ON KUIPER'S CONJECTURE

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Abstract. We prove that any connected proper Dupin hypersurface in $\mathbf{R}^n$ is analytic algebraic and is an open subset of a connected component of an irreducible algebraic set. We prove the same result for any connected non-proper Dupin hypersurface in $\mathbf{R}^n$ that satisfies a certain finiteness condition. Hence any taut submanifold M in $\mathbf{R}^n$, whose tube M_ϵ satisfies this finiteness condition, is analytic algebraic and is a connected component of an irreducible algebraic set. In particular, we prove that every taut submanifold of dimension $m \leq 4$ is algebraic.

1. Introduction

An embedding f of a compact, connected manifold M into Euclidean space $\mathbf{R}^n$ is *taut* if there is a field $\mathbf{F}$ such that every nondegenerate (Morse) Euclidean distance function,

$$L_p : M \to \mathbf{R}, \quad L_p(x) = d(f(x), p)^2, \quad p \in \mathbf{R}^n,$$

has $\beta(M, \mathbf{F})$ critical points on M, where $\beta(M, \mathbf{F})$ is the sum of the $\mathbf{F}$-Betti numbers of M. That is, L_p is a perfect Morse function on M.

We can also consider taut embeddings into S^n, in which case we use spherical distance functions instead of Euclidean distance functions. Tautness is preserved by stereographic projection, and so the theories in the two ambient spaces are essentially the same, and we will use whichever is most appropriate for the particular question at hand.

Examples of taut submanifolds are metric spheres, cyclides of Dupin (Banchoff [1]), isoparametric hypersurfaces in spheres (Cecil-Ryan [5]) isoparametric submanifolds of higher codimension (Terng [25]), and compact proper Dupin hypersurfaces (Thorbergsson [27]).

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In a paper published in 1984, Kuiper [13] asked whether all taut submanifolds of $\mathbf{R}^n$ must be real algebraic. The affirmative answer to this question is now known as *Kuiper's Conjecture*. In the 1980's, Kuiper's Conjecture was widely thought to be true, but a proof was never published. In particular, in 1984 Ulrich Pinkall sent a sketch [17] of a proof that proper Dupin hypersurfaces are algebraic in a letter to Tom Cecil, and we wish to thank Professor Pinkall for allowing us to use his approach to the problem in our work in this paper.

Pinkall's sketch led us to the local parametrization (5.10) of a general proper Dupin hypersurface. We then used ideas from real algebraic geometry to show that a connected proper Dupin hypersurface is contained in a connected component of an irreducible algebraic subset of $\mathbf{R}^n$. There are still issues to be resolved to prove Kuiper's Conjecture, however, because arbitrary taut submanifolds are Dupin, but not necessarily proper Dupin.

In this paper, we prove Kuiper's Conjecture for manifolds of dimension $m \leq 4$, and we provide a criterion for algebraicity that may be useful in proving the conjecture in its entirety. We now discuss the problem in more detail.

Let M be an immersed hypersurface in $\mathbf{R}^n$ or the unit sphere S^n in $\mathbf{R}^{n+1}$. A *curvature surface* of M is a smooth connected submanifold S such that for each point $x \in S$, the tangent space T_xS is equal to a principal space of the shape operator A of M at x. The hypersurface M is said to be *Dupin* if it satisfies the condition

(a) along each curvature surface, the corresponding principal curvature is constant.

The hypersurface M is called *proper Dupin* if, in addition to condition (a), it also satisfies the condition

(b) the number g of distinct principal curvatures is constant on M.

Since both of these conditions are invariant under conformal transformations, stereographic projection allows us to consider our hypersurfaces in $\mathbf{R}^n$ or S^n, whichever is more convenient.

A primary result in this paper, contained in Theorem 31 and Corollary 32, is a proof that any connected proper Dupin hypersurface M in $\mathbf{R}^n$ is analytic algebraic and thus is an open subset of a connected component of an irreducible algebraic set in $\mathbf{R}^n$. If M is complete, then it equals the connected component.

Pinkall [19] extended the notions of Dupin and proper Dupin to submanifolds M of $\mathbf{R}^n$ of codimension greater than one. In that case, the principal curvatures are defined on the unit normal bundle $UN(M)$ of

M in $\mathbf{R}^n$. Using Pinkall's definition, one can show that a submanifold M of $\mathbf{R}^n$ of codimension greater than one is proper Dupin if and only if a tube W^{n-1} over M of sufficiently small radius is a proper Dupin hypersurface. Since algebraicity is also preserved by the tube construction, Theorem 31 also implies that a proper Dupin submanifold of arbitrary codimension in $\mathbf{R}^n$ is algebraic.

These results can also be extended to non-proper Dupin hypersurfaces that satisfy what we call the local finiteness property (see Definition 34), which, for example, is implied by the condition that any point not in the open dense subset $\mathcal{G}$ on which the multiplicities of the principal curvatures are locally constant has a neighborhood that intersects $\mathcal{G}$ in only finitely many components. To our knowledge, this local finiteness property is satisfied by all known examples of Dupin hypersurfaces.

Thorbergsson [27] proved that a compact proper Dupin hypersurface embedded in $\mathbf{R}^n$ is taut. Thus, our Theorem 31 shows that these taut hypersurfaces are algebraic. Conversely, every taut submanifold in $\mathbf{R}^n$ is Dupin (see Miyaoka [14] for hypersurfaces and Pinkall [19] in general), but not necessarily proper Dupin (see Pinkall [18]). In [19], Pinkall also showed that the tube M_ϵ of radius ϵ around an embedded compact submanifold M of $\mathbf{R}^n$ is $\mathbf{Z}_2$-taut if and only if M is $\mathbf{Z}_2$-taut. (The field $\mathbf{F} = \mathbf{Z}_2$ has proven to work well in the theory of taut submanifolds, and we will use it exclusively in this paper.)

In Theorem 37, we prove: If M is a compact taut submanifold of $\mathbf{R}^n$ such that the tube M_ϵ over M, which is Dupin, satisfies the local finiteness property, then M is an analytic submanifold and a connected component of an irreducible algebraic subset of $\mathbf{R}^n$. The proof of this theorem shows that the conclusion holds for any analytic taut submanifold M, without the need to assume the local finiteness property on M_ϵ.

In Section 7, we develop the theory of Alexander cohomology, in conjunction with the notion of ends, for noncompact manifolds needed in Section 8 to prove that any taut submanifold of dimension 3 or 4 satisfies the local finiteness property given in Theorem 37 and is therefore algebraic. A key step in this approach is to show that the complement $\mathcal{Z}$ of the focal set $\mathcal{F}$ of a taut hypersurface M in S^n is connected, and that the normal exponential map E from the normal bundle of M to S^n restricts to a finite covering map on $E^{-1}(\mathcal{Z})$. These results, together with the classifications of taut submanifolds of dimensions one and two due to Banchoff [1], show that Kuiper's Conjecture is true for taut submanifolds of dimension $m \leq 4$ (Theorem 65). This approach may have applications to higher dimensional taut submanifolds, although we have not been able to extend it further at this time.

The paper is organized as follows. Section 2 outlines the facts needed about algebraic and semi-algebraic subsets of $\mathbf{R}^n$ and explains that Nash functions are the same as analytic algebraic functions. Our principal algebraic tool is contained in Lemma 17, which states that if a connected analytic submanifold $M \subset \mathbf{R}^n$ contains a connected open subset that is a semi-algebraic subset of $\mathbf{R}^n$, then M is an open subset of a connected component of an irreducible algebraic set.

Section 3 outlines how to calculate the center and radius of a spherical curvature surface in terms of the mean curvature normal.

In Section 4, we present the facts needed about jet spaces and show that the curvature surface through a point for a principal curvature of constant multiplicity is determined by the 3-jet at the point of the embedding.

The proof of Theorem 31 showing that proper Dupin hypersurfaces are algebraic is contained in Section 5.

In Section 6, we define the local finiteness property and extend the results of Section 5 to any connected Dupin hypersurface possessing this finiteness property. We then prove Theorem 37 concerning taut submanifolds mentioned above.

Finally, in Section 7, we develop the theory of Alexander cohomology and ends for noncompact manifolds, and in Section 8, we prove that any taut submanifold of dimension $m \leq 4$ in $\mathbf{R}^n$ is algebraic.

2. Algebraic Preliminaries

In this section we briefly review the material we need in real algebraic geometry, referring for details to the book by Bochnak, et al. [3].

Let $\mathbf{R}[X_1, \ldots, X_n]$ denote the ring of all real polynomials in n variables. If B is any subset of this ring, we denote the *set of zeros* of B by

$$\mathcal{Z}(B) = \{x \in \mathbf{R}^n : \forall f \in B,\ f(x) = 0\}.$$

Definition 1. An *algebraic subset* of $\mathbf{R}^n$ is the set of zeros of some $B \subset \mathbf{R}[X_1, \ldots, X_n]$.

Given any subset S of $\mathbf{R}^n$, the set of all polynomials vanishing on S, denoted

$$\mathcal{I}(S) = \{f \in \mathbf{R}[X_1, \ldots, X_n] : \forall x \in S,\ f(x) = 0\},$$

is an ideal of $\mathbf{R}[X_1, \ldots, X_n]$. Ideals of the ring of real polynomials are finitely generated, which implies that for any algebraic set V of $\mathbf{R}^n$, there exists a polynomial $f \in \mathbf{R}[X_1, \dots, X_n]$ such that $V = \mathcal{Z}(f)$. In fact, if $f_1, \dots, f_m$ generate $\mathcal{I}(V)$, take

$$f = f_1^2 + \cdots + f_m^2.$$

Definition 2. A *semi-algebraic subset* of $\mathbf{R}^n$ is one which is a finite union of sets of the form

$$\cap_j \{x \in \mathbf{R}^n : F_j(x) * 0\},$$

where * is either $<$ or $=$, $F_j \in \mathbf{R}[X_1, \dots, X_n]$, and the intersection is finite. ($F_j < 0$ is just $-F_j > 0$. So there is no need to introduce $>$ in the definition.)

All algebraic sets are certainly semi-algebraic, and $\mathbf{R}^n$ itself is given by the null relation. Another simple example popular in calculus is

$$\{(x, y) : y - 1 + x^2 \leq 0\} \cap \{(x, y) : x^2 - y \leq 0\},$$

the area enclosed by the two parabolas.

It follows from the definition that the complement of a semi-algebraic set is semi-algebraic, and hence a semi-algebraic set take away another semi-algebraic set leaves a semi-algebraic set.

The following result is proved in [3, Theorem 2.21, p. 26].

Proposition 3. *The projection $\pi : \mathbf{R}^n \to \mathbf{R}^k$ sending $x \in \mathbf{R}^n$ to its first k coordinates maps a semi-algebraic set to a semi-algebraic set.*

In particular, even though a linear projection does not map an algebraic set to an algebraic set, in general, it does map it to a semi-algebraic set.

Corollary 4. *The (topological) closure and interior of a semi-algebraic set are semi-algebraic.*

A proof is given in [3, Proposition 2.2.2, p. 27].

Definition 5. A map $f : S \subset \mathbf{R}^n \to \mathbf{R}^k$ is *semi-algebraic* if its graph in $\mathbf{R}^n \times \mathbf{R}^k$ is a semi-algebraic set.

Corollary 6. *The image of a semi-algebraic map $f : S \subset \mathbf{R}^n \to \mathbf{R}^k$ is semi-algebraic, via the composition*

$$\mathrm{graph}(f) \subset \mathbf{R}^n \times \mathbf{R}^k \to \mathbf{R}^k,$$

where the last map is the projection onto the second summand.

Definition 7. A *Nash function* is a C^∞ semi-algebraic map from an open semi-algebraic subset of $\mathbf{R}^n$ to $\mathbf{R}$.

Definition 8. A real analytic function f defined on an open semi-algebraic subset U of $\mathbf{R}^n$ is *analytic algebraic* if it is a solution of a polynomial equation on U of the form,

$$a_0(x)f^s(x) + a_1(x)f^{s-1}(x) + \cdots + a_s(x) = 0, \tag{2.1}$$

where $a_0(x) \neq 0, a_1(x), \cdots, a_s(x)$ are polynomials over $\mathbf{R}^n$.

A significant result in the subject is that these two concepts are equivalent (see [3, Proposition 8.1.8, p. 165]).

Proposition 9. *A function is Nash if and only if it is analytic algebraic.*

The following example will be useful to us later in the paper.

Example 10. For any number ϵ satisfying $0 < \epsilon < 1$, the open ball

$$B^n(\epsilon) = \{s = (s^1, \ldots, s^n) \in \mathbf{R}^n : |s| < \epsilon\} \tag{2.2}$$

is an open semi-algebraic subset of $\mathbf{R}^n$. The function

$$s^0 = \sqrt{1 - |s|^2} \tag{2.3}$$

on $B^n(\epsilon)$ is analytic algebraic, since $(s^0(x))^2 + a_0(x) = 0$ on $B^n(\epsilon)$, where $a_0(x)$ is the polynomial $|s|^2 - 1$ on $\mathbf{R}^n$. Partial derivatives of all orders of s^0 are analytic algebraic. In fact, an elementary calculation and induction argument shows that if D_i denotes the partial derivative with respect to s^i, then

$$D_{i_1 \ldots i_k} s^0 = \frac{a_k(s)}{(s^0)^m}$$

where $a_k(s)$ is a polynomial on $\mathbf{R}^n$ and m is a positive integer. Therefore,

$$(s^0)^{2m}(D_{i_1 \ldots i_k} s^0)^2 - a_k(s)^2 = 0$$

is an equation of the form (2.1), since $(s^0)^2$ is a polynomial on $\mathbf{R}^n$.

A slight generalization of the single-variable case in [3, Proposition 2.9.1, p. 54], shows that the partial derivatives of any Nash function are again Nash functions.

Proposition 11. *Let f be a Nash function defined on a connected semi-algebraic open set U in $\mathbf{R}^n$. Then a partial derivative of f of any order with respect to $x = (x^1, \cdots, x^n) \in \mathbf{R}^n$ is again a Nash function on U.*

Definition 12. Let S be a semi-algebraic subset of $\mathbf{R}^n$. The *dimension* of S, denoted $\dim S$, is the dimension of the ring $R = \mathbf{R}[x^1, \cdots, x^n]/\mathcal{I}(S)$, which is the maximal length of chains of prime ideals of R.

