Analytic Controllability of Time-dependent Quantum Control Systems

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Abstract

The question of controllability is investigated for a quantum control system in which the Hamiltonian operator components carry explicit time dependence which is not under the control of an external agent. We consider the general situation in which the state moves in an infinite-dimensional Hilbert space, a drift term is present, and the operators driving the state evolution may be unbounded. However, considerations are restricted by the assumption that there exists an analytic domain, dense in the state space, on which solutions of the controlled Schrödinger equation may be expressed globally in exponential form. The issue of controllability then naturally focuses on the ability to steer the quantum state on a finite-dimensional submanifold of the unit sphere in Hilbert space – and thus on analytic controllability. A relatively straightforward strategy allows the extension of Lie-algebraic conditions for strong analytic controllability derived earlier for the simpler, time-independent system in which the drift Hamiltonian and the interaction Hamiltonia have no intrinsic time dependence. Enlarging the state space by one dimension corresponding to the time variable, we construct an augmented control system that can be treated as time-independent. Methods developed by Kunita can then be implemented to establish controllability conditions for the one-dimension-reduced system defined by the original time-dependent Schrödinger control problem. The applicability of the resulting theorem is illustrated with selected examples.

1 Introduction

Over the last two decades, quantum control has played an important part in theoretical and experimental progress toward the realization of laser control of chemical reactions and the development of quantum computers [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. Essential to this contribution has been the integration of concepts and mathematical results from control engineering with the fundamental principles of quantum theory.

Geometric control, a treatment of differential equations rooted in differential geometry, unitary groups, and Lie algebras, provides a natural mathematical basis for quantum control theory. Explicitly or implicitly, its elements [14] pervade the manipulation of quantum states in both traditional and novel technologies.

Indeed, the field of nuclear magnetic resonance (NMR) is largely concerned with geometric control of collections of interacting nuclear spins [12, 15, 16, 17]. Geometric control is also a key ingredient in the theory of quantum computation, figuring prominently in the works of Lloyd [18], Deutsch [19], and Akulin [20].

In particular, Lloyd [18] was among the first to establish that almost all quantum logic gates are universal. More precisely, if one has available a gate that can operate on two qubits, plus a single-qubit operation, then an arbitrary unitary transformation on the variables of the system can be performed with arbitrary precision by implementing a finite sequence of local operations. Clark [21] and Ramakrishna and Rabitz [22, 23] called attention to the close relationship between open-loop geometric quantum control methods and the application of quantum logic gates [19, 18].

Following Ref. [23], let us consider differential system

$$\frac{dX(t)}{dt} = AX(t) + \sum_{i=1}^{m} B_i X(t) u_i(t) \, , \qquad X(0) = I \, , \tag{1}$$

which arises both in quantum computing and molecular control. Here, X is a $N \times N$ unitary matrix (I being the corresponding identity matrix), the matrices A and B_i, $i = 1, \ldots, m$ are $N \times N$ skew-Hermitian, and the functions $u_i(t)$ are controls. This equation is the law of motion of the evolution operators which govern time development of the N-dimensional vector representing a pure state of the system in its N-dimensional Hilbert space. A necessary and sufficient condition for (1) to be controllable is that the set of all matrices generated by $A, B_i, i = 1, \ldots, m$, and their commutators (i.e., the Lie algebra generated by A and B_i) equals the set of all $N \times N$ skew-Hermitian matrices. Additionally, when this condition is met, any X can be attained through some choice among the controls $u_i(t)$ restricted to piecewise constant functions of time. In fact, the formulation adopted by Lloyd [18] in his universality proof corresponds to the special case $A = 0$ and $m = 2$ of system (1). Already in the 1970s, Sussmann and Jurdjevic [24, 25] applied Lie-group theory to obtain rigorous results on controllability for finite-dimensional control problems corresponding to (1).

Quantum computation has mostly concerned itself with the manipulation of discrete systems with finite-dimensional state spaces. However, the fundamental quantum observables representing position and momentum, and functions thereof, are continuous in nature. In view of recent developments in quantum error correction [26, 27, 28] and quantum teleportation [29, 30] of continuous variables, the potential of quantum computation over continuous variables warrants serious investigation, thus reopening issues of controllability on infinite-dimensional Hilbert spaces. Continuous quantum computers may in fact be able to perform some tasks more efficiently than their discrete counterparts.

As early as 1983, Huang, Tarn, and Clark (HTC) [5, 31] proved a basic theorem on strong analytic controllability of quantum systems. This theorem explicitly embraces the case of quantum systems whose observables are continuous quantum variables acting on an infinite dimensional state space, but the essential finite-dimensional results may be extracted as special cases. Because of the difficulties caused by infinite-dimensionality and the unboundedness of operators, an analytic domain in the sense of Nelson [32] was introduced to deal with domain problems [5, 31] and maintain key features of the application of Lie algebraic methods to finite-dimensional problems.

Infinite-dimensional control systems have been widely if not systematically studied outside the quantum context. Brockett [14] addressed the problem of realization of infinite-dimensional bilinear systems. Sakawa [33] introduced a method for design of finite-dimensional $\mathcal{H}_\infty$ controllers for diffusion systems with bounded input and output operators by using residual model filters. Keulen [34] designed infinite-dimensional $\mathcal{H}_\infty$ controllers for infinite-dimensional systems with bounded input and output operators by using the solutions to two kinds of Riccati equations in an infinite-dimensional space. Based on gap topology, Morris [35] constructed finite-dimensional $\mathcal{H}_\infty$ controllers for infinite-dimensional systems with bounded input and output operators. Morris [36] also showed that approximations of Galerkin type can be used to design controllers for an infinite-dimensional system. Costa and Kubrusly [37] derived necessary and sufficient conditions for existence of a state feedback controller that stabilizes a discrete-time infinite-dimensional stochastic bilinear system and ensures that the influence of the additive disturbance on the output is smaller than some prescribed bound. In Ref. [38], optimizability and estimatability for infinite-dimensional linear systems are investigated; also, a theorem on the equivalence of input-output stability and exponential stability of well-posed infinite-dimensional linear systems is established. In Ref. [39], the Hilbert-space generalization of the circle criterion is used for finite-dimensional controller design of unstable infinite-dimensional systems. There is also literature on absolute stability problems and open-loop stability problems in infinite-dimensional systems [40, 41, 42, 43, 44]. In addition, the spectral factorization problem plays a central role in designing feedback control for the linear quadratic optimal control problem in infinite-dimensional state-space systems [45, 46, 47, 48]. In contrast to this body of work, very little has been published on controllability for time-dependent infinite-dimensional quantum control systems.

In the microscopic world ruled by quantum mechanics, most interesting phenomena involve change, and all real-world quantum systems are influenced to a greater or lesser extent by interactions with their environments. The environment changes with time, so the Hamiltonians used to describe these open quantum systems are explicitly time-dependent, as in Ref. [50, 51]. Tailored time-dependent perturbations are used to improve system performance [51] in high-resolution NMR spectroscopy, where versatile decoupling techniques are available to manipulate the overall spin Hamiltonian [16]. Colegrave and Abdalla studied quantum systems with a time-dependent mass to investigate the field intensities in a Fabry-Perot cavity [52]. They suggested possible applications to solid-state physics and quantum field theory [53]. Remaud and Hernandez [54] found that a time-dependent mass parameter offers a means of simulating input or removal of energy from the system. Implementation of controls on these time-dependent quantum systems requires guidance from mathematical studies of controllability for time-dependent Hamiltonian operators. Although the HTC theorem deals with controllability in infinite-dimensional Hilbert space, it is restricted to time-independent operators. This paper explores a more general case. We seek an extension of the HTC theorem that is applicable both to time-independent and time-dependent quantum systems, as well as to systems with discrete or continuous operators acting on finite- or infinite-dimensional state spaces.

