} \cdot \mathbf{r}}{2} \cos \zeta + \frac{1}{\sqrt{-2H}} (1 + 2Hr) \sin \zeta \quad (35)$$

$$D = \frac{1}{\sqrt{-2H}}, \quad (36)$$

where $\zeta = \sqrt{-2H}(\mathbf{r} \cdot \mathbf{p} + \mathbf{p} \cdot \mathbf{r})/2 + (-2H)^{\frac{3}{2}}t$.

Further, we compute the commutators between the Hamiltonian H and the generators of $so(4, 2)$ as follows:

$$[H, \mathbf{L}] = 0 \quad (37)$$

$$[H, \mathbf{A}] = 0 \quad (38)$$

$$[H, \mathbf{B}] = (-2H)^{\frac{3}{2}}i\mathbf{\Gamma} \quad (39)$$

$$[H, \mathbf{\Gamma}] = -(-2H)^{\frac{3}{2}}i\mathbf{B} \quad (40)$$

$$[H, S] = (-2H)^{\frac{3}{2}}iC \quad (41)$$

$$[H, C] = -(-2H)^{\frac{3}{2}}iS \quad (42)$$

$$[H, D] = 0. \quad (43)$$

Obviously these $so(4, 2)$ generators satisfy the condition

$$i\frac{\partial G}{\partial t} - [H, G] = 0. \quad (44)$$

In [26], the relation (44) tells us that if ψ is an eigenstate of the Schrödinger equation

$$i\frac{\partial \psi}{\partial t} = H\psi, \quad (45)$$

then $G\psi$ is an eigenstate of the above Schrödinger equation when G satisfies the condition (44). So this realization of $so(4, 2)$ forms a finite-dimensional spectrum-generating algebra [26]. All of the energy eigenfunctions of the physical problem form a basis for a single unitary irreducible representation.

Remark 3.1 We know that D , \mathbf{L} , and \mathbf{A} commute with the Hamiltonian H , so the multiplets produced by the action of the transformations generated by D , \mathbf{L} , and \mathbf{A} are degenerate eigenfunctions of H . The other generators of $so(4, 2)$ do not commute with Hamiltonian H , so the multiplets produced by the transformation generated by these operators cannot be degenerate eigenfunctions of H . Rather, these transformations generate a spectrum of eigenfunctions of H .

For the positive energy, it is known [24] that the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ and the Runge-Lenz vector $\mathbf{B} = \frac{1}{\sqrt{2H}}\frac{1}{2}(\mathbf{L} \times \mathbf{p} - \mathbf{p} \times \mathbf{L}) + \frac{\mathbf{r}}{r}$ generate the Lie algebra $so(3, 1)$, which is part of the Lie algebra of $so(4, 2)$. Similar to the negative energy case, we give

the positive-energy realization of $so(4, 2)$ as follows:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \quad (46)$$

$$\mathbf{A} = \frac{\mathbf{pr} + r\mathbf{p}}{2} \sinh \zeta - \frac{1}{\sqrt{2H}} \frac{\mathbf{p}(\mathbf{r} \cdot \mathbf{p}) + (\mathbf{p} \cdot \mathbf{r})\mathbf{p}}{2} \cosh \zeta + \frac{1}{\sqrt{2H}} \frac{\mathbf{r}}{r} \cosh \zeta \quad (47)$$

$$\mathbf{B} = \frac{1}{\sqrt{2H}} [(\mathbf{L} \times \mathbf{p} - \mathbf{p} \times \mathbf{L})/2 - \frac{\mathbf{r}}{r}] \quad (48)$$

$$\mathbf{\Gamma} = \frac{\mathbf{pr} + r\mathbf{p}}{2} \cosh \zeta - \frac{1}{\sqrt{2H}} \frac{\mathbf{p}(\mathbf{r} \cdot \mathbf{p}) + (\mathbf{p} \cdot \mathbf{r})\mathbf{p}}{2} \sinh \zeta + \frac{1}{\sqrt{2H}} \frac{\mathbf{r}}{r} \sinh \zeta \quad (49)$$

$$S = \frac{1}{\sqrt{2H}} (2Hr + 1) \sinh \zeta - \frac{\mathbf{r} \cdot \mathbf{p} + \mathbf{p} \cdot \mathbf{r}}{2} \cosh \zeta \quad (50)$$

$$D = \frac{\mathbf{r} \cdot \mathbf{p} + \mathbf{p} \cdot \mathbf{r}}{2} \sinh \zeta - \frac{1}{\sqrt{2H}} (2Hr + 1) \cosh \zeta \quad (51)$$

$$C = \frac{1}{\sqrt{2H}}, \quad (52)$$

where $\zeta = \sqrt{2H}(\mathbf{r} \cdot \mathbf{p} + \mathbf{p} \cdot \mathbf{r})/2 - (2H)^{\frac{3}{2}}t$.