In [3, Proposition 2.8.14, p. 54] it is proved that

Proposition 13. *If S is a semi-algebraic subset of $\mathbf{R}^n$ that is a C^∞ submanifold of $\mathbf{R}^n$ of dimension d, then $dim\, S = d$.*

We denote the topological closure of S by $\overline{S}^{\mathrm{top}}$. Denote the Zariski closure of S by $\overline{S}^{\mathrm{zar}}$. It is the smallest algebraic set containing S, so

$$\overline{S}^{\mathrm{zar}} = \mathcal{Z}(\mathcal{I}(S)).$$

The following plays a central role in our arguments to follow (see [3, Proposition 2.8.2, p. 50]).

Proposition 14. *If S is a semi-algebraic subset of $\mathbf{R}^n$, then*

$$dim\, S = dim\,(\overline{S}^{top}) = dim\,(\overline{S}^{zar}).$$

Definition 15. An algebraic set V in $\mathbf{R}^n$ is *irreducible* if, whenever $V = F_1 \cup F_2$, where F_1 and F_2 are algebraic sets, then $V = F_1$ or $V = F_2$.

The following facts are in [3, Theorem 2.8.3, p. 50]: Every algebraic set V is the union, in a unique way, of a finite number of irreducible algebraic sets $V_1, \dots, V_p$, such that V_i is not a subset of $\cup_{j \neq i} V_j$ for $i = 1, \dots, p$. The V_i are the *irreducible components* of V and

$$\dim V = \max\,(\dim V_1, \dots, \dim V_p).$$

An algebraic set $V \subset \mathbf{R}^n$ is irreducible if and only if the ideal $\mathcal{I}(V)$ is prime.

Remark 16. Let X be an irreducible real algebraic set in $\mathbf{R}^n$. By [9], X can be successively blown up to reach its desingularization X^*. Let $\pi : X^* \to X$ be the projection. Then the proper transform of X in X^*, that is, the Zariski closure of the preimage of X with its singular set removed, is a smooth algebraic set with disjoint smooth connected (algebraic) components $X_1^*, \dots, X_s^*$, whose projections in X are called the *connected components* of X. We will show these connected components are semi-algebraic as follows.

$X_1^*, \dots, X_s^*$ are exactly the irreducible components of the proper transform of X under π. The morphism π is algebraic. Hence it maps the algebraic sets X_i^* to the connected components of X, which are therefore semi-algebraic. Note that the proper transform itself is not affine, but we can always cover it by a finite number of affine charts and then proceed with our arguments in each chart and take a finite union.

That the connected components of X are closed follows from the fact that π is a proper map.

A real irreducible algebraic set may have several connected components of varying dimensions. A good example is a degenerate torus given by

$$y^2 = x^2(x-1).$$

The solution set is irreducible since the polynomial is. This irreducible algebraic set consists of two connected components. One is the curve that opens toward positive infinity, with vertex at $(1,0)$. The other is $\{(0,0)\}$.

The name connected component is standard in real algebraic geometry, but we should realize that these are not always the connected components in the topological sense. For example, the irreducible algebraic set known as the Cartan umbrella (see [3, pp. 60-61]).

$$z(x^2+y^2) - x^3 = 0$$

is topologically connected, but has two connected components: the surface $z = x^3/(x^2+y^2)$ (including the origin) and the z-axis.

If an irreducible algebraic set X in $\mathbf{R}^n$ contains a connected analytic hypersurface M in $\mathbf{R}^n$, then M must be an open subset of one of the connected components of X. If M is closed, then it must be the whole connected component.

We can now prove our principal algebraic tool.

Lemma 17. *Let $M \subset \mathbf{R}^n$ be a connected analytic submanifold and let $U \subset M$ be a connected open subset of M. If U is a semi-algebraic subset of $\mathbf{R}^n$, then M is an open subset of a connected component of the irreducible algebraic set $\overline{U}^{zar}$.*

Proof. The Zariski closure $\overline{U}^{\mathrm{zar}}$ of U is irreducible, because $\mathcal{I}(\overline{U}^{\mathrm{zar}}) = \mathcal{I}(U)$ is a prime ideal. In fact, suppose f and g are polynomials such that $fg \in \mathcal{I}(U)$. If $f \notin \mathcal{I}(U)$, then there exists a point $p \in U$ such that $f(p) \neq 0$. By continuity, there exists an open subset O of U, containing p, on which f is never zero. But fg is identically zero on O, so g must be identically zero on O. Since O is an open subset of U, and since g is an analytic function on U, it follows that g is identically zero on the connected set U, that is, $g \in \mathcal{I}(U)$.

By the same sort of argument, if $f \in \mathcal{I}(U)$, then f is an analytic function on M, identically zero on the open subset U, so must be identically zero on M. Hence $M \subset \mathcal{Z}(\mathcal{I}(U)) = \overline{U}^{\mathrm{zar}}$. □

Definition 18. A semi-algebraic subset M of $\mathbf{R}^m$ is a *Nash submanifold* of $\mathbf{R}^m$ of dimension n if for every point p of M, there exists a Nash diffeomorphism ψ from an open semi-algebraic neighborhood U of the origin in $\mathbf{R}^m$ into an open semi-algebraic neighborhood V of p in $\mathbf{R}^m$

such that $\psi(0) = p$ and $\psi((\mathbf{R}^n \times \{0\}) \cap U) = M \cap V$. Here, by a *Nash diffeomorphism* ψ we mean the coordinate functions of ψ and ψ^{-1} are Nash functions.

Definition 19. Let M be a Nash submanifold of $\mathbf{R}^m$. A mapping $f : M \to \mathbf{R}$ is a *Nash mapping* if it is semi-algebraic, and for every ψ in the preceding definition, $f \circ \psi|_{\mathbf{R}^n \cap U}$ is a Nash function.

Semi-algebraic subsets of $\mathbf{R}^n$ are, in a sense, piecewise algebraically analytic. This is made precise in the following result (see [3, Proposition 2.9.0, p. 57]).

Proposition 20. *Let S be a semi-algebraic subset of $\mathbf{R}^n$. Then S is the disjoint union of a finite number of Nash submanifolds M_i of $\mathbf{R}^n$, each Nash diffeomorphic to an open cube $(0,1)^{dim M_i}$.*

The following lemma is a kind of converse to Lemma 17, in that it implies that if a C^∞ submanifold U^d of $\mathbf{R}^n$ is contained in an irreducible algebraic subset of dimension d in $\mathbf{R}^n$, then U is a real analytic submanifold.

Lemma 21. *Let X be a closed semi-algebraic subset of $\mathbf{R}^n$ of dimension d. If there is an open set U of X such that U is a C^∞ submanifold of dimension d in $\mathbf{R}^n$, then around each point p of U there is an open neighborhood $B \subset U$ that is a semi-algebraic subset of $\mathbf{R}^n$.*

Proof. By Proposition 20, X is the finite disjoint union of Nash submanifolds, each Nash diffeomorphic to an open cube of some dimension. Let $M_1, \ldots, M_k$ be those Nash submanifolds of dimension d that intersect U. There are such Nash submanifolds, for otherwise, U would be contained in a finite union of Nash submanifolds of smaller dimensions. Let Y be the topological closure of the union of $M_1, \ldots, M_k$. Then Y is semi-algebraic and $Y \subset X$.

U is contained in Y, for otherwise, there would be a point $z \in U$ that is not in Y, so that z stays some distance away from Y. Therefore, there would be a small open ball D around z in U disjoint from Y, which would imply that D is contained in Nash manifolds of smaller dimensions, because Y contains all the Nash submanifolds of dimension d not disjoint from U. This is absurd.

The part Y^* where Y is not in U is closed. Since $p \in U$, there is a nonzero shortest distance δ from p to Y^*. Choose a Euclidean ball B of radius $\delta' < \delta$ centered at p such that $B \cap U$ is a submanifold of $\mathbf{R}^n$ diffeomorphic to a cube of dimension d, which is possible since U is a submanifold of $\mathbf{R}^n$.

We claim that $B \cap U = B \cap Y$. It is clear that $B \cap U \subset B \cap Y$ since $U \subset Y$. Suppose $w \in B \cap Y$. Then w must be in U, because otherwise

$w \in Y^*$, so that the distance from p to w would be $> \delta'$ and so w would not be in B, a contradiction. Hence, $w \in B \cap U$, proving the claim.

Now that $B \cap U = B \cap Y$, we see that $B \cap U$ on the one hand is diffeomorphic to an open cube of dimension d contained in U, and on the other hand it is semi-algebraic since so are B and Y. □

3. Mean curvature normal

Definition 22. Following Cecil-Ryan [6, p. 140], we say that a submanifold $\mathbf{x} : V^m \to \mathbf{R}^n$, for any $m < n$, is *umbilic* if there exists a linear map $\omega_{(x)} : T_x^\perp V \to \mathbf{R}$, for every $x \in V$, such that for every $Z \in T_x^\perp V$, the shape operator

$$B_Z : T_xV \to T_xV \quad \text{is} \quad B_Z = \omega(Z)I. \tag{3.1}$$

Let $\mathbf{x}, e_i, e_\alpha$ be a *first order* frame field on an open set $U \subset V$ along $\mathbf{x}$, which means that the vector fields e_i, for $i = 1, \ldots, m$, are tangent to $\mathbf{x}(V)$ and the e_α, for $\alpha = m+1, \ldots, n$, are normal to $\mathbf{x}(V)$ at each point of U. Then

$$d\mathbf{x} = \sum_i \theta^i e_i, \quad de_i = \sum_j \omega_i^j e_j + \sum_\alpha \omega_i^\alpha e_\alpha, \quad de_\alpha = \sum_j \omega_\alpha^j e_j + \sum_\beta \omega_\alpha^\beta e_\beta \tag{3.2}$$

where the 1-forms $\omega_b^a = -\omega_a^b$, for $a, b = 1, \ldots, n$, and

$$\omega_i^\alpha = \sum_{i,j} h_{ij}^\alpha \theta^j \tag{3.3}$$

for functions $h_{ij}^\alpha = h_{ji}^\alpha$. Then

$$B_{e_\alpha} = \sum_i \omega_i^\alpha e_i = \sum_{i,j} h_{ij}^\alpha \theta^j e_i. \tag{3.4}$$

A normal vector is given by $Z = \sum_\alpha z^\alpha e_\alpha$, so

$$B_Z = \sum_\alpha z^\alpha B_{e_\alpha} = \sum_{i,j,\alpha} z^\alpha h_{ij}^\alpha \theta^j e_i.$$

Definition 23. The *mean curvature normal* vector field of $\mathbf{x}$ is the normal vector field on V

$$H = \sum_\alpha (\frac{1}{m} \operatorname{trace} B_{e_\alpha}) e_\alpha = \frac{1}{m} \sum_\alpha \left(\sum_i h_{ii}^\alpha \right) e_\alpha,$$

which is independent of the choice of orthonormal frame e_α in the normal bundle.

We see that $\mathbf{x}$ is umbilic if and only if $B_{e_\alpha} = h^\alpha I$, for some function h^α, for all α; that is,

$$\omega_i^\alpha = h^\alpha \theta^i,$$

for all i, α. When $\mathbf{x}$ is umbilic, then the mean curvature normal is

$$H = \sum_\alpha h^\alpha e_\alpha.$$

Let $\mathbf{x} : M^{n-1} \to \mathbf{R}^n$ be a connected immersed hypersurface. Let ξ be a field of unit normals to $\mathbf{x}(M)$ defined on an open subset U of M, and let A denote the shape operator corresponding to ξ. If the principal curvatures are ordered

$$\kappa_1(p) \leq \cdots \leq \kappa_{n-1}(p)$$

for each $p \in U$, then the κ_i are continuous functions on U (see Ryan [21, Lemma 2.1, p.271]). If a continuous principal curvature function κ has constant multiplicity m on U, then κ is smooth on U, as is its m-dimensional principal distribution T_κ of eigenvectors of A corresponding to the eigenvalue κ (see Nomizu [15] or Singley [23]). Furthermore, T_κ is integrable and we will refer to T_κ as the *principal foliation* corresponding to the principal curvature κ.

Proposition 24. *Suppose $\mathbf{x} : M^{n-1} \to \mathbf{R}^n$ is a connected immersed hypersurface with a principal curvature function κ of constant multiplicity m. If $m = 1$, suppose further that κ is constant on any leaf V^1 of the principal foliation T_κ of M. For any leaf V^m of T_κ,*

i). If $\kappa = 0$ and $d\kappa = 0$ at each point of V, then $\mathbf{x}(V)$ is an open subset of an m-plane in $\mathbf{R}^n$.

ii). If κ or $d\kappa$ is nonzero at some point of V, then $\mathbf{x}(V)$ is an open subset of an m-sphere in $\mathbf{R}^n$ of radius $1/|H|$ and center at

$$\mathbf{x}(p_0) + H(p_0)/|H|^2, \tag{3.5}$$

where p_0 is any point in V and H is the mean curvature normal of $\mathbf{x}(V^m) \subset \mathbf{R}^n$.