Since this paper is aimed at an interdisciplinary readership that includes pure quantum theorists as well as control engineers, it is well to draw a clear distinction between time dependence of the system arising solely from influences that are directly under the control of an external, purposeful agent, and time dependence that is intrinsic to the physical system either in isolation or as embedded in a natural environment. In the accepted terminology of control theory, which we adopt, the former case defines a time-independent control system, and the latter, a time-dependent system. The issue of controllability has received considerable attention in the time-independent situation so identified (e.g., in Refs. [5, 8, 22, 12]); whereas relevant results for the time-dependent case are very limited.

The time-dependent quantum control problem that we shall address is stated formally in Sec. 2. To cope with the unboundedness of operators involved in the Schrödinger equation, an analytic domain is introduced in Sec. 3, such that solutions of the Schrödinger equation can be expressed globally in exponential form on this domain. In Sec. 4, we define an augmented system in a space enlarged by one dimension, enabling its description within the framework of time-independent control systems. Following the pattern of Kunita's proof [55] of strong controllability of a time-independent system, we then establish conditions for controllability of this kind for the one-dimension-reduced system defined by the original time-dependent Schrödinger equation. Three illustrative applications of the theorem are presented in Sec. 5, and our findings are reviewed in Sec. 6.

2 Problem Formulation

The following quantum control system is derived by applying the geometric quantization method [56] to a classical bilinear control system [57, 31]:

$$
\begin{aligned}
i\hbar\frac{\partial}{\partial t}\psi(t) &= \left[H_0'(t) + \sum_l u_l(t)H_l'(t)\right]\psi(t), \\
\psi(t_0) &= \psi_0\,.
\end{aligned}
\tag{2}
$$

Here, $H_0'(t)$, and the $H_l'(t)$ with $l = 1, 2, \ldots, r$, are Hermitian operators on a unit sphere $S_{\mathcal{H}}$ of Hilbert space, the $u_l(t)$, $l = 1, \ldots, r$ are restricted to piecewise-constant real functions of time, and $\psi(t)$ denotes a quantum state belonging to $S_{\mathcal{H}}$. In physical language, H_0' is the unperturbed or autonomous Hamiltonian, and the H_l' are interaction Hamiltonians. It is the coefficients $u_l(t)$ that are subject to purposeful control by an agent external to the system, within the specified class of functions. Setting $\hbar = 1$ and dividing $H_0'(t)$ and the $H_l'(t)$ by i, we arrive at a more familiar control form,

$$\frac{\partial}{\partial t}\psi(t) = \left[H_0(t) + \sum_l u_l(t) H_l(t)\right]\psi(t)\,, \qquad \psi(t_0) = \psi_0 \in S_{\mathcal{H}}\,, \tag{3}$$

where the $H_i(t)$, $i = 0, 1, 2, \ldots, r$, are skew-Hermitian operators on $S_{\mathcal{H}}$. From the standpoint of systems engineering, $H_0(t)$ is called the drift term in Eq. (3) because no control function directly modifies its action. Importantly, we depart from previous studies of quantum controllability in allowing the Hamiltonian operators $H_i(t)$ to their own carry explicit time dependence, which is assumed to be inherent in the physical structure of the system and therefore beyond the control of any external agent. The operators $H_i(t)$ are the counterparts of the structural matrices involved in standard formulations of linear control theory.

For the system (3), we know from arguments presented in Ref. [5] that the transitivity of states on $S_{\mathcal{H}}$ requires an infinite sequence of control manipulations within the control set $\{u_l(t)\}$ of piecewise-constant real functions. Clearly, such a process is strictly meaningless in practice, although under certain conditions it may be possible to find a finite series of control operations that approach the desired target state arbitrarily closely. Even so, we are naturally directed to consider the issue of controllability on a *finite-dimensional* submanifold of the unit sphere $S_{\mathcal{H}}$, for which in turn a finite-dimensional tangent space is generated by $H_0(t)\psi(t), \ldots, H_r(t)\psi(t)$.

Accordingly, our attention focuses on a finite-dimensional submanifold $M \subset S_{\mathcal{H}}$, on which the following dynamics prevail

$$\frac{\partial}{\partial t}\psi(t) = \left[H_0(t) + \sum_l u_l(t) H_l(t)\right]\psi(t)\,,$$
$$\psi(t_0) = \psi_0,\ \psi(t) \in M,\ \forall t \geq t_0\,, \tag{4}$$

Thus, instead of studying controllability on $S_{\mathcal{H}}$, we consider controllability on $M \subset S_{\mathcal{H}}$. On the submanifold M, the inherited topology of $S_{\mathcal{H}}$ still applies; hence it is paracompact and connected.

For system (4), we have available a set of vector fields $O(M)$ composed of skew-Hermitian operators on M with Lie algebra defined by $O(M) = \mathcal{L}\{H_0, \ldots, H_r\}$. Let V be a subset of $O(M)$. The Lie algebra generated by V is denoted by $\mathcal{L}(V)$. The restriction of $\mathcal{L}(V)$ to a point ψ on M, which is a tangent subspace of TM_ψ at ψ, is written as

$$\mathcal{L}(V)(\psi) = \{Y\psi | Y \in \mathcal{L}(V)\} \subset TM_\psi\,, \tag{5}$$

while

$$\tilde{\mathcal{L}}(V) = \{\mathcal{L}(V)\psi | \psi \in M\} \tag{6}$$

defines an involutive differential system. A vector field X is said to belong to $\tilde{\mathcal{L}}(V)$ if $X(\psi) \in \tilde{\mathcal{L}}(V)(\psi)$ holds for all $\psi \in M$.

3 Selecting the Domain

Recognizing that operators in quantum mechanics are in general unbounded, we need to find a domain on which exponentiations of the operators entering the system (4) converge. To this end, we introduce the so-called analytic domain conceived by Nelson [32], a dense domain invariant under the action of the operators in system (3). The solution of the Schrödinger equation can be expressed globally in exponential form on this domain, which is also invariant under the action of the exponentiations of the operators H_i.

Definition 3.1 *If H is an operator on the state space $\mathcal{H}$, we call an element ω of $\mathcal{H}$ an analytic vector for H in case the series expansion of* $\exp(Ht)\omega$ *has a positive radius of absolute convergence, that is, provided*

$$\sum_{n=0}^{\infty} \frac{||H^n \omega||}{n!} s^n < \infty \tag{7}$$

for some $s > 0$.

If H is a bounded operator, then every vector in $\mathcal{H}$ is trivially an analytic vector for H.