Further, we compute the commutation relations between the Hamiltonian H and these fifteen positive-energy generators of $so(4, 2)$ as follows:

$$[H, \mathbf{L}] = 0 \quad (53)$$

$$[H, \mathbf{A}] = -(2H)^{\frac{3}{2}}i\mathbf{\Gamma} \quad (54)$$

$$[H, \mathbf{B}] = 0 \quad (55)$$

$$[H, \mathbf{\Gamma}] = -(2H)^{\frac{3}{2}}i\mathbf{A} \quad (56)$$

$$[H, S] = (2H)^{\frac{3}{2}}iD \quad (57)$$

$$[H, D] = (2H)^{\frac{3}{2}}iS \quad (58)$$

$$[H, C] = 0. \quad (59)$$

It is easily verified that the realization of $so(4, 2)$ for positive energy satisfies the relationship (44), based on the relationships (53) — (59).

Remark 3.2 In fact, relation (44) is a special case of the following relation. Let $Q = i\partial_t - H$, where H is the Hamiltonian. Then the sufficient condition for operator Z to belong to a symmetry algebra is [29]

$$[Z, Q] = R_L(t, x)Q, \quad (60)$$

where $R_L(t, x)$ is a operator have consistent orders of ∂t and ∂x , and $x \in R^n$. When $R_L(t, x) = 0$, relation (60) is reduced to relation (44). In [29], Miller uses relation (60) to find the basis of the symmetry algebra, then diagonalize different operators in this symmetry algebra to separate variables in different coordinate systems, thereby finding solutions for differential equations.

Now let us study the matrix representation of \mathcal{L} . If $\{|n\rangle : n = 1, \dots, N\}$, where N can be infinity, is a basis for vector space \mathcal{V} , and if $P_i = T(h_i), i = 1, \dots, l$, where $h_i \in \mathcal{L}$, h_i is a basis for \mathcal{L} , and T is a homomorphism, then $P_i|n\rangle = \sum_m |m\rangle \langle m|P_i|n\rangle$, $m, n = 1, \dots, N$, where $\langle m|P_i|n\rangle$ denotes matrix element (m, n) of P_i . These matrices are a basis for a matrix representation of the Lie algebra \mathcal{L} . Now we already got the homomorphism of abstract $so(4, 2)$ —the realization of $so(4, 2)$, so we need to look for the representation states $\{|n\rangle : n = 1, \dots, N\}$. First, we decompose $so(4, 2)$ as follows

$$so(4, 2) \supset so(4) \otimes so(2, 1)$$

By expanding the representation states of $so(4)$ [24] and the representation states of $so(2, 1)$ [24], the representation states of $so(4, 2)$ for negative energy case are obtained as follows:

$$L_3|nlm\rangle = m|nlm\rangle \quad (61)$$

$$A_3|nlm\rangle = \alpha_m^l c_l^n |n, l-1, m\rangle + \alpha_m^{l+1} c_{l+1}^n |n, l+1, m\rangle \quad (62)$$

$$\begin{aligned} B_3|nlm\rangle &= \alpha_m^l u_l^n |n-1, l-1, m\rangle + \alpha_m^l v_l^n \\ &\cdot |n+1, l-1, m\rangle + \alpha_m^{l+1} v_{l+1}^{n-1} |n-1, l+1, m\rangle \\ &+ \alpha_m^{l+1} u_{l+1}^{n+1} |n+1, l+1, m\rangle \end{aligned} \quad (63)$$

$$\begin{aligned} \Gamma_3|nlm\rangle &= -i\alpha_m^l u_l^n |n-1, l-1, m\rangle + i\alpha_m^l v_l^n \\ &\cdot |n+1, l-1, m\rangle - i\alpha_m^{l+1} v_{l+1}^{n-1} |n-1, l+1, m\rangle \\ &+ i\alpha_m^{l+1} u_{l+1}^{n+1} |n+1, l+1, m\rangle \end{aligned} \quad (64)$$

$$D|nlm\rangle = n|nlm\rangle \quad (65)$$

where

$$\alpha_m^l = \sqrt{(l-m)(l+m)}, \quad u_l^n = \frac{1}{2} \sqrt{\frac{(n+l-1)(n+l)}{(2l-1)(2l+1)}},$$

$$c_l^n = \sqrt{\frac{(n-l)(n+l)}{(2l-1)(2l+1)}}, \quad v_l^n = \frac{1}{2} \sqrt{\frac{(n-l)(n-l+1)}{(2l-1)(2l+1)}},$$

and $m = -l, \dots, l-1, l, \quad l = 0, 1, \dots, n-1, \quad n = 1, 2, 3, \dots$

Remark 3.3 Vector operators \mathbf{L} and \mathbf{A} act on the angular portion of the basis kets $|nlm\rangle$ (affecting only the magnetic quantum number m and orbital angular momentum quantum number l), while operators S, C , and D act only on the remaining part. Further, vector operators \mathbf{B} and $\mathbf{\Gamma}$ act on both parts.