Proof. See [6, Theorem 4.5, pp. 140-141 and Theorem 4.8, pp. 145-147]. □

We shall need an explicit formula for the mean curvature vector H of V in Proposition 24. Let $\mathbf{x}, e_i, e_\alpha, e_n$ be a first order frame field along $\mathbf{x}$, for which e_n is a unit normal vector field, each vector field $e_i \in T_\kappa$, where $i = 1, \ldots, m$, and each vector field $e_\alpha \in T_\kappa^\perp \subset T_\mathbf{x}M$, where $\alpha = m+1, \ldots, n-1$. Then $d\mathbf{x} = \sum_i \theta^i e_i + \sum_\alpha \theta^\alpha e_\alpha$ and $de_n = \sum_{a=1}^{n-1} \omega_n^a e_a$ and the shape operator $B_{e_n} : T_xM \to T_xM$ satisfies

$B_{e_n} e_i = \kappa e_i$, for all i, and $B_{e_n} e_\alpha \in \operatorname{span}\{e_\beta\}$, for all α, since B_{e_n} is self-adjoint. But

$$B_{e_n} = \sum_{a=1}^{n-1} \omega_a^n e_a = \sum_i \omega_i^n e_i + \sum_\alpha \omega_\alpha^n e_\alpha, \tag{3.6}$$

where $\omega_a^n = \sum_{b=1}^{n-1} h_{ab}\theta^b$. Therefore,

$$\omega_i^n = \kappa\theta^i, \quad \omega_\alpha^n = \sum_\beta h_{\alpha\beta}\theta^\beta, \tag{3.7}$$

for all i, β; that is, $h_{i\alpha} = h_{\alpha i} = 0$, for all i, α. In addition, the $(n-m-1) \times (n-m-1)$ symmetric matrix of functions

$$c = (c_{\alpha\beta}) = (h_{\alpha\beta} - \kappa\delta_{\alpha\beta}) \tag{3.8}$$

is non-singular at every point of V, so has inverse

$$c^{-1} = (c^{\alpha\beta}). \tag{3.9}$$

Set

$$d\kappa = \sum_i \kappa_i\theta^i + \sum_\alpha \kappa_\alpha\theta^\alpha. \tag{3.10}$$

Under the hypothesis of Proposition 24, the $\kappa_i = 0$. The κ_α are the derivatives of κ normal to the leaves of T_κ. At any point of a leaf V^m of T_κ, the mean curvature normal of V at the point is

$$H = \kappa e_n - \sum_{\alpha,\beta} c^{\alpha\beta}\kappa_\beta\, e_\alpha. \tag{3.11}$$

4. Jets

Definition 25. Smooth maps $f, g : M^m \to N^n$ have the same k-jet at $p \in M$ means $f(p) = g(p) = q$, and for any smooth curve $\gamma : \mathbf{R} \to M$ such that $\gamma(0) = p$ and any smooth function $F : N \to \mathbf{R}$ such that $F(q) = 0$, the derivatives

$$(F \circ f \circ \gamma)^{(j)}(0) = (F \circ g \circ \gamma)^{(j)}(0) \tag{4.1}$$

for all $j \leq k$. This defines an equivalence relation on the set of smooth maps $C^\infty(M, N)$. If $f \in C^\infty(M, N)$, denote its k-jet at $p \in M$ by $j_p^k(f)$.

Proposition 26. *If $f, g \in C^\infty(M, N)$, then $j_p^k(f) = j_p^k(g)$ if and only if for any local coordinate charts $U, x = (x^1, \ldots, x^m)$ about p in M and $V, y = (y^1, \ldots, y^n)$ about $f(p) = g(p) = q$ in N, all partial derivatives*

$$D_{i_1 \ldots i_j} f^\alpha(p) = D_{i_1 \ldots i_j} g^\alpha(p), \quad \text{for all } j \leq k,\ \alpha = 1, \ldots, n, \tag{4.2}$$

where $f^\alpha = y^\alpha \circ f$ and $g^\alpha = y^\alpha \circ g$.

Proof. See [4, pp. 20-22]. □

If $(p,q) \in M \times N$, let

$$J^k_{p,q}(M,N) = \{j^k_p(f) : f \in C^\infty(M,N),\ f(p) = q\}, \tag{4.3}$$

and let

$$J^k(M,N) = \cup_{(p,q)\in M\times N} J^k_{p,q}(M,N) \tag{4.4}$$

for each whole number k. This space has a natural C^∞ structure with C^∞ projections

$$\begin{array}{rlrl} J^k(M,N) & \to M, & J^k(M,N) & \to N \\ j^k_p(f) & \mapsto p & j^k_p(f) & \mapsto f(p). \end{array} \tag{4.5}$$

A map $f \in C^\infty(M,N)$ defines a C^∞ section

$$\begin{array}{rl} j^k(f) : M & \to J^k(M,N) \\ p & \mapsto j^k_p(f). \end{array} \tag{4.6}$$

Lemma 27. *For any $f \in C^\infty(M,N)$, and $p \in M$, the jet $j^{k+l}_p(f) \in J^{k+l}(M,N)$ can be identified with $j^k_p(j^l(f)) \in J^k(M, J^l(M,N))$.*

Proof. Local coordinates in M and N determine local coordinates in $J^l(M,N)$ (see [4]), and these give the local expression

$$j^l(f) = (p, f, D_{i_1\dots i_j} f), \quad j \le l. \tag{4.7}$$

Then $j^k_p(j^l(f))$ is given by the first k partial derivatives of these component functions at p, which are, ignoring redundancies, the first $k+l$ partial derivatives of f at p. □

It seems justified to abuse notation to express this lemma as

$$j^{k+l}_p(f) = j^k_p(j^l(f)), \tag{4.8}$$

for any $f \in C^\infty(M,N)$, and $p \in M$.

Remark 28. Let $\mathbf{y} : V^m \to \mathbf{R}^n$ be an immersed submanifold. Any first order frame field

$$\mathbf{y}, e_i, e_\alpha \tag{4.9}$$

along $\mathbf{y}$, where e_i are tangent, $i = 1,\dots,m$, and e_α are normal, $\alpha = m+1,\dots n$, is determined by the 1-jet $j^1_p(\mathbf{y})$ at each point $p \in V$. Now

$$d\mathbf{y} = \sum_i \theta^i e_i, \quad de_a = \sum_{b=1}^n \omega^b_a e_b,\ a = 1,\dots,n, \tag{4.10}$$

so the 1-forms θ^i are determined by the 1-jet of $\mathbf{y}$ at each point and the 1-forms ω^b_a are determined by the 2-jet of $\mathbf{y}$ at each point, since each e_a is determined by the 1-jet of $\mathbf{y}$. Then

$$\omega^\alpha_i = \sum_j h^\alpha_{ij}\theta^j, \tag{4.11}$$

where the functions $h^\alpha_{ij} = h^\alpha_{ji}$ are determined by the 2-jet of $\mathbf{y}$ at each point. Hence, the shape operators (3.4) are determined by the 2-jet of $\mathbf{y}$ at each point, and thus the shape operator B_Z is determined by the 2-jet of $\mathbf{y}$, for any normal vector $Z = \sum_\alpha z^\alpha e_\alpha$.

In terms of our first order frame field (4.9) on $U \subset V$, the characteristic polynomial of the shape operator B_{e_α} of (3.4) is

$$F^\alpha(p,z) = \det(h^\alpha(p) - zI), \tag{4.12}$$

for $p \in U$, $z \in \mathbf{C}$, where $h^\alpha(p) = (h^\alpha_{ij}(p))$ is the $m \times m$ symmetric matrix defined in (4.11). Thus, $F(p,z)$ is a polynomial function of $h^\alpha_{ij}(p)$ and z, and is determined by the 2-jet of y. The eigenvalues of B_{e_α} are necessarily real, so can be arranged as

$$\kappa^\alpha_1(p) \leq \cdots \leq \kappa^\alpha_m(p), \tag{4.13}$$

for each $p \in U$. These are the roots of $F^\alpha(p,z)$. They are continuous functions on U (see Ryan [21, Lemma 2.1, p. 271]). A slight variation on Ryan's argument shows that if κ^α_i has constant multiplicity m_i on U, for some i, then it is a rational function of the entries of h^α on U. In particular, κ^α_i is determined by the 2-jet of $\mathbf{y}$ at each point, so its derivative $d\kappa^\alpha_i$ is determined by the 3-jet of y.

Suppose now that $\mathbf{x} : M^{n-1} \to \mathbf{R}^n$ is a hypersurface, so that we can drop the superscript α in the above notation. Suppose that κ is a principal curvature of constant multiplicity m on $U \subset M$. Then the functions $c_{\alpha\beta}$ in (3.8) and $c^{\alpha\beta}$ of (3.9) depend rationally on the 2-jet of $\mathbf{x}$ at each point and the normal derivatives $d\kappa(e_\alpha) = \kappa_\alpha$ depend rationally on the 3-jet of $\mathbf{x}$ at each point. Therefore, the mean curvature vector H in (3.11) depends rationally on the 3-jet of $\mathbf{x}$ at each point.

5. Proper Dupin hypersurfaces are algebraic

We are ready to prove that a proper Dupin hypersurface $f : M^{n-1} \to \mathbf{R}^n$ is analytic algebraic in the sense that its differentiable structure has an analytic subatlas with respect to which the immersion into $\mathbf{R}^n$ is analytic algebraic. It follows that each point of M is contained in an open neighborhood of M that is a semi-algebraic subset of $\mathbf{R}^n$. From the analyticity and the semi-algebraic neighborhood we can conclude that if M is connected, then it is an open subset of an irreducible algebraic set in $\mathbf{R}^n$ of dimension $n-1$.

Suppose f has g principal curvatures $\kappa_1, \ldots, \kappa_g$, with multiplicities $m_1, \ldots, m_g$. Let $\{1\}, \ldots, \{g\}$ be a partition of the set $\{1, \ldots, n-1\}$ into disjoint subsets for which the cardinality of $\{i\}$ is the multiplicity m_i, for $i = 1, \ldots, g$. By Proposition 24, for any point $p \in M$, the i^{th} curvature surface through p is either an open subset of a sphere of dimension m_i in $\mathbf{R}^n$ or an open subset of a plane of dimension m_i. Inversion in a sphere whose center does not lie on the union of all these spheres and planes will transform all curvature surfaces of $f(M)$ to open subsets of spheres. Thus, we may assume that all curvature surfaces of f are open subsets of spheres of the appropriate dimensions.

Denote the center of the sphere of dimension m_i containing the i^{th} curvature surface through $f(p)$ by

$$c_i(p) \in \mathbf{R}^n, \tag{5.1}$$

and its radius by

$$r_i(p) = |f(p) - c_i(p)|. \tag{5.2}$$

By Proposition 24 and Remark 28, $c_i(p)$ is determined by $j^3_p(f)$. If $e_a(p)$, for $a \in \{i\}$, form an orthonormal basis of principal vectors for the i^{th} principal curvature, then they are determined by $j^2_p(f)$.

Lemma 29. *For any point $p \in M$, there is an analytic algebraic parametrization of an open subset about $f(p)$ of the i^{th} curvature surface through $f(p)$ given by*

$$\mathbf{x}(s_i) = c_i(p) + s_i^0(s_i)(f(p) - c_i(p)) + r_i(p) \sum_{a \in \{i\}} s_i^a e_a(p), \tag{5.3}$$

where

$$s_i^0(s_i) = \sqrt{1 - |s_i|^2}, \tag{5.4}$$

for all s_i in the open ball $B^{m_i}(\epsilon_i) \subset \mathbf{R}^{m_i}$, for some number ϵ_i satisfying $0 < \epsilon_i < 1$. The components of s_i are denoted s_i^a, for each $a \in \{i\}$, so

$$|s_i|^2 = 1 - \sum_{a \in \{i\}} (s_i^a)^2.$$

The vectors $c_i(p)$, $f(p)$, and $e_a(p)$, for all $a \in \{i\}$, are determined by $j^3_p(f)$.

The proof is contained in Example 10, which shows that the map $\mathbf{x}(s_i)$ in (5.3) is algebraic analytic. This lemma is essentially showing that a sphere of any radius, dimension, and center in $\mathbf{R}^n$ is a Nash submanifold (see [3, Definition 2.9.9, p. 57]).

Since the map $\mathbf{x}(s_i)$ in (5.3) is an embedding, there exists an embedding $x_i : B^{m_i}(\epsilon_i) \to M$ such that $f(x_i(s_i)) = \mathbf{x}(s_i)$, for all $s_i \in B^{m_i}(\epsilon_i)$.

Lemma 30. *For any $s_i \in B^{m_i}(\epsilon_i)$ in the parametrization (5.3), $j^k_{x_i(s_i)}(f)$ depends analytically algebraically on s_i and on $j^{k+3}_p(f)$, for any $k \geq 0$.*

Proof. For some $\delta > 0$, there exists $y : B^{n-1-m_i}(\delta) \to Y \subset M$, an embedded submanifold through $y(0) = p$ transverse to the i^{th} curvature surface through each of its points. Let $c_i(y)$ be the center of the i^{th} curvature surface through $f(y)$. Let e_j, $j = 1, \ldots, n-1$, be a smooth orthonormal frame field of principal vectors on a neighborhood of p. For each $y \in Y$,

$$f(x_i(s_i), y) = c_i(y) + s_i^0(s_i)(f(y) - c_i(y)) + r_i(y) \sum_{a \in \{i\}} s_i^a e_a(y) \tag{5.5}$$

is a parametrization of a neighborhood of the i^{th} curvature surface through $f(y)$. Thus, for ϵ_i and δ sufficiently small, (x_i, y) is a parametrization of a neighborhood of p in M.

From (5.5) we see that the partial derivatives of f with respect to the s_i variables at (s_i, y) depend on $j^3_{(0,y)}(f)$ and analytically algebraically on s_i, since all partial derivatives of $s_i^0(s_i)$ are analytic algebraic (see Example 10). Again from (5.5) we see that the partial derivatives of f with respect to the y variables at (s_i, y) depend analytically algebraically on s_i and on $j^1_{(0,y)}(c_i)$, $j^1_{(0,y)}(f)$, $j^1_{(0,y)}(r_i)$, and $j^1_{(0,y)}(e_a)$, for all $a \in \{i\}$. Since $c_i(0, y)$, $f(0, y)$, $r_i(0, y)$, and the $e_a(0, y)$ are determined by $j^3_{(0,y)}(f)$, it follows that their 1-jets are determined by

$$j^1_{(0,y)}(j^3(f)) = j^4_{(0,y)}(f). \tag{5.6}$$

Taking higher derivatives of (5.5) in this way, we see that $j^k_{(x_i(s_i),p)}(f)$ depends analytically algebraically on s_i and on

$$j^k_p(j^3(f)) = j^{k+3}_p(f). \tag{5.7}$$

$\square$

Theorem 31. *For any point $p \in M$, there exists an analytic algebraic parametrization of a neighborhood of $f(p)$ in $f(M) \subset \mathbf{R}^n$, where analytic algebraic means that the component functions are analytic algebraic. This collection of analytic algebraic parametrizations defines an analytic structure on M with respect to which f is analytic. In addition, it shows that any point $p \in M$ has an open neighborhood $U \subset M$ such that $f(U)$ is a semi-algebraic subset of $\mathbf{R}^n$.*

Proof. The idea is to iterate parametrizations along curvature surfaces, starting with the first. Choose a point $p \in M$. Setting $i = 1$ in equation (5.3) gives an analytic algebraic parametrization of a neighborhood of $f(p)$ of the first curvature surface through p. To simplify the following notation, we will write $f(s_1)$ in place of $f(x_1(s_1))$, and likewise $c_2(s_1))$, $r_2(s_1))$, and $e_b(s_1)$ without showing explicitly the composition with the map $x_1(s_1)$, etc.