The corresponding definition of analytic vectors for a Lie algebra of operators runs as follows [32, 58]:

Definition 3.2 *A vector $\omega \in \mathcal{H}$ is said to be an analytic vector for the whole Lie algebra $\mathcal{L}$ if for some $s > 0$ and some linear basis $\{H_1, \ldots, H_d\}$ of the Lie algebra, the series*

$$\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{1 \leq i_1, \ldots, i_n \leq d} ||H_{i_1} \ldots H_{i_n} \omega|| s^n \tag{8}$$

is absolutely convergent.

The concept of analytic vectors is especially useful for our purposes, since for certain types of unbounded operators they form a dense set in the Hilbert space. In fact, the set of all analytic vectors for a Lie algebra $\mathcal{L}$ forms an analytic domain in the following sense [32, 58].

Definition 3.3 *Let $\mathcal{L}$ be the Lie algebra generated by the skew-Hermitian operators $H_0, \ldots, H_r$ on a unit sphere $S_{\mathcal{H}}$ of Hilbert space. An analytic domain $\mathcal{D}_A$ is said to exist for the H_i, $i = 0, 1, \ldots r$, if (i) there exists a common dense invariant subspace $\mathcal{D}_A \subset \mathcal{H}$ on which the corresponding unitary Lie group G can be expressed locally in exponential form with Lie algebra $\mathcal{L}$, (ii) $\mathcal{D}_A$ is invariant under G and $\mathcal{L}$, and (iii) on $\mathcal{D}_A$, elements of G can be extended globally to all $t \in \mathbb{R}^+$.*

We now state Nelson's fundamental theorem, which provides conditions under which a Lie algebra $\mathcal{L}$ defined by a set of skew-Hermitian operators can be associated with a unitary group G having $\mathcal{L}$ as its Lie algebra.

Theorem 3.1 *(Nelson) Let $\mathcal{L}$ be a Lie algebra of skew-Hermitian operators in a Hilbert space $\mathcal{H}$ which have a common invariant dense domain $\mathcal{D}_A$. Let $X_1, \ldots, X_d$ be an operator basis for $\mathcal{L}$. If $T = X_1^2 + \ldots + X_d^2$ is essentially self-adjoint, then there is a unique unitary group G in $\mathcal{H}$ with Lie algebra $\mathcal{L}$. Let $\overline{T}$ denote the unique self-adjoint extension of T. Then the analytic vectors of $\overline{T}$ are analytic vectors for the whole Lie algebra $\mathcal{L}$ and form a set invariant under G and dense in $\mathcal{H}$.*

Accordingly, on the analytic domain $\mathcal{D}_A$, the Lie algebra and its unitary Lie group are related through the familiar exponential formula. The Lie algebra is composed of skew-Hermitian operators which are vector fields defined on $\mathcal{D}_A \cap S_{\mathcal{H}}$. By property (iii) of the definition 3.3 of the analytic domain, these vector fields on $\mathcal{D}_A \cap S_{\mathcal{H}}$ are complete. Moreover, owing to the skew-Hermiticity of the operators H_i of system (3), the corresponding transformation groups, taking a given state on $S_{\mathcal{H}}$ to another state on $S_{\mathcal{H}}$, are unitary. This feature guarantees preservation of the norm of quantum states, as required for the statistical interpretation of quantum mechanics.

In fact, Nelson's theorem only provides sufficient conditions for the important properties it yields. With this in mind, we shall assume an analytic domain $\mathcal{D}_A$ exists *without* explicitly imposing the conditions stated in this theorem, a stance also adopted in Ref. [5] This strategy clearly implies that the existence of such a domain must be established explicitly prior to application of the controllability results to be derived in the following sections.

We are now prepared to adapt the concept of controllability to problems involving unbounded operators.

Definition 3.4 *For system (3), if $\mathcal{D}_A$ exists for $\mathcal{L}$, and if for any ψ_0 and $\psi_f \in \mathcal{D}_A \cap S_{\mathcal{H}}$ there exist control functions $u_1(t), \ldots, u_r(t)$, and a time t_f [resp. $\forall t_f$] such that the solution of control system (3) satisfies $\psi(t_0) = \psi_0$, $\psi(t_f) = \psi_f$, and $\psi(t) \in \mathcal{D}_A \cap S_{\mathcal{H}}$, where $t_0 \leq t \leq t_f$, then the system is called analytically controllable [resp. strongly analytically controllable] on $S_{\mathcal{H}}$; moreover we then say that the corresponding unitary Lie group is analytically transitive on $S_{\mathcal{H}}$.*

As has been argued, the more pertinent concept is controllability on the submanifold M of $S_{\mathcal{H}}$. By assumption, $M \cap \mathcal{D}_A$ is dense in M, while $\dim(M \cap \mathcal{D}_A) = \dim M = m$. Denoting the tangent space of $M \cap \mathcal{D}_A$ at ψ by $TM_\psi = \mathcal{L}\{H_0, \ldots, H_r\}\psi$, the tangent bundle of the system (4) is given by $T(M \cap \mathcal{D}_A) = \cup_{\psi \in M \cap \mathcal{D}_A} TM_\psi$.

Let $R_t(\psi)$ denote the set of all points that are reachable from ψ at time t. The set $R(\psi) = \bigcup_{t > t_0} R_t(\psi)$ is then reachable from ψ at some time greater than t_0. We say that system (4) is analytically controllable on M if $R(\psi) = M \cap \mathcal{D}_A$, $\forall \psi \in M \cap \mathcal{D}_A$, and that the system is strongly analytically controllable on M if $R_t(\psi) = M \cap \mathcal{D}_A, \forall t > t_0$, $\forall \psi \in M \cap \mathcal{D}_A$.

4 Controllability of Time-dependent Quantum Control Systems

4.1 Reformulation as a Time-independent Augmented System

Most of the methods developed for determining controllability of time-independent bilinear or nonlinear systems [59, 60, 5, 31, 61, 62] cannot be applied directly to the time-dependent bilinear control problem studied here, since these approaches rely upon the following property. Let $Y_t(\varphi)$ be an integral curve of the time-independent tangent vector Y starting from point φ and $t \in [t_0, t_0 + t_f]$, and let $cY_t(\varphi)$ be an integral curve of the tangent vector cY starting from φ and $t \in [t_0, t_0 + t_f/||c||]$; then the integral curves $Y_t(\varphi)$ and $cY_t(\varphi)$ coincide. This property holds for all time-independent tangent vectors, but it generally fails for time-dependent tangent vectors.

However, recognizing that this feature has been instrumental to controllability proofs for nonlinear systems, we recast the system (4) as a time-independent problem so that it can once again be exploited. Reformulation of the original problem is accomplished by regarding the time variable t as an additional parameter in the specification of the system state, supplementing the state vector ψ. Thus the state of the extended system is expressed as

$$\xi = \left(\begin{array}{c} t + t_0 \\ \psi \end{array} \right) . \tag{9}$$

Making the corresponding extension of the manifold M, we form an augmented $(m+1)$-dimensional manifold defined by

$$N = \left\{ \begin{array}{c} \mathbb{R} \\ M \cap \mathcal{D}_A \end{array} \right\} , \tag{10}$$

where $\mathbb{R}$ is the real line. Next we define augmented vector fields W_l by

$$\begin{aligned} W_0(\xi) &= \left[\begin{array}{c} 1 \\ H_0(t+t_0)\psi(t+t_0) \end{array} \right] , \\ W_l(\xi) &= \left[\begin{array}{c} 0 \\ H_l(t+t_0)\psi(t+t_0) \end{array} \right] , \end{aligned} \tag{11}$$

with $l = 1, 2, \ldots, r$. Obviously, the W_l, with $l = 0, 1, \ldots, r$, depend on both t and ψ, i.e., the W_l now depend on the state ξ defined by Eq. (9).