Construction of representation of $so(4, 2)$ for positive energy follows a procedure similar to that used above for the negative energy case, based on the following alternative decomposition

$$so(4, 2) \supset so(3, 1) \otimes so(2, 1).$$

For the radial part S, C, D of this realization we need to diagonalize C , which has a continuous spectrum, instead of diagonalizing D in the negative energy case.

$$C|vlm\rangle = v|vlm\rangle, \quad v \in (-\infty, \infty),$$

where the $|vlm\rangle$ denote the continuous eigenstates. Thus, using $|vlm\rangle$ to denote the continuous eigenstates, the representation of $so(4, 2)$ for positive energy is similar to that for negative energy with the following exceptions: replace quantum number n in the negative energy case with $iv, l = 0, 1, \dots, \infty$, and note that $v \in (-\infty, \infty)$ is continuous [19, 23].

4 A strongly and analytically controllable hydrogen atom system

Now let us consider the hydrogen atom control system

$$\begin{aligned} i \frac{\partial}{\partial t} \psi(t, x) &= [H + u_1(t)L_1 + u_2(t)L_2 + u_3(t)L_3 \\ &+ u_4(t)A_1 + u_5(t)A_2 + u_6(t)A_3 + u_7(t)B_1 \\ &+ u_8(t)B_2 + u_9(t)B_3 + u_{10}(t)\Gamma_1 + u_{11}(t)\Gamma_2 \\ &+ u_{12}(t)\Gamma_3 + u_{13}(t)S + u_{14}(t)C + u_{15}(t)D] \psi(t, x), \end{aligned} \quad (66)$$

where $H, L_i, A_i, B_i, \Gamma_i, S, C, D, i = 1, 2, 3$ are Hermitian operators defined as before, and $u_j(t), j = 1, 2, \dots, 15$ are piecewise constant control functions. Divide the operators in the right hand side of equation (66) by i , converting the control system (66) to the standard form

$$\begin{aligned} \frac{\partial}{\partial t} \psi(t, x) = & [H' + u_1(t)L'_1 + u_2(t)L'_2 + u_3(t)L'_3 \\ & + u_4(t)A'_1 + u_5(t)A'_2 + u_6(t)A'_3 + u_7(t)B'_1 \\ & + u_8(t)B'_2 + u_9(t)B'_3 + u_{10}(t)\Gamma'_1 + u_{11}(t)\Gamma'_2 \\ & + u_{12}(t)\Gamma'_3 + u_{13}(t)S' + u_{14}(t)C' + u_{15}(t)D'] \psi(t, x), \end{aligned} \quad (67)$$

where $H', L'_i, A'_i, B'_i, \Gamma'_i, S', C', D', i = 1, 2, 3$ are skew-Hermitian operators, and the $u_j(t)$ with $j = 1, 2, \dots, 15$, are piecewise constant functions. Corresponding to the commutators (4)–(28), we have

$$[\mathbf{L}', \mathbf{L}'] = \mathbf{L}' \quad (68)$$

$$[\mathbf{L}', \mathbf{A}'] = \mathbf{A}' \quad (69)$$

$$[\mathbf{L}', \mathbf{B}'] = \mathbf{B}' \quad (70)$$

$$[\mathbf{L}', S'] = 0 \quad (71)$$

$$[\mathbf{A}', \mathbf{A}'] = \mathbf{L}' \quad (72)$$

$$[S, \mathbf{A}'] = \mathbf{B}' \quad (73)$$

$$[\mathbf{A}', \mathbf{B}'] = S' \quad (74)$$

$$[\mathbf{B}', \mathbf{B}'] = -\mathbf{L}' \quad (75)$$

$$[S', \mathbf{B}'] = \mathbf{A}' \quad (76)$$

$$[C', \mathbf{A}'] = \Gamma' \quad (77)$$

$$[D', \mathbf{A}'] = 0 \quad (78)$$

$$[C', \mathbf{B}'] = 0 \quad (79)$$

$$[D', \mathbf{B}'] = -\Gamma' \quad (80)$$

$$[C', \mathbf{L}'] = 0 \quad (81)$$

$$[D', \mathbf{L}'] = 0 \quad (82)$$

$$[\mathbf{L}', \Gamma'] = \Gamma' \quad (83)$$

$$[C', S'] = -D' \quad (84)$$

$$[S', D'] = C' \quad (85)$$

$$[D', C'] = S' \quad (86)$$

$$[\Gamma', \mathbf{A}'] = -C' \quad (87)$$

$$[\Gamma', \mathbf{B}'] = -D' \quad (88)$$

$$[\Gamma', C'] = -\mathbf{A}' \quad (89)$$

$$[\Gamma', S'] = 0 \quad (90)$$

$$[\Gamma', D'] = -\mathbf{B}' \quad (91)$$

$$[\Gamma', \Gamma'] = -\mathbf{L}'. \quad (92)$$

From the commutators (68)–(92), we can see that $\mathbf{L}', \mathbf{A}', \mathbf{B}', \Gamma', S', C', D'$ form a real Lie algebra, which we denote as $\bar{\mathcal{B}}$. In addition, it has already been shown that both of the realizations of $so(4, 2)$ for positive and negative energy satisfy relation (44), so we can rewrite relation (44) as