For each s_1, parametrize a neighborhood of the second curvature surface through $f(s_1)$ by

$$f(s_1, s_2) = c_2(s_1) + s_2^0(s_2)(f(s_1) - c_2(s_1)) + r_2(s_1) \sum_{b\in\{2\}} s_2^b e_b(s_1) \tag{5.8}$$

where $c_2(s_1)$, $f(s_1)$, and $e_b(s_1)$, $b \in \{2\}$, are determined by $j^3_{s_1}(f)$, which in turn depends analytically algebraically on s_1 and $j^3_p(j^3(f)) = j^6_p(f)$, by Lemma 30.

If $g > 2$, then one more step should make the iteration clear. For each $s_1, s_2 \in B_1(\epsilon_1) \times B_2(\epsilon_2)$, parametrize a neighborhood of the third curvature surface through $f(s_1, s_2)$ by

$$\begin{aligned} f(s_1, s_2, s_3) &= c_3(s_1, s_2) + s_3^0(s_3)(f(s_1, s_2) - c_3(s_1, s_2)) \\ &+r_3(s_1, s_2) \textstyle\sum_{c\in\{3\}} s_3^c e_c(s_1, s_2). \end{aligned} \tag{5.9}$$

Now $c_3(s_1, s_2)$, $f(s_1, s_2)$, and $e_c(s_1, s_2)$ are determined by $j^3_{(s_1,s_2)}(f)$, which in turn depends analytically algebraically on s_2 and

$$j^3_{(s_1,0)}(j^3(f)) = j^6_{(s_1,0)}(f),$$

which in turn depends analytically algebraically on s_1 and

$$j^3_p(j^6(f)) = j^9_p(f).$$

Continuing in this way, we parametrize a neighborhood of the g^{th} curvature surface through $f(s_1, \ldots, s_{g-1})$ by

$$\begin{aligned} f(s_1, \ldots, s_g) &= c_g(s_1, \ldots, s_{g-1}) \\ &+s_g^0(s_g)(f(s_1, \ldots, s_{g-1}) - c_g(s_1, \ldots, s_{g-1})) \\ &+r_g(s_1, \ldots, s_{g-1}) \textstyle\sum_{d\in\{g\}} s_g^d e_d(s_1, \ldots, s_{g-1}), \end{aligned} \tag{5.10}$$

which is analytic algebraic in $(s_1, \ldots, s_g) \in B^{m_1}(\epsilon_1) \times \cdots \times B^{m_g}(\epsilon_g)$ and depends on the finite set of constants determined by $j^{3g}_p(f)$. By a standard argument (see, for example, Thorbergsson [27, p. 497]), $f(s_1, \ldots, s_g)$ parametrizes a neighborhood of $p \in M$.

Finally, since a finite product of open balls is a semi-algebraic subset of $\mathbf{R}^{n-1}$, its image under the analytic algebraic map (5.10) is a semi-algebraic subset of $\mathbf{R}^n$. $\square$

Corollary 32. *A connected proper Dupin hypersurface M in $\mathbf{R}^n$ is an open subset of a connected component of the irreducible algebraic set $\overline{M}^{zar}$ of dimension $n-1$.*

Proof. By the theorem, for any point $p \in M$, there is an open neighborhood $U \subset M$ of the point that is a semi-algebraic subset of $\mathbf{R}^n$. The result now follows from Lemma 17. □

A slightly more general result holds that we shall apply to Dupin hypersurfaces.

Corollary 33. *If a connected analytic hypersurface $N \subset \mathbf{R}^n$ contains a connected proper Dupin hypersurface M, then N is an open subset of the connected component of an irreducible algebraic set that contains M.*

Proof. An open subset of M is open in N, so the result follows from Lemma 17. □

6. Kuiper's Conjecture

As observed in the Introduction, Kuiper's Conjecture would be proved if we could prove that any connected non-proper Dupin hypersurface in $\mathbf{R}^n$ is algebraic.

A connected Dupin hypersurface M in $\mathbf{R}^n$ has an open dense subset $\mathcal{G}$ such that each connected component of $\mathcal{G}$ is proper Dupin. In fact, $\mathcal{G}$ is the set on which the multiplicities of the principal curvatures are locally constant ([23]). Decompose $\mathcal{G}$ into its at most countably many disjoint connected components $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \ldots$. By Corollary 33, if the Dupin hypersurface M were an analytic submanifold of $\mathbf{R}^n$, then, since it contains the connected proper Dupin hypersurface $\mathcal{G}_1$, it would be contained in an irreducible algebraic set of dimension $n-1$. Hence, our goal is to prove that M is an analytic hypersurface of $\mathbf{R}^n$.

Let $\mathcal{G}^c$ denote the complement of $\mathcal{G}$ in M. It is the set of points in M that do not have a proper Dupin neighborhood. For example, let $T^2 \subset \mathbf{R}^3$ be a torus of revolution, and let $M^3 \subset \mathbf{R}^4 = \mathbf{R}^3 \times \mathbf{R}$ be a tube over T^2 in $\mathbf{R}^4$ of sufficiently small radius ϵ so that M^3 is an embedded hypersurface in $\mathbf{R}^4$. Then M^3 is a Dupin hypersurface that is not proper Dupin. M^3 has three distinct principal curvatures at all points except for the two tori $T^2 \times \{\pm\epsilon\}$ on which there are only two distinct principal curvatures. In this case, the set $\mathcal{G}$ is all of M^3 except $T^2 \times \{\pm\epsilon\}$, and $\mathcal{G}^c$ equals $T^2 \times \{\pm\epsilon\}$. See [6, p. 188].

Definition 34. A connected Dupin hypersurface M of $\mathbf{R}^n$ has the *local finiteness property* if there is a subset $S \subset \mathcal{G}^c$, closed in M, such that

S disconnects M into only a finite number of connected components, and for each point $p \in \mathcal{G}^c$ not in S, there is an open neighborhood W of p in M such that $W \cap \mathcal{G}$ contains a finite number of connected open sets whose union is dense in W. We call S the set of *bad points* in $\mathcal{G}^c$.

For instance, if $\mathcal{G}^c$ is a finite union of compact, connected submanifolds, or more generally, is a locally finite CW complex, then M has the local finiteness property. An example with nonempty S is a set $\mathcal{G}^c$ that consists of the boundaries of an infinite nested sequence $T_1, T_2, \ldots,$ of open neighborhoods, with the closure of T_{n+1} properly contained in T_n, such that the intersection $S = \cap_n T_n$ is a submanifold of codimension greater than one in M.

Theorem 35. *Let M be a connected Dupin hypersurface in $\mathbf{R}^n$. If M has the local finiteness property, then it is an analytic submanifold of $\mathbf{R}^n$, and is therefore contained in a connected component of dimension $n-1$ of an irreducible algebraic set.*

Proof. Let $\mathcal{G} \subset M$ be the open dense subset of M whose connected components are proper Dupin hypersurfaces in $\mathbf{R}^n$. Let $S \subset \mathcal{G}^c$ be the set of bad points in $\mathcal{G}^c$. By the local finiteness property, each point $p \in \mathcal{G}^c$ not in S has a neighborhood W, open in M, such that $W \cap \mathcal{G}$ contains a finite number of connected open sets $U_1, \ldots, U_s$, whose union is dense in W. Note that $\mathcal{G}$ dense in M implies that

$$W \subset \cup_1^s \overline{U}_i,$$

where $\overline{U}_i$ is the topological closure of U_i in M. By Corollary 33, each U_i is contained in an irreducible algebraic set C_i. Then $\overline{U}_i$ is a subset of C_i since C_i is closed. Hence,

$$W \subset \cup_1^s \overline{U}_i \subset \cup_1^s C_i,$$

which is a semi-algebraic subset of $\mathbf{R}^n$.

By Lemma 21, since W is a C^∞-manifold, there is a connected open semi-algebraic subset U of $\cup_1^s C_i$ contained in W with $p \in U$. The intersection B of U with any open ball of $\mathbf{R}^n$ centered at p is still an open subset of M and a semi-algebraic subset. Thus, we may assume B is so small that it is the graph of $h : D \to \mathbf{R}$ for some open set $D \subset \mathbf{R}^{n-1}$ by performing a linear change of coordinates $(x^1, \ldots, x^n)$ in $\mathbf{R}^n$ if necessary. Since the projection

$$\pi : (x^1, \ldots, x^n) \mapsto (x^1, \ldots, x^{n-1}),$$

which is semi-algebraic, sends B to D, we see D is a semi-algebraic subset of $\mathbf{R}^{n-1}$. Thus, the map h is semi-algebraic, because its graph

$$\{(x, h(x)) : x \in D\} = B$$

is a semi-algebraic subset of $\mathbf{R}^n$. Therefore, h is a Nash function and so it must be analytic algebraic as noted in Proposition 9, and the map

$$H : D \to B,\ H(x) = (x, h(x))$$

is an analytic algebraic parametrization of the open neighborhood $B \subset M$ about p. Since $p \in \mathcal{G}^c \setminus S$ was arbitrary, and since every point of $\mathcal{G}$ has a neighborhood with an analytic algebraic parametrization by Theorem 31, it follows that $M \setminus S$ is an analytic submanifold of $\mathbf{R}^n$. Since $M \setminus S$ has a finite number of connected components, each of which contains an open subset that is a semi-algebraic subset of $\mathbf{R}^n$, it follows that $M \setminus S$ is contained in the union C of finitely many irreducible algebraic sets. Then M is contained in the topological closure of $M \setminus S$ which in turn must be contained in the closed set C.

Now let q be any point in S and consider M itself as an open neighborhood about q. Then M is a C^∞ manifold contained in the semi-algebraic set C. Apply the above argument to conclude that some open neighborhood of q in M is the image of an analytic algebraic parametrization. Hence, every point of M has an analytic parametrization, and M is an analytic hypersurface of $\mathbf{R}^n$. □

Question 36. Is the local finiteness property true for any connected Dupin hypersurface in $\mathbf{R}^n$?

If the answer is yes, then we could resolve Kuiper's conjecture for taut submanifolds of $\mathbf{R}^n$ in the following manner.

First note that Pinkall [19] extended the notion of Dupin to submanifolds in $\mathbf{R}^n$ of codimension greater than one as follows. Let M be a submanifold in $\mathbf{R}^n$ of codimension greater than one, and let $UN(M)$ denote the unit normal bundle of M in $\mathbf{R}^n$. A *curvature surface* of M is a connected submanifold $S \subset M$ for which there is a parallel (with respect to the normal connection) section $\eta : S \to UN(M)$ such that for each $x \in S$, the tangent space T_xS is equal to some principal space of the shape operator $A_{\eta(x)}$. The corresponding principal curvature $\kappa : S \to \mathbf{R}$ is then a smooth function on S. Pinkall calls a submanifold M of codimension greater than one *Dupin* if along each curvature surface, the corresponding principal curvature is constant. This definition is equivalent to Pinkall's [18] Lie sphere geometric definition of the Dupin property in terms of the Legendre lift of M, and of course, it agrees with the usual definition of a Dupin hypersurface in the case of codimension one.

Pinkall [19] then proved that a taut submanifold of $\mathbf{R}^n$ must be Dupin, although not necessarily proper Dupin. He also proved that if M is an embedded taut compact submanifold in $\mathbf{R}^n$ and M_ϵ is a tube over M of sufficiently small radius that M_ϵ is an embedded hypersurface in $\mathbf{R}^n$, then M is Dupin if and only if M_ϵ is Dupin (since their Legendre lifts are Lie equivalent [18]), and M is taut (with respect to $\mathbf{Z}_2$-homology) if and only if M_ϵ is taut.

We now prove that a taut submanifold M in $\mathbf{R}^n$ is algebraic if the local finiteness property holds on the tube M_ϵ.

Theorem 37. (a) *If M^{n-1} is a connected, compact taut hypersurface in $\mathbf{R}^n$ that satisfies the local finiteness property, then M^{n-1} is an analytic submanifold and a connected component of an irreducible algebraic subset of $\mathbf{R}^n$.*
(b) *If M^m is a connected, compact taut submanifold of codimension greater than one in $\mathbf{R}^n$ such that the tube M_ϵ over M, which is Dupin, satisfies the local finiteness property, then M is an analytic submanifold and a connected component of an irreducible algebraic subset of $\mathbf{R}^n$.*

Proof. (a) Since a taut hypersurface must be Dupin, this follows immediately from Theorem 35.
(b) Let M be an embedded taut connected, compact submanifold in $\mathbf{R}^n$ and let M_ϵ be a tube over M of sufficiently small radius that M_ϵ is an embedded hypersurface in $\mathbf{R}^n$. Then M_ϵ is a Dupin hypersurface by the work of Pinkall [19] mentioned above. Thus $M_\epsilon \subset \mathbf{R}^n$ is semi-algebraic by Theorem 35, since M_ϵ satisfies the local finiteness property. Consider the focal map $F_\epsilon : M_\epsilon \to M \subset \mathbf{R}^n$ given by

$$F_\epsilon(x) = x - \epsilon\xi,$$

where ξ is the outward field of unit normals to the tube M_ϵ. Any point of M_ϵ has an open neighborhood U parametrized by an analytic algebraic map. By Proposition 11, the first derivatives of this parametrization are also analytic algebraic, and thus the Gram-Schmidt process applied to these first derivatives and some constant non-tangential vector produces the vector field ξ and shows that ξ is analytic algebraic on U. Hence F_ϵ is analytic algebraic on U and so the image $F_\epsilon(U) \subset M$ is a semi-algebraic subset of $\mathbf{R}^n$. Covering M_ϵ by finitely many sets of this form U, we see that M, being the union of their images under F_ϵ, is a semi-algebraic subset of $\mathbf{R}^n$. Then $\overline{M}^{\mathrm{zar}}$ is an irreducible algebraic of the same dimension as M and contains M. □

The preceding proof uses the finiteness condition on the tube M_ϵ to prove that M_ϵ is analytic via Theorem 35. This will also follow from the assumption that M itself is analytic. Thus we have the following result.