The time-dependent control system (4) has thereby been reformulated as an augmented system of time-independent form. Explicitly,

$$\begin{aligned} &\frac{\partial \xi(t)}{\partial t} = \left[W_0(\xi) + \sum_l u_l(t) W_l(\xi) \right] , \\ &\xi(0) = \eta = \left(\begin{array}{c} t_0 \\ \psi(t_0) \end{array} \right) = \left(\begin{array}{c} t_0 \\ \psi_0 \end{array} \right) , \\ &\forall t \geq 0,\, \psi_0 \in M \cap \mathcal{D}_A,\, \xi \in N , \end{aligned} \tag{12}$$

where N is the $n = (m+1)$-dimensional manifold constructed in Eq. (10) and M is now viewed as a one-dimension-reduced manifold of the augmented system. As always, the controls $u_l(t)$, with $l = 1, \ldots, r$, are piecewise-constant real functions of time t.

It is convenient to employ $t + t_0$ instead of t in definitions (9) and (11), thereby setting the starting time at zero for the augmented system (12). Since the latter system is time-independent by construction, this can be done without affecting its trajectory. Thus, if the time for the augmented system is t, then the time for the original system (4) is $t + t_0$. Standard differential equation techniques can evidently be employed to analyze the behavior of the augmented system on the manifold N, and the results will reflect the behavior of the original system on manifold M.

We note peripherally that system (12) is in a decomposed form in the sense of Ref. [60], where several theorems were developed for decomposition of nonlinear control systems. However, these theorems do not specify reachable sets, so they cannot be applied here to obtain controllability results.

Reachable sets $\hat{R}_t(\eta)$ and $\hat{R}(\eta)$ are defined for the augmented system (12) in just the same manner as for system (4). From the work of Huang, Tarn, and Clark [5] based on the results of Chow [63], Sussmann and Jurdjevic [24], and Kunita [55, 59], it is to be expected that the issue of analytic controllability will hinge on the relationships among certain Lie algebras generated by the vector fields involved in the control system (4) or its augmented counterpart (12). For the latter problem, these Lie algebras are specified by $\hat{\mathcal{A}} = \mathcal{L}\{W_0, \ldots, W_r\}$, $\hat{\mathcal{B}} = \mathcal{L}\{W_1, \ldots, W_r\}$, and $\hat{\mathcal{C}} = \mathcal{L}\{\mathrm{ad}^m_{W_0} W_l, \, l = 1, \ldots, r, \, m = 0, \ldots, \infty\}$. By definition, $\mathrm{ad}^m_{W_0} W_l$ is built from repeated commutators of W_0, present in $\hat{\mathcal{A}}$ but not $\hat{\mathcal{B}}$, with any and all of the W_l present in $\hat{\mathcal{A}}$ or $\hat{\mathcal{B}}$; clearly,

$$\hat{\mathcal{B}} \subset \hat{\mathcal{C}} \subset \hat{\mathcal{A}} \,. \tag{13}$$

For future reference we note (in particular) that the restriction of $\hat{\mathcal{B}}$ to a point ψ on N, which is a tangent subspace of TN_ψ at ψ, is written as

$$\hat{\mathcal{B}}(\psi) = \{Y(\psi) | Y \in \hat{\mathcal{B}}\} \subset TN_\psi , \tag{14}$$

and in turn that

$$\widetilde{\hat{\mathcal{B}}} = \{\hat{\mathcal{B}}(\psi) | \psi \in N\} \tag{15}$$

is an involutive differential system.

4.2 Controllability of the Augmented System

We must still face the situation that standard controllability results [59, 60, 5, 31, 61, 62], derived for time-independent systems, cannot be carried over directly to our problem as reformulated in the preceding subsection, since derivation of these results employs the vector-space property of the tangent space. Specifically, it is required that if Y is an acceptable tangent vector, then so is cY, where c is an arbitrary constant. But in our case, once the first component of a tangent vector of the augmented manifold is fixed at unity, it is not possible for both Y and cY, with $c \neq 1$, to be available tangent vectors. However, with the aid of a result of Kunita [55], we may nevertheless establish one-dimension-reduced controllability of the augmented system; that is, we may prove strong analytic controllability of the original system since it is not necessary to control the time dimension.

First, let us identify certain properties of the reachable set $\hat{R}_t(\eta)$ that will be useful in proving strong analytic controllability.

Theorem 4.1 *[24, 55] Assume that the Lie algebra $\hat{\mathcal{C}}$ is locally finitely generated, and let $I(\eta)$ be the maximal connected integral manifold of $\hat{\mathcal{C}}$ containing the point η. Then $\hat{R}_t(\eta) \subset \alpha_t^0(I(\eta))$, where α_t^0 is the integral curve whose vector field is W_0. Furthermore, the interior of $\hat{R}_t(\eta)$ with respect to the topology of $\alpha_t^0(I(\eta))$ is dense in $\hat{R}_t(\eta)$.*

A key relationship between the interior of the reachable set $\hat{R}_t(\eta)$ of the augmented system at time t and the interior of its closure is provided by the following lemma.

Lemma 4.2

$$\mathrm{int}(\mathrm{cl}\,\hat{R}_t(\eta)) = \mathrm{int}\,\hat{R}_t(\eta)\,. \tag{16}$$

Proof: Let $\chi \in \mathrm{int}(\mathrm{cl}\,\hat{R}_t(\eta))$ and let $S_\epsilon(\chi)$ be the set of all χ' such that χ is reachable from χ' within time $\epsilon > 0$. Then $S_\epsilon(\chi)$ is the reachable set within time $\epsilon > 0$ for the dual control system

$$\frac{\partial v}{\partial t} = -\left[W_0(v) + \sum_l u_l(t) W_l(v)\right]\,. \tag{17}$$

Theorem 4.1 implies that $\mathrm{int}\,S_\epsilon(\chi)$ is dense in $\mathrm{cl}\,S_\epsilon(\chi)$, and $\mathrm{int}\,\hat{R}_t(\eta)$ is dense in $\mathrm{cl}\,\hat{R}_t(\eta)$. Since $\chi \in \mathrm{cl}\,S_\epsilon(\chi)$, we know that

$$\mathrm{cl}\,S_\epsilon(\chi) \cap \mathrm{int}(\mathrm{cl}\,\hat{R}_t(\eta)) \neq \emptyset \tag{18}$$

and hence that

$$\mathrm{int}\,S_\epsilon(\chi) \cap \mathrm{int}(\mathrm{cl}\,\hat{R}_t(\eta)) \cap \hat{R}_t(\eta) \neq \emptyset\,. \tag{19}$$

If ζ belongs to the latter intersection, then ζ is reachable from η using time t, and χ is reachable from ζ in elapsed time less than or equal to ϵ. Therefore, χ is reachable from η in elapsed time between t and $t + \epsilon$. This argument holds for any $t > 0$ and any $\epsilon > 0$. Letting $\epsilon \to 0$, we conclude that χ is reachable from η in time t, so $\chi \in \hat{R}_t(\eta)$. Thus,

$$\mathrm{int}(\mathrm{cl}\,\hat{R}_t(\eta)) \subset \hat{R}_t(\eta) \Longrightarrow \mathrm{int}(\mathrm{cl}\,\hat{R}_t(\eta)) \subset \mathrm{int}\,\hat{R}_t(\eta)\,.$$

But clearly $\mathrm{int}\,\hat{R}_t(\eta) \subset \mathrm{int}(\mathrm{cl}\,\hat{R}_t(\eta))$ and the statement (16) follows.