$$\frac{\partial G'}{\partial t} - [H', G'] = 0, \quad \text{where } G' \in \bar{\mathcal{B}}. \quad (93)$$

Based on the strong analytic controllability theorem given in section 2, the following holds:

$$B_1 = \mathcal{L}\{-[H', G'] + \frac{\partial G'}{\partial t}, \quad G' \in \bar{\mathcal{B}}\} = 0,$$

so $\bar{\mathcal{C}} = \mathcal{L}\{\bar{\mathcal{B}}, B_1, \dots\} = \bar{\mathcal{B}}$ and $[\bar{\mathcal{B}}, \bar{\mathcal{C}}] \subset \bar{\mathcal{B}}$. In addition, the representation space of $so(4, 2)$ spanned by $|nlm\rangle$ and $|vlm\rangle$ for negative and positive energies respectively, is indeed an analytic domain, because this space is invariant both under the Lie algebra $so(4, 2)$ and under the Lie group $SO(4, 2)$. That is, exponentiation of any operator from $so(4, 2)$ acting on this representation space converges. Many examples can be found in [30]. Let M denote the closure of a manifold spanned by a finite set of eigenfunctions of $so(4, 2)$ as is the case in [10, 11, 31, 32]. It is not hard to see that $\dim \bar{\mathcal{C}}\psi(x, t) = \dim M$ [10, 11, 31, 32] for any (x, t) according to the representation of $so(4, 2)$ in the last section. Hence, by the Theorem 2.1, the control system (66) is strongly analytically controllable on M . In fact, the effect of the drift term $H\psi$ on the control system can be compensated by the effect of $\bar{\mathcal{B}}\psi$.

Remark 4.1 In control system (67), we do not need 15 generators to appear in the control system. We can selectively pick up at least five generators, for example, L'_1, L'_2, A'_3, S' , and C' , and then by the commutation relations (68)–(92) we can generate the other generators so as to make the system (67) strongly analytically controllable as 15 generators do. Because system (67) corresponds to system

(66), we immediately conclude that the system

$$i\frac{\partial}{\partial t}\psi(t, x) = [H + u_1(t)L_1 + u_2(t)L_2 + u_3(t)A_3 + u_4(t)S + u_5(t)C]\psi(t, x) \quad (94)$$

is strongly analytically controllable.

The detailed analysis presented in this section provides theoretical guidance for choosing and imposing controls and interacting Hamiltonians on an original unperturbed system described by the Schrödinger equation for the hydrogen atom, such that the constructed control system is strongly analytically controllable. In addition, we can see that the unperturbed Hamiltonian (29) and the interacting Hamiltonians from the realization of $so(4, 2)$ act on the same state space. That is, addition of interacting Hamiltonians does not affect the state space of the original hydrogen atom system. The controllability problem can still be analyzed on the original state space, but with imposing controls and the interacting Hamiltonians.

Note that the angular momentum and the Runge-Lenz vector commute with the hydrogen atom Hamiltonian, so they cannot cause a transition from one energy level quantum state to another. However, the rest of operators of the realization of $so(4, 2)$ can produce such transitions. Hence, at least one operator besides the angular momentum and the Runge-Lenz vector must be chosen as interacting Hamiltonians (as we did in control systems (66) and (94)) in order that the system is strongly analytically controllable among different quantum states of different energies.

5 Conclusion

The hydrogen atom is the simplest atom, and studying it can help us understanding the other atoms. It is known that $so(4, 2)$ is represented fully on the energy states of the hydrogen atom [24, 26] — discrete and continuous, and energy-degenerate and non-degenerate. Moreover, representation states of $so(4, 2)$ can be brought in one-to-one correspondence with the states of the hydrogen atom. In this paper, we first have given the realization and representation of the Lie algebra of $so(4, 2)$. Then Picking up

operators from the realization of $so(4, 2)$ as interacting Hamiltonians, and choosing piecewise constant functions as controls, we have constructed a hydrogen atom control system which is strongly analytically controllable on a manifold consisting of all representation states. This construction demonstrates a promising theoretical approach for building control systems on the basis of original systems described by the Schrödinger equations.

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