Corollary 38. *If M is an analytic connected, compact taut submanifold of $\mathbf{R}^n$, then M is a connected component of an irreducible algebraic subset of $\mathbf{R}^n$.*

7. Alexander Cohomology

We now begin our work to show that the local finiteness property is satisfied by taut hypersurfaces satisfying certain additional restrictions on their dimensions and the multiplicities of their principal curvatures. A principal tool in our work is the theory of Alexander cohomology for noncompact manifolds. We begin with the following definition.

Definition 39. An *end* of a noncompact manifold X is an equivalence class of sequences of connected open neighborhoods $X, U_1, U_2, \ldots$, where $U_{n+1} \subset U_n$, such that the intersection of the closures of these sets is the empty set, subject to the equivalence relation $(X, U_1, U_2, \ldots) \sim (X, W_1, W_2, \ldots)$ if for each U_i there is a j with $U_i \subset W_j$, and for each W_k there is an m such that $W_k \subset U_m$.

Though the definition is intuitive, another equivalent intrinsic definition is often more convenient to work with.

Definition 40. An *end* of a noncompact manifold X is a function ϵ which assigns each compact set $K \subset X$ to a subset of X such that $\epsilon(K)$ is a connected component of $X \setminus K$ for each K, and if $K \subset L$ then $\epsilon(L) \subset \epsilon(K)$.

For a proof that these two definitions are equivalent, see [11, Proposition 9, p. 9].

We can "glue" the ends of X to X to make the resulting space compact (see [20]). In particular, the set of ends is not empty by the noncompactness of X.

Definition 41. Let S be a closed subset of a manifold M. The k-th *Alexander cohomology group* is defined to be

$$\overline{H}^k(S) := \lim_{\rightarrow} H^k(V),$$

where $H^k(V)$ is the usual k-th singular cohomology group of V (with, say, the coefficient ring $\mathbf{Z}_2$; in fact, all commutative rings with identity suffice), and $\lim_{\rightarrow}$ denotes taking the direct limit over the directed system of open sets V containing S in the ambient space M.

In other words, we define $\overline{H}^k(S)$ to be the set of equivalence classes of the disjoint union of $H^k(V)$, for all open V containing S, where $a_U \in H^k(U)$ and $a_V \in H^k(V)$ are considered equivalent if there is a third open set W containing S, where W is contained in both U and V, with the inclusion maps $\iota_W^U : W \to U$ and $\iota_W^V : W \to V$, and the pullback morphisms $(\iota_W^U)^* : H^k(U) \to H^k(W)$ and $(\iota_W^V)^* : H^k(V) \to H^k(W)$, such that $(\iota_W^U)^*(a_U) = (\iota_W^V)^*(a_V)$.

Let us look at $\overline{H}^0(S)$ to gain some insight. First observe that if $W \subset U$ are two open neighborhoods of S, then the morphism $(r_W^U)^* : H^0(U) \to H^0(W)$ induced by the inclusion $r_W^U : W \to U$ is nothing other than restricting an element $a_U \in H^0(U)$, which is a $\mathbf{Z}_2$-valued locally constant function, to the same function on W. Hence, $a_U \in H^0(U)$ and $a_V \in H^0(V)$ are equivalent in the definition of the Alexander cohomology group $\overline{H}^0(S)$ if and only if there is a smaller open neighborhood W of S contained in both U and V such that the two $\mathbf{Z}_2$-valued functions a_U and a_V restrict to the same function on W. In other words, $\overline{H}^0(S)$ is the ring of germs of locally constant $\mathbf{Z}_2$-valued functions around S.

Here, for any topological space X, a function f from X to $\mathbf{Z}_2$ is said to be *locally constant* if there is an open covering $\mathcal{U}$ of X such that f is constant on each element of $\mathcal{U}$.

Theorem 42. *$\overline{H}^0(S)$ is the $\mathbf{Z}_2$-module of locally constant functions from S to $\mathbf{Z}_2$. In particular, if all the (topological) connected components of S are open in S, then the rank of $\overline{H}^0(S)$ is identified with the cardinality of the (topological) connected components.*

For a proof, see Theorem 5 and Corollary 6 in [24, p. 309-310].

Corollary 43. *Let M be a compact and simply connected manifold. Let $S \subset M$ be a compact subset which disconnects $M \setminus S$ into α connected components. Suppose all the connected components of S are open (whose cardinality is necessarily a finite number m) in S. Then the number of ends ϵ of $X = M \setminus S$ is finite, and*

$$\epsilon = \alpha + m - 1. \tag{7.1}$$

Proof. First of all, observe that the number of (topological) connected components of S is finite, since S is the disjoint union of these open components and S is compact; let this number be m. Therefore, by Theorem 42, the rank of $\overline{H}^0(S)$ is m.

We claim that the number ϵ of ends of X is finite. Suppose $\epsilon = \infty$. Then there is a decreasing sequence of open sets $W_1, W_2, \ldots$ in X such that $X \setminus W_j$ is compact, $\cap_j \overline{W}_j = \emptyset$, and the number of unbounded connected components of W_j is an increasing sequence diverging to infinity ($\overline{W}$ is the closure of W). (By [10, Theorem 3.9, p. 111], each of W_j has finitely many unbounded connected components, since $X \setminus W_j$

is compact.) Let n_j be the number of connected components of W_j. Let $U_j = W_j \cup S$. Then U_j is a neighborhood of S in M since $U_j = M \setminus (X \setminus W_j)$. Let m_j be the number of connected components of U_j in M.

Observe that since M is covered by the open sets $M \setminus S$ and U_j, the Mayer-Vietoris sequence, with $H^1(M) = 0$ by simple connectedness, says

$$0 \leftarrow H^0(U_j \setminus S) \leftarrow H^0(M \setminus S) \oplus H^0(U_j) \leftarrow H^0(M) \leftarrow 0$$

so that

$$\begin{aligned}(7.2)\ n_j &= \dim H^0(U_j \setminus S) = \dim H^0(M \setminus S) + \dim H^0(U_j) - 1 \\ &= \alpha + m_j - 1.\end{aligned}$$

It follows that the sequence $\{m_j\}$ also diverges to ∞. Now the direct limit in the definition of Alexander cohomology can be taken over the directed system $U_1, U_2, \ldots$ in place of the system of all open neighborhoods of S. This is because for each open neighborhood U of S, there is a $W_j \subset U \setminus S$, and hence a $U_j \subset U$, such that if we define $a_j \in H^0(U_j)$ to be the restriction of $a_U \in H^0(U)$ on U_j, then $[a_U] = [a_{U_j}] \in \overline{H}^0(S)$.

The restriction maps $r_j : U_{j+1} \to U_j$ induce the restriction morphisms

$$(r_j)^* : H^0(U_j) \to H^0(U_{j+1}).$$

Moreover, $a_j \in H^0(U_j)$ is identified with $a_{j+1} \in H^0(U_{j+1})$ if and only if $a_{j+1} = (r_j)^*(a_j)$. Then

$$(r_j)^* : \oplus \mathbf{Z}_2 \to \oplus \mathbf{Z}_2,$$

where the domain ring consists of m_j copies of $\mathbf{Z}_2$ and the target ring of m_{j+1} copies with $m_j < m_{j+1}$.

Starting with $H^0(U_1)$, which has cardinality 2^{m_1}, $(r_1)^*(U_1)$ is of cardinality no greater than 2^{m_1} in the space $H^0(U_2)$ of cardinality $2^{m_2} > 2^{m_1}$, $(r_2)^*(r_1)^*(U_1)$ is of cardinality at most 2^{m_1} in the space $H^0(U_3)$ of cardinality $2^{m_3} > 2^{m_1}$, etc. Now pick one element a_2 in $H^0(U_2)$ not in the image of $H^0(U_1)$ under $(r_1)^*$, an element a_3 not in the image of $H^0(U_1)$ under $(r_2)^*(r_1)^*$, etc. Then $(a_2, a_3, \cdots)$ is not equal to any element in $\overline{H}^0(S)$ whose representative comes from $H^0(U_1)$. Continuing in this fashion, we can construct an infinite number of distinct elements in $\overline{H}^0(S)$, since $m_1 < m_2 < \cdots$ increasingly diverge to infinity. The upshot is that this implies that the cardinality of $\overline{H}^0(S)$ is infinite, which contradicts the fact that it is equal to 2^m.

The contradiction proves that the number ϵ of ends is finite. It follows that, since n_j is the number of connected components of W_j, then $n_j = \epsilon$ for sufficiently large j, so that (7.2) gives

$$m_j = 1 + \epsilon - \alpha$$

for sufficiently large j.

In the case when $m = 1$, i.e., when S is connected, we have $m_j = 1$ for large j, since the closure of each of the finite ends intersects S (or else it would not be an end), so that in fact the union of each end with S is connected, and so U_j, being the union of all W_j and S, with W_j composed of the ends, is connected. In general, since all the components of S are open, we can separate them by disjoint open sets in M and hence can work on each component to conclude that in fact $m_j = m$ for large j, from which we derive

$$\epsilon = \alpha + m - 1.$$

□

We assume in the preceding corollary that $H^1(M) = 0$ only for simplicity. When $H^1(M) \neq 0$, we have the connecting homomorphism

$$\leftarrow H^1(M) \leftarrow H^0(U_j \setminus S) \leftarrow$$

when chasing upward the Mayer-Vietoris sequence, so that in fact

$$n_j = \alpha + m_j - 1 + \delta_j \tag{7.3}$$

with δ_j the dimension of the image of $H^0(U_j \setminus S)$ via the connecting homomorphism. Clearly

$$\delta_j \leq b_1,$$

where b_1 is the first Betti number of M.

By the same reasoning as above we see that n_j, m_j, δ_j eventually stabilize, so that we have

$$\epsilon = \alpha + m - 1 + \delta \tag{7.4}$$

with $\delta \leq b_1$. We conclude the following.

Corollary 44. *Let S be a closed set in a compact manifold M. Suppose all the (topological) connected components of S are open (and so necessarily of finite cardinality m) in S, and suppose the number of ends of $X = M \setminus S$ is finite with cardinality ϵ. Then the number of connected components of X is no greater than $\epsilon - m + 1$.*

Proof. In Equation (7.4)

$$\alpha = \epsilon - m + 1 - \delta \leq \epsilon - m + 1.$$

□

Remark 45. It is important to observe that we calculated Alexander cohomology under the assumption that the connected components of S are all open in S. Conversely, when we know in advance that the number of ends of X is finite, we can assert that the (topological) connected components of S are all open in S. (As long as S is closed, M need not be compact.) This is because (7.3) and the finite cardinality of ends imply that the numbers m_j are always bounded above by a constant ρ. (Note, in particular, that the number of connected components of X, which is α in (7.3), is then always finite.) Fix a large j in the process and suppose the open set U_j decomposes into β connected components $U_{j1}, \cdots, U_{j\beta}$. We may assume all $S_k := U_{jk} \cap S \neq \emptyset, 1 \leq k \leq \beta$; or else we ignore those having empty intersection with S and redefine U_j to be the union of those intersecting S. Then for $U_{j+1} \subset U_j$, since $S \subset U_{j+1}$, none of the $U_{j+1,k} := U_{j+1} \cap U_{jk}, 1 \leq k \leq \beta$, are empty. It follows that U_{j+1} consists of at least β connected components. Hence $m_j \leq m_{j+1} \leq \cdots \leq \rho$. This implies that eventually all large m_t are equal to a fixed number m. Fix such t. The m connected components of U_t intersect S in m nonempty disjoint sets $S_1, \ldots, S_m$, which in turn are contained in the m connected components of all $U_r, r \geq t$, by the process. Note that the restriction morphisms $(r_t)^*$ are all isomorphisms for large t. Hence the rank of $\overline{H}^0(S)$ is m, which is the rank of the module of locally constant functions on S. Now the sets $S_1, \ldots, S_m$ are both open and closed in S, and they form an open covering. It follows that the locally constant functions on $S_1, \ldots, S_m$ already exhaust $\overline{H}^0(S)$. Each S_j must be connected, or else, we could cut it up into two disjoint nonempty open sets to find a refined open covering with $m+1$ open sets, contradicting the rank of $\overline{H}^0(S)$ being m. Hence all $S_1, \ldots, S_m$ are open connected components.

Definition 46. The set S is called the *end set* of $X = M \setminus S$. If X has h ends, then there is an open neighborhood of U of S such that for any open neighborhood V of S contained in U, $V \setminus S$ has h (unbounded) connected components, each of which corresponds uniquely to an end. We call these components the *end components* of X.

8. Application to Compact Taut Submanifolds

We begin this section by recalling some fundamental facts about taut embeddings. An embedded compact, connected submanifold $M \subset \mathbf{R}^n$ is taut if every nondegenerate Euclidean distance function,

$$L_p : M \to \mathbf{R}, \quad L_p(x) = d(x,p)^2, \quad p \in \mathbf{R}^n,$$

has $\beta(M, \mathbf{Z}_2)$ critical points on M, where $\beta(M, \mathbf{Z}_2)$ is the sum of the $\mathbf{Z}_2$-Betti numbers of M. That is, L_p is a perfect Morse function on M.