From the control-theoretic perspective, the drift term is undesirable because no control is present to influence or remove its effect. It is therefore of strategic value to consider a suitably modified control system, called the auxiliary system, that will serve as a bridge to an effective controllability analysis of the augmented system. Let $e_0, e_1, \ldots, e_r$ be unit vectors in $\mathbb{R}^{r+1}$; in particular, let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, in which only the $(i+1)^{th}$ element is unity and the others are zero. Denote by $\mathcal{U}_0$ the set of controls $u(t) = (u_0(t), \ldots, u_r(t))$ composed of piecewise-constant functions $u_i(t)$ taking the values $e_0, \pm e_1, \ldots, \pm e_r$ only. Consider then the control system expressed in the form

$$\frac{\partial \xi}{\partial t} = u_0(t) W_0(\xi) + \sum_l u_l(t) W_l(\xi)\,, \quad \xi(t_0) = \eta\,, \tag{20}$$

where $u(t) \in \mathcal{U}_0$. The solution of this system may be written as

$$\alpha_t = \alpha_{t_k}^{i_k} \cdots \alpha_{t_j}^{i_j} \cdots \alpha_{t_1}^{i_1}, \tag{21}$$

where k is a positive integer and where $\alpha_{t_j}^{i_j}$ is the integral curve of W_{i_j} with $i_j = 0, 1, \ldots, r$, $j = 1, \ldots, k$, and k a positive integer. The times t_j satisfy $t_j \geq 0$ if $i_j = 0$, $t_j \in \mathbb{R}$ if $i_j = l \neq 0$. We denote by $\hat{R}_t^0(\eta)$ the reachable set of the auxiliary system corresponding to the total time t since time zero, over which the control function $u_0(\cdot)$ is nonzero; the reachable set of the auxiliary system is then $\hat{R}^0(\eta) = \bigcup_{t>0} \hat{R}_t^0(\eta)$. Theorem 4.1 is valid for this control system [24].

The following notations are convenient:

$\operatorname{Exp}\hat{\mathcal{L}}$	$=$ the group of diffeomorphisms generated by the α_t^i, $t \in \mathbb{R}$, $i = 0, \ldots, r$, where α_t^i is an integral curve of W_i ,
$(\operatorname{Exp}\hat{\mathcal{L}})_+$	$=$ the semigroup of diffeomorphisms generated by α_t^0, $t \geq 0$, and the α_t^l, with $t \in \mathbb{R}$ and $l = 1, \ldots, r$,
$(\operatorname{Exp}\hat{\mathcal{L}})_t$	$=$ the subset of $(\operatorname{Exp}\hat{\mathcal{L}})_+$ generated by $\alpha_{t_k}^{i_k} \cdot \ldots \cdot \alpha_{t_1}^{i_1}$, with $\sum_{j=1}^{k} t_j \cdot 1_{\{i_j=0\}} = t$.

To clarify the meaning of the last line, we note that when the index j is such that $i_j = 0$, we have $u_0 = 1$ (and all the other $u_i = 0$), so W_0 is "turned on" and does play a role as an active vector field or tangent vector. Conversely, for indices j such that $i_j \neq 0$, the factor u_0 multiplying W_0 in system (20) vanishes, and W_0 plays no role. The sum appearing in the definition of $(\operatorname{Exp}\hat{\mathcal{L}})_t$ gives the total time over which W_0 is active in the system dynamics.

From Chow's theorem [63, 24], it is known that the group $\operatorname{Exp}\hat{\mathcal{L}}$ acts transitively on the manifold N when $\dim \hat{\mathcal{L}}\{W_0, W_1, \ldots, W_r\} = \dim N$, i.e., we know that $\{\alpha(\eta) | \alpha \in \operatorname{Exp}\hat{\mathcal{L}}\} = N$ for any $\eta \in N$. On the other hand, the reachable set at time t for the auxiliary system (20) is $\hat{R}_t^0(\eta) = \{\alpha(\eta) | \alpha \in (\operatorname{Exp}\hat{\mathcal{L}})_t\}$. (It is to be noted that in the present context t is the total time over which W_0 has been active since time zero, which is generally not equal to the actual elapsed time, since W_0 may be turned off over certain intervals.)

Lemma 4.3

$$\operatorname{cl} \hat{R}_t(\eta) = \operatorname{cl} \hat{R}_t^0(\eta)\,. \tag{22}$$

We may gain intuitive understanding of this lemma by analyzing a simple example.

Example. Let us compare the control system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + u\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{23}$$

wherein $u \in \mathbb{R}$, with the system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = u_0\begin{pmatrix} 1 \\ 0 \end{pmatrix} + u_1\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{24}$$

wherein $(u_0, u_1) \in \{(0, \pm 1), (1, 0)\}$. Clearly, the first of these corresponds to the augmented system, and the second to the auxiliary system. Let $\hat{R}_t(\eta)$ and $\hat{R}^0_t(\eta)$ denote respectively the reachable sets of systems (23) and (24), staring from the state η. While stopping short of rigorous argument, explicit computation will be used to reveal the pertinent relationship between $\mathrm{cl}\,\hat{R}_t(\eta)$ and $\mathrm{cl}\,\hat{R}^0_t(\eta)$.

First consider the integral curve

$$\alpha_t(\eta) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{t_1} \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix}_{t_2} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{t} \in \hat{R}^0_t(\eta), \tag{25}$$

and for $n = 1, 2, 3, \ldots$ form a series of integral curves $\beta^n_t(\eta) \in \hat{R}_t(\eta)$ defined by

$$\beta^n_t(\eta) = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + n\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)_{\frac{t_1}{n}} \cdot \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + n\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right)_{\frac{t_2}{n}} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{t-\frac{t_1}{n}-\frac{t_2}{n}}. \tag{26}$$

As n goes to ∞, we find

$$\beta^n_t(\eta) \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{t_1} \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix}_{t_2} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{t}, \tag{27}$$

that is, $\beta^n_t(\eta) \to \alpha_t(\eta)$. Hence $\alpha_t(\eta) \in \mathrm{cl}\,\hat{R}_t(\eta)$.