Kuiper [12] showed that a compact, connected submanifold $M \subset \mathbf{R}^n$ is taut if and only if M satisfies the following condition: for every closed ball $B \subset \mathbf{R}^n$ the homorphism on homology

$$H_*(B \cap M, \mathbf{Z}_2) \to H_*(M, \mathbf{Z}_2) \tag{8.1}$$

induced by the inclusion of $(B \cap M) \subset M$ is injective for all $r \in \mathbf{R}$.

Tautness is invariant under Möbius transformations of $\mathbf{R}^n \cup \{\infty\}$. Further, a compact submanifold $M \subset \mathbf{R}^n$ is taut if and only if the embedding $\sigma(M) \subset S^n$, where $\sigma : \mathbf{R}^n \to S^n - \{P\}, P \in S^n$, is stereographic projection, is taut in S^n, where spherical distance functions d_p are used instead of Euclidean distance functions L_p. In this section, we will consider submanifolds of S^n instead of $\mathbf{R}^n$ for various reasons, including the fact that a focal point cannot vanish to infinity in S^n.

Since a spherical distance function $d_p(q) = \cos^{-1}(p \cdot q)$ has the same critical points as the Euclidean height function $\ell_p(q) = p \cdot q$, for $p, q \in S^n$, a compact submanifold $M \subset S^n$ is taut if and only if it is *tight*, i.e., every nondegenerate height function ℓ_p has $\beta(M, \mathbf{Z}_2)$ critical points on M. It is often simpler to use height functions rather than spherical distance functions when studying tautness for submanifolds of S^n, and we will use whichever type of function is most convenient for our argument.

We now come to a fundamental result on taut submanifolds due to Ozawa [16].

Theorem 47 (Ozawa)**.** *Let M be a taut compact connected submanifold embedded in S^n, and let $\ell_p, p \in S^n$, be a linear height function on M. Let $x \in M$ be a critical point of ℓ_p, and let S be the connected component of the critical set of ℓ_p that contains x. Then S is*
(a) *a smooth compact manifold of dimension equal to the nullity of the Hessian of ℓ_p at x;*
(b) *nondegenerate as a critical manifold;*
(c) *taut in S^n.*

We call such a connected component of the critical set of ℓ_p a *critical submanifold* of ℓ_p.

Remark 48. Terng and Thorbergsson [26, p.190] generalized the notion of tautness to submanifolds of arbitrary complete Riemannian manifolds and proved an analogue of Ozawa's Theorem in that context.

Let M be a compact taut hypersurface in S^n. Consider the normal exponential map $E : M \times (-\pi, \pi) \to S^n$, where

$$E : (p, t) \mapsto \cos(t)p + \sin(t)\mathbf{n},$$

with $\mathbf{n}$ the chosen unit normal field of M. Here, E is smooth at points where $t = -\pi, \pi$.

A point $q = E(p, t)$ is called a *focal point of multiplicity* $m > 0$ *of* M *at* p if the nullity of the derivative E_* is equal to m at (p, t). The set of all focal points is the *focal set* $\mathcal{F}$ of M. The focal points at p are antipodally symmetric on the circle $E(p, t)$ with each principal curvature of the form $\cot(t)$ for some $t \neq 0$. Let $\mathcal{Z}$ be the complement of $\mathcal{F}$ in S^n.

Lemma 49. *$\mathcal{Z}$ is connected in S^n.*

Proof. By Federer's version of Sard's theorem [8, Theorem 3.4.3, p.316], the image of the critical points of a given smooth function $f : \mathbf{R}^t \to \mathbf{R}^s$, at which the rank of the derivative is less than or equal to ν, is of $\mathcal{H}^\nu$-measure 0, where $\mathcal{H}^\nu$ denotes the Hausdorff ν-dimensional measure.

Label the principal curvature functions by $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1}$. It is known that λ_j are Lipschitz-continuous, because the principal curvature functions on the linear space $\mathcal{S}$ of all symmetric matrices are Lipschitz-continuous by general matrix theory (see, for example, [2, p. 64]) and the fact that the Hessian of M is a smooth function from M into $\mathcal{S}$. Let $\cot(t_j) := \lambda_j, 0 < t_j < \pi$. We know that the focal maps

$$f_j(p) = \cos(t_j)p + \sin(t_j)\mathbf{n} \tag{8.2}$$

are Lipschitz-continuous and are smooth on a dense open set.

For each i, let O_i be the open subset of M on which λ_i has multiplicity 1. (O_i could be empty.) Then O_i consists of countably many open components $O_{ij}, j = 1, 2, \ldots$, such that the restriction of f_i to (O_{ij}) is an embedded submanifold of dimension $n - 2$. This follows from the fact that λ_i is constant on its lines of curvature on each O_{ij} by the Dupin condition.

Let Z_i be the complement of O_i in M. (Z_i could be empty.) At each point p of Z_i, the principal curvature $\cot(t_i)$ must have multiplicity at least 2, and so the normal exponential map E has rank $\leq n - 2$ at the point (p, t_i). Thus the focal point $f_i(p)$ lies in the singular value set K of points for which the derivative of E has rank $\leq n - 2$.

We conclude that the entire focal set $\mathcal{F}$ is composed of the countably many embedded submanifolds $f_i(O_{ij})$ of dimension $n-2$, their antipodal sets and the set K which has Hausdorff $(n-2)$-measure zero by Federer's theorem quoted above. Therefore, the Hausdorff $(n-1)$-measure of the whole focal set $\mathcal{F}$ is zero, which implies that the complement $\mathcal{Z}$ of $\mathcal{F}$ is connected (see [22, p. 269]). $\square$

Away from $\mathcal{F}$, the map E is a local diffeomorphism. For each $0 \leq m \leq n-1$, we let W_m be the set of points (p,t) in $E^{-1}(\mathcal{Z})$ for which the spherical distance function d_q, where $q = E(p,t)$, has index m at p. Then the tautness of M implies that

$$E : W_m \to \mathcal{Z}$$

is a degree b_m (regular) covering map, where b_m is the m-th Betti number. Therefore, by the connectedness of $\mathcal{Z}$, the set W_m decomposes into finitely many (connected) covering sheets $W_{m1}, W_{m2}, \ldots, W_{ms_m}$ onto $\mathcal{Z}$, where $s_m \leq b_m$. Note, in particular, that $W_m \neq \emptyset$ if and only if $b_m \neq 0$. Further, since W_0 and W_{n-1} are both strips around the 0-section $M \times \{0\}$, we will not consider $m = 0$ or $m = n-1$. Thus, we will consider only those m with $0 < m < n-1$, such that $b_m \neq 0$. For such values of m, W_m will be composed of connected components either in $M \times (0,\pi)$ or $M \times (-\pi, 0)$. We denote by W^+_{mj} those W_{mj} contained in $M \times (0,\pi)$. We set $W^+_m := \cup_j W^+_{mj}$.

Lemma 50. *All the (topological) connected components of $\mathcal{F}$ are open in $\mathcal{F}$. In particular, the number of ends of $\mathcal{Z}$ is finite.*

Proof. Since each $f_j(M), 1 \leq j \leq n-1$, defined in (8.2) is compact and path-connected in S^n, we can group them and the sets antipodal to them into classes where the union of those $f_j(M)$ in each class is path-connected whereas the unions from different classes are disjoint. Call these disjoint unions $X_1, \ldots, X_r$, each of which, being a finite union of compact sets, is closed in S^n and path-connected. Hence by the Urysohn separation lemma, there are disjoint open sets $O_1, \ldots, O_r$ that contain $X_1, \ldots, X_r$, respectively. This means that each (topological) component, now being just the path-connected components, are open in the relative topology. The conclusion follows by Corollary 43 since the ambient sphere is simply connected and $\mathcal{F}$ does not disconnect S^n so that $\alpha = 1$ in (7.1). The number of ends of $\mathcal{Z}$ equals the number of path-connected components of $\mathcal{F}$. $\square$

Corollary 51. *Each W^+_{mj} has a finite number of ends.*

Proof. This follows since $E : W^+_{mj} \to \mathcal{Z}$ is a proper map because it is a covering map of finite degree and $\mathcal{Z}$ has a finite number of ends. $\square$

Let $pr : M \times (-\pi, \pi) \to M$ be the projection. Since pr is an open map, the sets

$$U^+_{mj} := pr(W^+_{mj}) \tag{8.3}$$

are open and connected in M. We also set

$$U_m^+ := \cup_j U_{mj}^+ = pr(W_m^+). \tag{8.4}$$

It is clear that W_m^- and U_m^- can be similarly defined on $M \times (-\pi, 0)$.

Definition 52. We define $(U_m^*)^+$ to be the collection of all x for which there exists a $t > 0$ such that (x, t) is a regular point of the normal exponential map E, and the spherical distance function d_y, where $y = E(x, t)$, has index m at x.

Note that $U_m^+ \subset (U_m^*)^+$ by definition. Further, any point p in the complement $(U_m^*)^+ \setminus U_m^+$ satisfies the condition that every $q = E(p, t), t > 0$, on the normal exponential circle, such that d_q has a nondegenerate critical point of index m at p, is a focal point of some other point in M.

Lemma 53. *$(U_m^*)^+$ is open in M.*

Proof. This follows from the property that E is a local diffeomorphism around $(x, t), t > 0$, so that $E(x, t)$ is nonfocal along the normal exponential circle. Hence, for a point (x', t') near (x, t), the point $E(x', t')$ is also nonfocal along the respective normal exponential circle. Now, d_y has index m at the nondegenerate critical point x. When x' is sufficiently close to x, the function $d_{y'}, y' = E(x', t')$, is a slight perturbation of d_y. Since x' is a nondegenerate critical point of $d_{y'}$ and since nondegenerate critical points are locally structurally stable, $d_{y'}$ must also have index m at x', so that $x' \in (U_m^*)^+$. $\square$

We introduce a space slightly larger than W_{mj}^+ as follows. Observe that any two points $(p, t_1), (p, t_2)$ in W_{mj} lie on an interval in the set $\{p\} \times (0, \pi)$ between two adjacent critical points of the map E. However, it should be noted that, in general, there might exist points (p, t) on the same interval not belonging to W_{mj}^+, because $E(p, t)$ is a focal point of some other point in M. In accordance, for (p, t_1) and (p, t_2) in $M \times (0, \pi)$, we say (p, t_1) is equivalent to (p, t_2), denoted $(p, t_1) \sim (p, t_2)$, if the distance functions d_{q_1}, for $q_1 = E(p, t_1)$, and d_{q_2}, for $q_2 = E(p, t_2)$, have nondegenerate critical points of the same index m at p. We then let L_{mj} be the (trivial) line bundle

$$L_{mj} := \{(p, t) \in M \times (0, \pi) : (p, t) \sim (p, t_0) \text{ for some } (p, t_0) \in W_{mj}\} \tag{8.5}$$

over U_{mj}^+.

Lemma 54. *$L_{mj} \setminus W_{mj}^+$ is of Hausdorff codimension at least 2 in L_{mj}. In particular, L_{mj} and U_{mj}^+ are of finite number of ends.*

Proof. By the openness of W_{mj}^+, we know $L_{mj} \setminus W_{mj}^+$ is closed in L_{mj}. We next show that $L_{mj} \setminus W_{mj}^+$ is of Hausdorff codimension at least 2. Consider the restriction of the map E on $L_{mj} \setminus W_{mj}^+$ given by

$$f := (p,t) \in L_{mj} \setminus W_{mj}^+ \mapsto E(p,t) \in \mathcal{F}.$$

Note that f is a finite-to-one map. This is because for any $z \in \mathcal{F}$, the height function ℓ_z is a perfect Morse-Bott function, and each p, for which some point of the form (p,t) is in $f^{-1}(z)$, is a nondegenerate critical point of index m of ℓ_z on the taut hypersurface M. Such a nondegenerate critical point is a critical submanifold. Hence the total number of such points is no more than b_m, the m-th Betti number of M.

The image of f is a closed subset of $\mathcal{F}$ of Hausdorff codimension at least 2 by Lemma 49. E is a local diffeomorphism at $(p,t) \in L_{mj}$ when it is restricted to L_{mj}. Hence, there is a neighborhood $Y_{(p,t)} \subset L_{mj}$ around each $(p,t) \in L_{mj} \setminus W_{mj}^+$ such that f restricted to $Y_{(p,t)} \cap (L_{mj} \setminus W_{mj}^+)$ is a homeomorphism into $\mathcal{F}$. It follows that the Hausdorff $(n-2)$-measure of $L_{mj} \setminus W_{mj}^+$ is at most b_m times that of $\mathcal{F}$, which is null in the ambient sphere S^n. So, $L_{mj} \setminus W_{mj}^+$ is of Hausdorff codimension at least 2 in L_{mj}. In particular, $L_{mj} \setminus W_{mj}^+$ does not disconnect L_{mj}.

We now apply Remark 45 with $S = L_{mj} \setminus W_{mj}^+$, $M = L_{mj}$ and $X = M \setminus S = W_{mj}^+$. Since X has a finite number of ends, it follows that S has finitely many components all of which are open in S. Now, suppose M has infinitely many ends. Then there is a decreasing sequence of open sets $X_1, X_2, \ldots$, in X such that $X \setminus X_j$ is compact, $\cap_j \overline{X}_j = \emptyset$ and the number of unbounded connected components of X_j increasingly diverges to infinity. As S has finitely many components $S_1, \ldots, S_t$, all of which are of Hausdorff codimension at least 2, $S_1, \ldots, S_t$ cannot disconnect the connected components of X_j. We see $X_j \setminus S$ form a decreasing sequence whose disconnected components diverge to infinity, so that $W_{mj}^+ = M \setminus S$ will have infinitely many ends. This contradiction establishes that L_{mj} has a finite number of ends. Since L_{mj} is a trivial line bundle over U_{mj}^+, so that $L_{mj} \simeq U_{mj}^+ \times \mathbf{R}$, it follows that U_{mj}^+ has a finite number of ends. □

Corollary 55. *W_{mj}^+ is disjoint from W_{ml}^+ if and only if U_{mj}^+ is disjoint from U_{ml}^+. In particular, U_m^+ has finitely many ends.*

Proof. The backward direction is clear. To prove the forward direction, suppose $p \in U_{mj}^+ \cap U_{ml}^+$. Let $X := U_{mj}^+ \cup U_{ml}^+$ and let $Y := W_{mj}^+ \cup W_{ml}^+$. Since X is open and connected, we can form the (trivial) line bundle L over X, where L consists of $(p,t), p \in X$, such that the distance function d_q, for $q = E(p,t)$, has a nondegenerate critical point of index m at p. Then $L = L_{mj} \cup L_{ml}$ and $Y \subset L$ by construction. However, the analysis in Lemma 54 shows that $L \setminus Y$ is of Hausdorff codimension at least 2; therefore, Y is connected, and so W^+_{mj} and W^+_{ml} cannot be disjoint.

Now that U^+_m is the finite disjoint union of all U^+_{mj}, each of which has finitely many ends, it follows that the same is true for U^+_m. □

Corollary 56. *U^+_m is dense in $(U^*_m)^+$. In particular, $(U^*_m)^+$ has a finite number of connected components.*

Proof. Suppose that U^+_m is not dense in $(U^*_m)^+$. Then there is an open set $X \subset (U^*_m)^+$ that is disjoint from U^+_m. Similar to Corollary 55, let us introduce the line bundle L over X, which consists of all points $(p,t), t > 0$, where $p \in X$ and d_q, for $q = E(p,t)$, has a nondegenerate critical point of index m at p.

Every point $(p,t) \in L$ is mapped to $\mathcal{F}$ via the map E, since $p \in (U^*_m)^+ \setminus U^+_m$. As we see in Lemma 54, the map f of Lemma 54 on L is finite-to-one and regular at every point of L. Thus, the image of f contains an open set of dimension $n = \dim L$. This contradicts the fact that the Hausdorff codimension of $\mathcal{F}$ is at least two. Thus, U^+_m is dense in $(U^*_m)^+$.

In particular, $(U^*_m)^+$ also has a finite number of connected components, because any of its components will contain at least one component of U^+_m. □

Recall that we say a point p is a good point in the taut hypersurface M if the multiplicities of the principal curvatures are locally constant around p. We denote the set of good points by $\mathcal{G}$. We know $\mathcal{G}$ is open and dense in M.

Our convention is that we label the principal curvature functions by $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1}$. When we say $(m_1, m_2, \ldots, m_g)$ is a given sequence of principal multiplicities, we mean that

$$\lambda_1 = \cdots = \lambda_{m_1}, \quad \lambda_{m_1+1} = \cdots = \lambda_{m_1+m_2}, \ldots,$$

so that m_i is the multiplicity of the i^{th} largest distinct principal curvature.

Lemma 57. *There is a dense open subset of $\mathcal{G}$ such that for any p in the subset, there is a point $q \in \mathcal{Z}$ between any two focal points on the normal exponential circle $E(p,t), t > 0$.*

Proof. Let X be a connected component of $\mathcal{G}$ assuming multiplicities $(m_1, m_2, \ldots, m_\mu)$. Let $\sigma_s = m_1 + \cdots + m_s$, for $1 \leq s \leq \mu$. Then

$X \subset (U^*_{\sigma_s})^+$. Then Corollary 56 implies that $X \cap U^+_{\sigma_s}$ is dense and open in X. $\square$

Definition 58. We let $\mathcal{G}^\circ$ be the largest dense open subset of $\mathcal{G}$ over which Lemma 57 is true.

Lemma 59. *Let g be the maximum number of distinct principal curvatures on M. Let $\mathcal{M} := (m_1, \ldots, m_g)$ be a given maximal sequence of multiplicities. Let $O_{\mathcal{M}} \subset \mathcal{G}^\circ$ be the (open) subset of $\mathcal{G}^\circ$ attaining these multiplicities. Let $\sigma_j = m_1 + \cdots + m_j$, for $1 \leq j \leq g$. Then $O_{\mathcal{M}}$ is the intersection of all $U^+_{\sigma_j}$.*

Proof. The intersection consists of all points p such that there is some $0 < t_j < \pi$ for which p is of index σ_j with respect to the distance function d_q, with $q = E(p, t_j)$, for each j. So, p must have g principal curvatures with multiplicities $m_1, m_2, \ldots, m_g$. The maximality of g implies that m_j cannot be broken further into smaller multiplicities. Moreover, since $p \in U^+_{\sigma_j}$, there is a t for which $E(p,t)$ between the two appropriate focal points is in $\mathcal{Z}$. We obtain $p \in \mathcal{G}^\circ$. So, $\cap U^+_{\sigma_j} \subset \mathcal{O}_M$.

Conversely, since $\mathcal{O}_M \subset \mathcal{G}^\circ$, it follows by the definition of $U^+_{\sigma_j}$ that $\mathcal{O}_M \subset \cap U^+_{\sigma_j}$. $\square$

We now proceed to handle the case of a taut submanifold of dimension 3 or 4. First we need the following lemma in homology that will be needed in our proof.

Lemma 60. *Let $B_1 \subset B_2$ be two closed disks. Then the k^{th} new topology of the taut submanifold M added between $B_1 \cap M$ and $B_2 \cap M$ is the relative homology*

$$H_k(B_2 \cap M, B_1 \cap M).$$

Proof. This follows from the exact sequence

$$\begin{aligned} \to H_k(B_1 \cap M) &\xrightarrow{i} H_k(B_2 \cap M) \to H_k(B_2 \cap M, B_1 \cap M) \\ &\xrightarrow{j} H_{k-1}(B_1 \cap M) \xrightarrow{i} H_{k-1}(B_2 \cap M) \to \end{aligned}$$

and the fact that i is injective by the injectivity property of tautness, so that j is surjective. It follows that

$$H_k(B_2 \cap M) = H_k(B_1 \cap M) \oplus H_k(B_2 \cap M, B_1 \cap M),$$

and hence the conclusion. $\square$

Corollary 61. *If $B_1 \subset B_2 \subset B_3$ are three closed disks, then*

$$H_k(B_3 \cap M, B_1 \cap M) = H_k(B_3 \cap M, B_2 \cap M) \oplus H_k(B_2 \cap M, B_1 \cap M).$$

Thus, $H_k(B_3 \cap M, B_1 \cap M)$ is surjective to both $H_k(B_3 \cap M, B_2 \cap M)$ and $H_k(B_2 \cap M, B_1 \cap M)$.

Proof. Given three abelian groups $A \to B \to C$, where each arrow is an embedding, then $C/B = (C/A)/(B/A)$. □

Lemma 62. *Suppose $\dim M = 4$.*
(a) *If the maximum number of principal curvatures is ≥ 3, then multiplicities $(1,3),(3,1)$ and $(2,2)$ cannot exist on open sets.*
(b)*If the maximum number of principal curvatures is 4, then the points with multiplicities $(1,1,2),(1,2,1)$ and $(2,1,1)$, at which the number of principal curvatures is not locally constant, cannot be approached by a sequence coming from an open set of points of the same multiplicities.*

Proof. (a) Suppose the multiplicities are $(1,3)$ on an open set O. Let $x \in O$ and let p be a boundary point of O. Let $c(t)$ be a smooth curve with $c(0) = x$ and $c(1) = p$. We can assume that $c(t) \in O$ for $0 \leq t < 1$. Otherwise, let t_0 be the first value of t such that $q = c(t_0)$ is not in O, and replace p by q.

Since p is a boundary point of O, there must be a sequence of points with multiplicities $(1,1,2)$, $(1,2,1)$ or $(1,1,1,1)$ that converges to p. The multiplicities must remain $(1,3)$ at p; otherwise, the list of multiplicities would drop to the single multiplicity (4), and this is impossible, because a taut hypersurface with a single umbilic point must be a totally umbilic sphere. We will handle the case where a sequence of points with multiplicities $(1,1,2)$ approaches p. The other cases are very similar.

At each point $c(t), 0 \leq t < 1$, the curvature surface $S(t)$ corresponding to the principal curvature λ of multiplicity 3 is a 3-dimensional metric sphere which is the intersection of the corresponding 4-dimensional curvature sphere $\Sigma(t)$ with a 4-dimensional plane $P(t)$. As we take the limit as t approaches 1, these 4-dimensional planes $P(t)$ approach a limiting 4-plane $P(1)$ that intersects the 4-dimensional curvature sphere $\Sigma(1)$ in a 3-dimensional sphere $S(1)$, which must be the 3-dimensional curvature surface corresponding to λ at p. On the other hand, consider a sequence of points $\{y_i\}$ where the multiplicities are $(1,1,2)$ that approaches p. For each y_i, the 2-dimensional curvature surface C_i through y_i corresponding to the principal curvature of multiplicity 2 is a topset for the spherical distance function centered at the corresponding focal point. Thus, by tautness, C_i represents a nontrivial 2-dimensional homology class in M. As y_i approaches p, these C_i approach a 2-cycle C in the curvature surface $S(1)$ that is nontrivial in the 2-dimensional homology of $S(1)$. This is a contradiction, since $S(1)$ is a 3-sphere and has trivial 2-dimensional homology. A similar proof shows that multiplicities $(3, 1)$ cannot exist on an open set.

Next suppose that the multiplicities are $(2, 2)$ on an open set O. As in the argument above, let $x \in O$ and let p be a boundary point of O. Let $c(t)$ be a smooth curve with $c(0) = x$, $c(1) = p$ and $c(t) \in O$ for $0 \leq t < 1$. Then the multiplicities must be $(2, 2)$ at p, because neither of the two multiplicities can drop to 1 at the limit point p and the single multiplicity (4) is impossible. Since p is a boundary point of O, there must be a sequence of points with multiplicities $(1, 1, 2)$, $(2, 1, 1)$ or $(1, 1, 1, 1)$ that converges to p. We will handle the case where the multiplicities are $(1, 1, 2)$, and the others are handled similarly.

Then as in the argument above, at each point $c(t), 0 \leq t < 1$, the curvature surface $S(t)$ corresponding to the first principal curvature λ of multiplicity 2 is a 2-dimensional metric sphere which is the intersection of the corresponding 4-dimensional curvature sphere $\Sigma(t)$ with a 3-dimensional plane $P(t)$. As we take the limit as t approaches 1, these 3-dimensional planes $P(t)$ approach a limiting 3-plane $P(1)$ that intersects the 4-dimensional curvature sphere $\Sigma(1)$ in a 2-dimensional sphere $S(1)$, which must be the 2-dimensional curvature surface corresponding to λ at p. On the other hand, there is a sequence $\{y_i\}$ approaching p such that the multiplicities are $(1, 1, 2)$ at y_i. For each y_i, the 1-dimensional curvature surface C_i through y_i corresponding to the first principal curvature of multiplicity 1 is a topset for the spherical distance function centered at the corresponding focal point. Thus, C_i represents a nontrivial 1-dimensional homology class in M. As y_i approaches p, these C_i approach a 1-cycle C in the curvature surface $S(1)$ that is nontrivial in the 1-dimensional homology of $S(1)$. This is a contradiction, since $S(1)$ is a 2-sphere and has trivial 1-dimensional homology.

(b) In the case of multiplicities $(1, 1, 2)$ or $(2, 1, 1)$, the same type of argument takes care of this statement, since one can produce a point p with the given multiplicities that is also a limit of a sequence of points with multiplicities $(1, 1, 1, 1)$. Then the nondegenerate critical manifold at p corresponding to the principal curvature of multiplicity two is a metric 2-sphere S. However, since p can be approached by a sequence of points with multiplicities $(1, 1, 1, 1)$, one can produce a top 1-cycle C in S that is nontrivial in the 1-dimensional homology of S, which is a contradiction.

In the case of multiplicities $(1, 2, 1)$, we need to modify the argument slightly. As above, the nondegenerate critical manifold at p corresponding to the principal curvature of multiplicity two is a metric 2-sphere S, and we can also approach p by a sequence of points with multiplicities

$(1,1,1,1)$. However, in this case, the 1-dimensional circles approaching p corresponding to the second and third multiplicities are not top sets. We can see that they still contribute in a nontrivial way to homology by the following argument.

At the point p, the multiplicities are $(1,2,1)$, and p is a limit point of the open connected set O on which the multiplicities are $(1,2,1)$. Let q be the second focal point of M at p corresponding to the principal curvature of multiplicity two. As above, we can also approach p by a sequence of points in the open set U with multiplicities $(1,1,1,1)$. Thus near p, we can find points $x \in O$ and $y \in U$, with corresponding second focal points u and v near q such that the height functions $\ell_u(x) = a$, $\ell_v(y) = b$, and positive numbers ϵ and δ such that

(8.6)
$$M_{a-\epsilon}(\ell_u) \subset M^-_{b-\delta}(\ell_v) \subset M_b(\ell_v), \quad M_a(\ell_u) \subset M^-_{b+\delta}(\ell_v) \subset M_{a+\epsilon}(\ell_u).$$

Furthermore, using a genericity argument, we can assume that the points x and y are chosen so that the critical submanifolds of the corresponding height functions ℓ_u and ℓ_v are at distinct levels, and we may choose ϵ and δ sufficiently small so that a (respectively b) is the only critical value between $a-\epsilon$ and $a+\epsilon$ (respectively, between $b-\delta$ and $b+\delta$).

Lemma 60 says that the new k^{th} homology between the levels $a-\epsilon$ and $a+\epsilon$ of ℓ_u is

$$H_k(M_{a+\epsilon}(\ell_u), M_{a-\epsilon}(\ell_u)). \tag{8.7}$$

Corollary 61 says that the group in equation 8.7 is surjective to

$$H_k(M_{a+\epsilon}(\ell_u), M_{b-\delta}(\ell_v)), \tag{8.8}$$

where in Lemma 60, we take

$$B_1 \cap M = M_{a-\epsilon}(\ell_u), B_2 \cap M = M_{b-\delta}(\ell_v), B_3 \cap M = M_{a+\epsilon}(\ell_u).$$

The same lemma says that the group in equation 8.8 is surjective to

$$H_k(M_{b+\delta}(\ell_v), M_{b-\delta}(\ell_v)), \tag{8.9}$$

where

$$B_1 \cap M = M_{b-\delta}(\ell_v), B_2 \cap M = M_{b+\delta}(\ell_v), B_3 \cap M = M_{a+\epsilon}(\ell_u).$$

It follows from these considerations that the group in equation 8.7 is surjective to the group in equation 8.9. However, by Morse-Bott critical point theory (see [7, Theorem 20.2, p. 239]), we have

$$H_k(M_{a+\epsilon}(\ell_u), M_{a-\epsilon}(\ell_u)) = H_{k-\mu}(W), \tag{8.10}$$

where W is the critical manifold of ℓ_u at x and μ is the index at x. Similarly, we have

$$H_k(M_{b+\delta}(\ell_v), M_{b-\delta}(\ell_v)) = H_{k-\nu}(V), \tag{8.11}$$

where V is the critical submanifold of ℓ_v at y and ν is the index at y.