On the other hand, consider

$$\beta_t(\eta) = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + m_1\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)_{t_1} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{t_2} \cdot \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + m_2\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right)_{t_3} \in \hat{R}_t(\eta), \tag{28}$$

where $m_1, m_2 \in \mathbb{R}$ and $t = t_1 + t_2 + t_3$, and construct

$$\alpha^n_1 = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\frac{t_1}{n}} \cdot m_1\begin{pmatrix} 0 \\ 1 \end{pmatrix}_{\frac{t_1}{n}}\right]^n, \tag{29}$$

again for $n = 1, 2, 3, \ldots$. Applying the Baker-Campbell-Hausdorff formula, it straightforward to show that

$$\begin{aligned}\lim_{n\to\infty} \alpha_1^n &= \lim_{n\to\infty} \left\{ \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + m_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)_{t_1} + \frac{t_1^2}{2n} m_1 \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] + O(\frac{1}{n^2}) \right\} \\ &= \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + m_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)_{t_1} .\end{aligned} \tag{30}$$

Similarly, let

$$\alpha_3^n = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\frac{t_3}{n}} \cdot m_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix}_{\frac{t_3}{n}} \right]^n \tag{31}$$

and employ the Baker-Campbell-Hausdorff formula to obtain

$$\begin{aligned}\lim_{n\to\infty} \alpha_3^n &= \lim_{n\to\infty} \left\{ \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + m_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right)_{t_3} + \frac{t_3^2}{2n} m_2 \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right] + O(\frac{1}{n^2}) \right\} \\ &= \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + m_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right)_{t_3} .\end{aligned} \tag{32}$$

Obviously

$$\alpha_1^n \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{t_2} \cdot \alpha_3^n \in \hat{R}_t^0(\eta) , \tag{33}$$

and we find that

$$\lim_{n\to\infty} \alpha_1^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{t_2} \alpha_3^n = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + m_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)_{t_1} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{t_2} \cdot \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + m_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right)_{t_3} = \beta_t(\eta) . \tag{34}$$

Therefore $\beta_t(\eta) \in \mathrm{cl}\, \hat{R}_t^0(\eta)$.

Now let us proceed with the proof of Lemma 4.3, showing first that $\mathrm{cl}\, \hat{R}_t^0(\eta) \subseteq \mathrm{cl}\, \hat{R}_t(\eta)$. Consider that $\alpha_t(\eta) \in \hat{R}_t^0(\eta)$ is expressible in the form of $\alpha_{t_k}^{i_k} \cdots \alpha_{t_1}^{i_1}(\eta)$, where $t = \sum_{j=1}^k t_j \cdot 1_{\{i_j = 0\}}$. With the guidance of the example above, a sequence of controls $u^{(n)}(\cdot)$ associated with the diffeomorphism of this form is constructed as follows. For an arbitrary positive integer n such that $nt_m \geq \sum_{i_j \neq 0} |t_j|$, where m is the last subscript j such that $i_j = 0$, let

$$t_m^{(n)} = t_m - \frac{\sum_{i_j \neq 0} |t_j|}{n} . \tag{35}$$

Define real numbers $s_1^{(n)}, \ldots, s_k^{(n)}$, ordered so that $0 \leq s_1^{(n)} \leq s_2^{(n)} \leq \ldots \leq s_k^{(n)}$, by

$$
\begin{aligned}
s_1^{(n)} &= |t_1| \quad \text{if} \quad i_1 = 0\,, \\
&= \tfrac{1}{n}|t_1| \quad \text{if} \quad i_1 \neq 0\,, \\
s_{j\geq 2}^{(n)} &= s_{j-1}^{(n)} + |t_j^{(n)}| \quad \text{if} \quad \text{last } j \text{ with } i_j = 0\,, \\
&= s_{j-1}^{(n)} + |t_j| \quad \text{if} \quad \text{other } j \text{ with } i_j = 0\,, \\
&= s_{j-1}^{(n)} + \tfrac{1}{n}|t_j| \quad \text{if} \quad i_j \neq 0\,.
\end{aligned}
\tag{36}
$$

Further, let

$$
\begin{aligned}
u^{(n)}(\tau) &= n \cdot \mathrm{sgn}(t_j) e_{i_j} \quad \text{if} \quad s_{j-1}^{(n)} \leq \tau \leq s_j^{(n)} \text{ and } \quad i_j \neq 0\,, \\
&= 0 \quad \text{if} \quad s_{j-1}^{(n)} \leq \tau \leq s_j^{(n)} \text{ and } \quad i_j = 0\,, \\
&= 0 \quad \text{if} \quad \tau \geq s_k^{(n)}\,,
\end{aligned}
\tag{37}
$$

where $e_1, \ldots, e_r$ are unit vectors in $\mathbb{R}^r$. The solution $\beta_t^{(n)}$ of the system (12) associated with the control $u^{(n)}(\cdot)$ may be written

$$
\beta_{s_k^{(n)}}^{(n)} = \beta_{|t_k|}^{n,i_k} \cdots \beta_{|t_1|}^{n,i_1} \in \hat{R}_t(\eta)\,,
\tag{38}
$$

where $\beta_{|\tau|}^{n,i_j}$ is the integral curve of W_0 if $i_j = 0$, or the integral curve of $W_0 + n \cdot \mathrm{sgn}(\tau) W_{i_j}$ if $i_j \neq 0$, i.e.,

$$
\begin{aligned}
\beta_{|\tau|}^{n,i_j} &= (W_0)_\tau \quad \text{if} \quad i_j = 0\,, \\
&= (W_0 + n \cdot \mathrm{sgn}(\tau) W_{i_j})_{\frac{|\tau|}{n}} \quad \text{if} \quad i_j \neq 0\,.
\end{aligned}
\tag{39}
$$

We note that $(W_0 + n \cdot \mathrm{sgn}(\tau) W_{i_j})_{\frac{|\tau|}{n}}$ and $(\frac{1}{n} W_0 + \mathrm{sgn}(\tau) W_{i_j})_{|\tau|}$ describe the same integral curve on N, by virtue of the time-invariance property of system (12). Obviously, $\beta_{|t_p|}^{n,i_j} \to \alpha_{t_p}^{i_j}$ as $n \to \infty$. On the other hand,

$$
s_k^{(n)} = \sum_j t_j \cdot 1_{\{i_j = 0\}} - \frac{\sum_l |t_l| \cdot 1_{\{i_l \neq 0\}}}{n} + \frac{\sum_l |t_l| \cdot 1_{\{i_l \neq 0\}}}{n} = t\,.
\tag{40}
$$

Thus, as $n \to \infty$ we obtain

$$
\beta_{s_k^{(n)}}^{(n)}(\eta) \to \alpha_{t_k}^{i_k} \cdots \alpha_{t_1}^{i_1}(\eta) = \alpha_t(\eta)\,,
\tag{41}
$$

and hence $\alpha_t(\eta) \in \mathrm{cl}\, \hat{R}_t(\eta)$. Because $\alpha_t(\eta)$ is an arbitrary element in $\hat{R}_t^0(\eta)$, it follows that $\hat{R}_t^0(\eta) \subseteq \mathrm{cl}\, \hat{R}_t(\eta)$, and since $\mathrm{cl}\, \hat{R}_t(\eta)$ is closed, it follows in turn that $\mathrm{cl}\, \hat{R}_t^0(\eta) \subseteq \mathrm{cl}\, \hat{R}_t(\eta)$.

Next we show $\mathrm{cl}\, \hat{R}_t(\eta) \subseteq \mathrm{cl}\, \hat{R}_t^0(\eta)$. Consider $\beta(\eta) \in \hat{R}_t(\eta)$ of the form of $\beta_{u_k}^{c_k} \cdot \ldots \cdot \beta_{u_1}^{c_1}(\eta)$, with $\beta_{u_j}^{c_j} = \exp u_j (W_0 + c_j^1 W_1 + \ldots + c_j^r W_r)$ and $c_j = (c_j^1, \ldots, c_j^r)$. Here, c_j^l is the control applied to W_l during time period u_j, so c_j is the control set applied to $W_1, \ldots W_r$ during the corresponding time interval u_j, with $u_j \in \mathbb{R}^+$ and $c_j^l \in \mathbb{R}$. For each $\beta_{u_j}^{c_j}$, $j = 1, \ldots, k$, take α_j^n in the form

$$
\alpha_j^n = \left[\exp \frac{u_j}{n}(c_j^1 W_1) \cdots \exp \frac{u_j}{n}(c_j^r W_r) \exp \frac{u_j}{n} W_0\right]^n.
\tag{42}
$$

Invoking the Baker-Campbell-Hausdorff formula [64], we write

$$\begin{aligned}
\lim_{n\to\infty} \alpha_j^n &= \lim_{n\to\infty} \left[\exp \frac{u_j}{n}(c_j^1 W_1) \cdots \exp \frac{u_j}{n}(c_j^r W_r) \cdot \exp \frac{u_j}{n} W_0\right]^n \\
&= \lim_{n\to\infty} \exp \left[u_j(W_0 + c_j^1 W_1 + \cdots + c_j^r W_r) + \sum_{0\le p,q\le r} \frac{u_j^2}{2n} c_j^p c_j^q [W_p, W_q] + O\left(\frac{1}{n^2}\right)\right] \\
&= \exp u_j(W_0 + c_j^1 W_1 + \ldots + c_j^r W_r) = \beta_{u_j}^{c_j} \,. \qquad (43)
\end{aligned}$$

Constructing $\alpha_1^n \ldots \alpha_k^n \in \hat{R}_t^0(\eta)$ we then obtain

$$\lim_{n\to\infty} \alpha_k^n \cdots \alpha_1^n(\eta) = \beta_{u_k}^{c_k} \cdots \beta_{r_1}^{c_1}(\eta) = \beta(\eta)\,, \qquad (44)$$

so that $\beta(\eta) \in \mathrm{cl}\,\hat{R}_t^0(\eta)$. Since $\beta(\eta)$ is an arbitrary element of $\hat{R}_t(\eta)$, we arrive at $\hat{R}_t(\eta) \subseteq \mathrm{cl}\,\hat{R}_t^0(\eta)$ and hence $\mathrm{cl}\,\hat{R}_t(\eta) \subseteq \mathrm{cl}\,\hat{R}_t^0(\eta)$. We conclude that $\mathrm{cl}\,\hat{R}_t(\eta) = \mathrm{cl}\,\hat{R}_t^0(\eta)$.

The time t labeling these reachable sets is to be interpreted as the time interval over which the control operation represented by W_0 is in effect, or "turned on." In fact, W_0 is necessarily *always* "on" in the augmented system, so the total time elapsing in the augmented system is the same as the time interval over which W_0 is turned on; hence the reachable sets $\hat{R}_t$ corresponding to these two times are identical. Of course, the same coincidence does not hold for the auxiliary system. However, this is immaterial, since the auxiliary system was only introduced to exploit the key relationship (22). Further, we may observe that the reachable set $\hat{R}_t^0(\eta)$ of system (20), with the control $u(t) = (u_0(t), \ldots, u_r(t))$ assuming values $(e_0, \pm e_1, \ldots, \pm e_r)$, is the same as the corresponding set for which the control $u(t)$ assumes the values $e_0, \pm ce_1, \ldots, \pm ce_r$, with $c \in \mathbb{R}^+$.

Since we can take advantage of the result (22) in this manner, it is clearly preferable to study the properties of $\hat{R}_t^0(\eta)$. The auxiliary system is easier to control, and the state at time t can be expressed as a composition of integral curves of W_i in the same style as Eq. (21). To do so, let the set of subscripts j with $i_j = 0$ be written as $\{p, \ldots, q, s\}$ in increasing order, of course with $t_p + \ldots + t_q + t_s = t$. Then we have

$$\begin{aligned}
\alpha_t \;=\; & (\alpha_{t_k}^{i_k} \cdots \alpha_{t_{s+1}}^{i_{s+1}}) \cdot (\alpha_{t_s}^0 \cdot \alpha_{t_{s-1}}^{i_{s-1}} \cdot \alpha_{-t_s}^0) \cdot (\alpha_{t_s}^0 \cdot \alpha_{t_{s-2}}^{i_{s-2}} \cdot \alpha_{-t_s}^0) \cdots (\alpha_{t_s+t_q}^0 \cdot \alpha_{t_{q-1}}^{i_{q-1}} \cdot \alpha_{-(t_s+t_q)}^0) \\
& \cdot (\alpha_{t_s+t_q}^0 \cdot \alpha_{t_{q-2}}^{i_{q-2}} \cdot \alpha_{-(t_s+t_q)}^0) \cdots (\alpha_{t_s+t_q+\cdots+t_p}^0 \cdot \alpha_{t_{p-1}}^{i_{p-1}} \cdot \alpha_{-(t_s+t_q+\cdots+t_p)}^0) \\
& \cdots (\alpha_{t_s+t_q+\ldots+t_p}^0 \cdot \alpha_{t_1}^{i_1} \cdot \alpha_{-(t_s+t_q+\ldots+t_p)}^0) \cdot \alpha_t^0 \\
=\; & \beta_0(\alpha_{t_k}^{i_k}) \cdots \beta_0(\alpha_{t_{s+1}}^{i_{s+1}}) \cdot \beta_{t_s}(\alpha_{t_{s-1}}^{i_{s-1}}) \cdot \beta_{t_s}(\alpha_{t_{s-2}}^{i_{s-2}}) \cdots \beta_{t_s+t_q}(\alpha_{t_{q-1}}^{i_{q-1}}) \cdot \beta_{t_s+t_q}(\alpha_{t_{q-2}}^{i_{q-2}}) \\
& \cdots \beta_t(\alpha_{t_{p-1}}^{i_{p-1}}) \cdots \beta_t(\alpha_{t_1}^{i_1}) \cdot \alpha_t^0, \qquad (45)
\end{aligned}$$

where $\beta_t(\gamma) = \alpha_t^0 \cdot \gamma \cdot \alpha_{-t}^0$. This analysis stimulates us to define the following three sets of diffeomorphisms:

$\operatorname{Exp}\hat{\mathcal{B}}$ = the group generated by α_t^l, $t \in \mathbb{R}$, $l = 1, \ldots, r$, where α_t^l is the integral curve whose vector field is W_l ,

$$F_t = \cup_{k=1}^{\infty} \{\beta_{t_k}(\gamma_k) \cdot \ldots \cdot \beta_{t_1}(\gamma_1) | \, \gamma_j \in \operatorname{Exp}\hat{\mathcal{B}}, 0 \le t_k \le \ldots \le t_1 = t\},$$

$$G_t = \cup_{k=1}^{\infty} \{\beta_{t_k}(\gamma_k) \cdot \ldots \cdot \beta_{t_1}(\gamma_1) | \, \gamma_j \in \operatorname{Exp}\hat{\mathcal{B}}, \, \min_j t_j \ge 0, \, \max_j t_j = t\}.$$

By construction,

$$\hat{R}_t^0(\eta) = F_t \alpha_t^0(\eta) \,. \tag{46}$$

We observe that F_t is a semi-group of diffeomorphisms included in the the group G_t, whose properties are established in the following lemma.