Now the critical submanifold W at x is a 2-sphere of index $\mu = 1$, whereas the critical submanifold V at y is a circle of index $\nu = 1$. Using $k = 2$ in equations 8.10 and 8.11, we see that $0 = H_1(W, \mathbf{Z}_2)$ is surjective to $H_1(V, \mathbf{Z}_2) = \mathbf{Z}_2$. This is a contradiction. $\square$

Theorem 63. *Let M be a compact taut hypersurface in S^n that is not a hypersphere.*
(I) *If $\dim M = 3$, then M is algebraic.*
(II) *If $\dim M = 4$, then M is algebraic.*

Proof. (I) We first establish that generically, i.e., on a dense open set, there is only one sequence of multiplicities. For $\dim M = 3$, if no points assume multiplicities $(1, 1, 1)$, then we have the proper Dupin case of two distinct principal curvatures at every point. This follows from the fact that no umbilic points can exist, since M is not a hypersphere. In this case, M is algebraic by Corollary 32.

In the case where there is an open set on which the multiplicities are $(1, 1, 1)$, the same type of argument used in the multiplicities $(1, 3)$ case in Lemma 62 shows that the multiplicities $(2, 1)$ and $(1, 2)$ cannot exist on open sets. Thus, the multiplicities are $(1, 1, 1)$ on the dense open subset $\mathcal{G}$ of M.

Now, consider $A := (U_1^*)^+$ and $B := (U_1^*)^-$. The set A consists of points of multiplicities $(1, 1, 1)$ or $(1, 2)$ and B with multiplicities $(1, 1, 1)$ or $(2, 1)$. The sets A and B both have finitely many connected components, and their union is all of M. Let $W := A \cap B$ be the set on which the multiplicities are $(1, 1, 1)$. The Mayer-Vietoris sequence

$$\leftarrow H^1(M) \leftarrow H^0(W) \leftarrow H^0(A) \oplus H^0(B) \leftarrow H^0(M) \leftarrow 0$$

gives that $W = \mathcal{G}$ has finitely many connected components. Hence, the local finiteness property holds on M, and M is algebraic by Theorem 37.

(II)(a) Let $\dim M = 4$. First consider the case where the maximum number of distinct principal curvatures is two. Since M is not a hypersphere, there cannot be an umbilic point, and therefore the number of distinct principal curvatures must be two at all points. Thus, M is proper Dupin, and M is algebraic by Corollary 32.

Next consider the case when the maximum number of distinct principal curvatures is three. Note, in general, that the set assuming multiplicities $(1,1,2)$, $(1,2,1)$ or $(2,1,1)$ is automatically open by the maximality of the number of distinct principal curvatures. As a consequence, only the multiplicities $(1,1,2),(1,2,1),(2,1,1)$ can possibly exist on open sets by Lemma 62.

Consider $A := (U_1^*)^+, B := (U_2^*)^+$ and $C := (U_3^*)^+$. The set A consists of points with multiplicities $(1,1,2),(1,2,1),(1,3)$, B with multiplicities $(1,1,2),(2,1,1),(2,2)$, and C with multiplicities $(1,2,1)$, $(2,1,1)$, $(3,1)$. Each of A,B,C has finitely many connected components, and M is the union of these three sets.

Consider A and $D := B \cup C$. Being a union, D also has a finite number of connected components. Now $M = A \cup D$ and $W := A \cap D = U \cup V$, where U is the open set of points with multiplicities $(1,1,2)$ and V is the open set of points with multiplicities $(1,2,1)$. Hence, the Mayer-Vietoris sequence applied to A and D implies that W has finitely many connected components. As the open sets U and V are disjoint, it follows that each of U and V has finitely many connected components. A similar consideration establishes that the set of points with multiplicities $(2,1,1)$ also has a finite number of components by considering B and $A \cup C$. Hence, the set $\mathcal{G}$ has finitely many connected components. Therefore, the local finiteness property holds on M, and M is algebraic by Theorem 37.

(II)(b) Assume now that the maximum number of multiplicities is 4. By Lemma 62, multiplicities $(1,3),(3,1)$ and $(2,2)$ cannot exist on open sets. Meanwhile, the set of points with multiplicities $(1,1,1,1)$ is open.

Since by Lemma 62, a point with multiplicities $(1,1,2)$ (or $(1,2,1)$ or $(2,1,1)$) at which the number of principal curvatures is not locally constant cannot be approached by a sequence coming from an open set of points of the same multiplicities, we see that such points must be entirely surrounded by points with multiplicities $(1,1,1,1)$. As a result, an open set of points with multiplicities $(1,1,2)$ will approach boundary points with multiplicities either $(1,3)$ or $(2,2)$.

The set $\mathcal{G}$ of good points are those with multiplicities $(1,1,1,1)$ and the points with multiplicities $(1,1,2),(1,2,1)$, or $(2,1,1)$ that exist on open sets. $\mathcal{G}$ is open and dense in M. We let S be the subset of $\mathcal{G}^c$ consisting of points of multiplicities $(1,3),(3,1)$ or $(2,2)$. We wish to establish that the local finiteness property is true.

Firstly, S is closed in M. This is because any converging sequence of points of the indicated multiplicities must maintain the same type of multiplicities.

We next show that $M \setminus S$ has finitely many connected components. It comes down to showing that $A := (U_1^*)^+$ (respectively, $B := (U_2^*)^+$

or $C := (U_3^*)^+$), with points of multiplicities $(1,3)$ (respectively, $(2,2)$ or $(3,1)$) removed, has only finitely many connected components; for then the union of the three resulting sets is exactly $M \setminus S$, which, being a union, must have finitely many connected components. However, as before, this follows from the Mayer-Vietoris sequence applied to the open covers A and $B \cup C$ of M, etc. Note that A is composed of points with multiplicities $(1,1,1,1), (1,1,2), (1,2,1)$ or $(1,3)$, B with multiplicities $(1,1,1,1), (1,1,2), (2,1,1)$ or $(2,2)$, and C with multiplicities $(1,1,1,1), (1,2,1), (2,1,1)$ or $(3,1)$. Hence, $A \cap (B \cup C)$ is exactly A with points of multiplicities $(1,3)$ removed.

Lastly, we verify that each point in $\mathcal{G}^c \setminus S$ has a open neighborhood W in M such that $W \cap \mathcal{G}$ contains finitely many connected open sets whose union is dense in W.

As mentioned earlier, $\mathcal{G}^c \setminus S$ is completely surrounded by points of multiplicities $(1,1,1,1)$. For each point $p \in \mathcal{G}^c \setminus S$ of multiplicities $(1,1,2)$, there is a small neighborhood of it which contains no points of multiplicities $(1,2,1)$ or $(2,1,1)$, etc. In other words, The sets T_1, T_2, T_3 of points of multiplicities $(1,1,2), (1,2,1), (2,1,1)$, respectively, are contained in disjoint open sets O_1, O_2, O_3, respectively, where $O_i \setminus T_i, 1 \leq i \leq 3$, consists of only points of multiplicities $(1,1,1,1)$.

Let $X \subset (U_3)^+$ be the subset of points of multiplicities $(1,1,1,1)$. X is dense in the set of points of multiplicities $(1,1,1,1)$ by Lemmas 57 and 59. It follows that T_1 is contained in the end sets of the finitely many end components $E_1, \ldots, E_s$ of U_3^+ on which the multiplicities remain $(1,1,1,1)$. Let W be the union of O_1 above and $E_1, \ldots, E_s$. (O_1 is the union of neighborhoods of points of multiplicities $(1,1,2)$. We make sure each of these neighborhoods is so small that its intersection with U_3^+ is contained in the end components $E_1, \ldots, E_s$.) Then $W \cap \mathcal{G}$ consists of only points of multiplicities $(1,1,1,1)$, and moreover contains open sets $E_1, \ldots, E_s$ whose union is dense in W.

Similarly, the same conclusion is true for points of multiplicities $(2,1,1)$ and $(1,2,1)$ in $\mathcal{G}^c$ with U_3^+ replaced by U_1^+ and U_2^+, respectively. Hence, M is algebraic. $\square$

Remark 64. In fact, the proof of (I) and (II)(a) are the same as that of (II)(b) in a hidden way. For (II)(a), $\mathcal{G}^c$ is the set of points with multiplicities $(1,3), (3,1)$ or $(2,2)$, which coincides with S, so that the set $\mathcal{G}^c \setminus S = \emptyset$, and the local finiteness condition is automatically satisfied on $\mathcal{G}^c \setminus S$. Similarly for (I), we have $\mathcal{G}^c = S$, which is the set of points with multiplicities $(1,2)$ or $(2,1)$.

Recall that an embedding $f : M \to \mathbf{R}^n$ is said to be *substantial* if the image of f does not lie in any affine hyperplane in $\mathbf{R}^n$.

Theorem 65. *Let M be a compact taut m-dimensional submanifold in $S^n \subset \mathbf{R}^{n+1}$. If $m \leq 4$, then M is algebraic.*

Proof. Banchoff [1] showed that a taut compact 1-dimensional submanifold of S^n must be a metric circle in S^n, which is certainly algebraic. In the same paper, he also showed that if M is a taut compact 2-dimensional surface substantially embedded in S^n, then M is a metric 2-sphere, a cyclide of Dupin in S^3, or a spherical Veronese surface $V \subset S^4$. All of these surfaces are algebraic.

Next let M be a compact taut 3-dimensional submanifold of codimension $k+1$ in $S^n \subset \mathbf{R}^{n+1}$. Without loss of generality, we may assume that M is substantially embedded in $\mathbf{R}^{n+1}$. We want to consider the tube M_ϵ of radius $\epsilon > 0$ over M in S^n, where ϵ is sufficiently small so that M_ϵ is an embedded hypersurface in S^n. By a theorem of Pinkall [19], we know that M_ϵ is a taut hypersurface in S^n. We can parametrize M_ϵ by the map $\phi_\epsilon : UN(M) \to S^n$, where $UN(M)$ is the unit normal bundle of M in S^n, given by

$$\phi_\epsilon(x, \xi) = \cos \epsilon x + \sin \epsilon \xi,$$

where ξ is a unit normal vector to M at x. Then

$$\eta = -\sin \epsilon x + \cos \epsilon \xi,$$

is a unit normal vector to the tube M_ϵ at the point $y = \phi_\epsilon(x, \xi)$. In [6, pp. 131-132] a formula is given for the principal curvatures of the shape operator A_η of M_ϵ at y in terms of the shape operator A_ξ of M at x. There it is shown that A_η has a principal curvature $\mu_0 = \cot(-\epsilon)$ of multiplicity k and principal curvatures

$$\mu_i = \cot(\theta_i - \epsilon), \quad 1 \leq i \leq 3,$$

where $\lambda_i = \cot \theta_i, 1 \leq i \leq 3$, for $0 < \theta_3 < \theta_2 < \theta_1 < \pi$, are the principal curvatures of A_ξ at x. Thus each $\mu_i, 1 \leq i \leq 3$, has the same multiplicity as the corresponding λ_i. Therefore, on M_ϵ the multiplicities of the principal curvatures must take the form $(k, 1, 1, 1), (k, 1, 2)$ or $(k, 2, 1)$. Note that $(k, 3)$ is not possible, for then A_ξ would have a principal curvature of multiplicity 3 at x, i.e., all the principal curvatures of A_ξ would be equal at x. Then, if p is the first focal point along the normal geodesic in S^n from x in the direction ξ, tautness implies that the height function ℓ_p must have both an absolute maximum and an absolute minimum at x (see [6, Lemma 1.24, p. 122]). Thus all of M lies in in hyperplane in $\mathbf{R}^{n+1}$ orthogonal to p. This contradicts the assumption that M is substantial in $\mathbf{R}^{n+1}$, and so the multiplicities $(k, 3)$ cannot occur on M_ϵ.

As in the 3-dimensional hypersurface case in Theorem 63, multiplicities $(k, 1, 2)$, respectively, $(k, 2, 1)$, cannot exist on open sets in M_ϵ, unless the multiplicities have the constant values $(1, 2)$, respectively, $(2, 1)$, on the unit normal bundle $UN(M)$ of M. In that case, the tube M_ϵ is a proper Dupin hypersurface, and so M_ϵ and M are algebraic by Theorem 37. Hence, the only remaining case is when multiplicities have the values $(1, 1, 1)$ on a dense open subset of $UN(M)$.

Then another application of the Mayer-Vietoris sequence applied to the sets $(U_1^*)^+$ and $(U_2^*)^+$ for the tube M_ϵ shows that the generic multiplicities $(k, 1, 1, 1)$ exist on only finitely many connected components. Thus, M_ϵ satisfies the local finiteness property and is algebraic, and so M is algebraic by Theorem 37.

Suppose now that M is a compact taut 4-dimensional submanifold substantially embedded in $S^n \subset \mathbf{R}^{n+1}$ with codimension $k + 1$ in S^n. By the same construction as above, the possible multiplicities on M_ϵ are

$$(k, 1, 1, 1, 1), (k, 1, 1, 2), (k, 1, 2, 1), (k, 2, 1, 1), (k, 1, 3), (k, 3, 1), (k, 2, 2).$$

Then the same arguments used to prove Lemma 62 and Theorem 63 (II) can be applied here by just adjoining the multiplicity k at the beginning, and thus we conclude that M_ϵ is algebraic, and so M is algebraic.

□

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