Lemma 4.4 *First, the set G_t is a group. Furthermore, if $\dim \hat{\mathcal{C}}(\eta) = n-1 = m$ holds for all $\eta \in N$, then $\{\alpha(\eta) | \alpha \in G_t\} = \alpha_t^0(I(\alpha_{-t}^0(\eta)))$ is true for all η, where $I(\nu)$ is the maximal connected integral manifold containing $\nu \in N$, whose associated Lie algebra is $\hat{\mathcal{C}}$.*

Proof: For $\alpha_1, \alpha_2 \in G_t$, it is easily seen that $\alpha_1 \cdot \alpha_2 \in G_t$. Writing $\alpha \in G_t$ as $\alpha = \beta_{t_k}(\gamma_k) \cdot \ldots \cdot \beta_{t_1}(\gamma_1)$, we also see that $\alpha^{-1} = \beta_{t_1}(\gamma_1^{-1}) \cdot \ldots \cdot \beta_{t_k}(\gamma_k^{-1})$. Therefore G_t is a group.

Now, denote the set $\{\alpha(\eta) | \alpha \in G_t\}$ by $B_t(\eta)$. It is straightforward to show that (i) $B_t(\eta) = B_t(\xi)$ if $\xi \in B_t(\eta)$ and (ii) $B_t(\eta) \cap B_t(\xi) = \emptyset$ if $\xi \notin B_t(\eta)$ [55]. We can demonstrate that (iii) $\eta \in \operatorname{int} B_t(\eta)$ under the topology of $\alpha_t^0(I(\alpha_{-t}^0(\eta)))$ as follows. By definition, $\hat{R}_t^0(\eta)$ is the reachable set for the system (20). By the same reasoning that leads to Eq. (46), we have $\hat{R}_t^0(\alpha_{-t}^0(\eta)) \subset B_t(\eta)$ because $\hat{R}_t^0(\alpha_{-t}^0(\eta)) = F_t \cdot \alpha_t^0 \cdot \alpha_{-t}^0(\eta)$. Since $\hat{R}_t^0(\alpha_{-t}^0(\eta))$ has a nonempty interior with respect to the topology of $\alpha_t^0(I(\alpha_{-t}^0(\eta)))$ by Theorem 4.1, we see that $B_t(\eta)$ contains a non-null open set U. Given $\mu \in U$, choose $\alpha \in G_t$ such that $\alpha(\eta) = \mu$. Since α is a continuous map, $\alpha^{-1}(U)$ is an open set containing η.

In fact, $\alpha^{-1}(U)$ is included in $B_t(\eta)$. We know that G_t is a group, so $\alpha^{-1} \in G_t$ if $\alpha \in G_t$. Letting $\zeta \in \alpha^{-1}(U)$, we can find $\chi \in U$, such that $\chi = \alpha(\zeta) \in U \subset B_t(\eta)$ and also $\chi \in B_t(\zeta)$. By properties (i) and (ii), we obtain $\chi \in B_t(\zeta) \cap B_t(\eta) \ne \emptyset$. Hence $B_t(\zeta) = B_t(\eta)$ and $\zeta \in B_t(\eta)$. Accordingly, $\alpha^{-1}(U) \subset B_t(\eta)$ and $\eta \in \operatorname{int} B_t(\eta)$ under the topology of $\alpha_t^0(I(\alpha_{-t}^0(\eta)))$.

The properties (i)-(iii) imply that $B_t(\eta)$ is maximally connected and open under the topology of $\alpha_t^0(I(\alpha_{-t}^0(\eta)))$. Thus we have $B_t(\eta) = \alpha_t^0(I(\alpha_{-t}^0(\eta)))$ for all $t > 0$ and $\eta \in N$. In addition, it is seen that $B_t(\eta) = \alpha_t^0(I(\alpha_{-t}^0(\eta))) = \begin{pmatrix} t_0 \\ M \cap \mathcal{D}_A \end{pmatrix}$. The proof of Lemma 4.4 is now complete.

Based on Lemmas 4.3 and 4.4, we could conclude that $\operatorname{cl}\hat{R}_t(\alpha_{-t}^0(\eta)) = \alpha_t^0(I(\alpha_{-t}^0(\eta)))$ if we could establish that $F_t = G_t$. The following proof takes a slightly different path. Let $\operatorname{Exp}\widetilde{\hat{\mathcal{B}}}$ denote the group of diffeomorphisms generated by all one parameter groups of transformations with respect to vector fields belonging to $\tilde{\hat{\mathcal{B}}}$. The sets $\widetilde{F_t}$ and $\widetilde{G_t}$ are defined in the same way as F_t and G_t, i.e. via Eq. (17), but with $\mathrm{Exp}\,\tilde{\hat{\mathcal{B}}}$ entering in place of $\mathrm{Exp}\,\hat{\mathcal{B}}$.

Obviously, $F_t \subset \widetilde{F_t}$ and $G_t \subset \widetilde{G_t}$ hold. We shall now establish that $\widetilde{F_t} = \widetilde{G_t}$.

Lemma 4.5 *Let X be a complete vector field belonging to $\tilde{\hat{\mathcal{B}}}$, and let γ_t be the one-parameter group of transformations generated by X. Assume $[\hat{\mathcal{B}}, \hat{\mathcal{C}}](\eta) \subset \hat{\mathcal{B}}(\eta)$ is satisfied for all η. Then $d\beta_s(\gamma_t)$ is an isomorphism between $\hat{\mathcal{B}}(\eta)$ and $\hat{\mathcal{B}}(\beta_s(\gamma_t)(\eta))$ for each η, and $\widetilde{F_t} = \widetilde{G_t}$ is true for all $t > 0$.*

Proof: Since $\beta_s(\gamma_{t_1}) \cdot \beta_s(\gamma_{t_2}) = \beta_s(\gamma_{t_1+t_2})$ holds, we have $d\beta_s(\gamma_{t_1+t_2}) = d\beta_s(\gamma_{t_1}) \cdot d\beta_s(\gamma_{t_2})$. Hence it is enough to prove the lemma's assertion for sufficiently small $|t|$. Let $Y_{t,s} = d\beta_s(\gamma_t)Z$, where $Z \in \tilde{\hat{\mathcal{B}}}$. For each value of s, $\beta_s(\gamma_t)$ with $t \in R$ is the one parameter group of transformations generated by $d\alpha_s^0 X$, while

$$\frac{\partial Y_{t,s}}{\partial t} = -d\beta_s(\gamma_t)[d\alpha_s^0 X, Z] = d\beta_s(\gamma_t)[Z, d\alpha_s^0 X]\,. \tag{47}$$

Therefore $[Z, d\alpha_s^0 X] \in \tilde{\hat{\mathcal{B}}}$ by assumption, because $d\alpha_s^0 X$ belongs to $\tilde{\hat{\mathcal{C}}} = \{\hat{\mathcal{C}}(\eta) | \eta \in N\}$ [65, 66].

Now we fix a point η of N and a value of $s \in R$. Let $Z^1, \ldots, Z^n$ provide a basis of $\hat{\mathcal{B}}$ in an open neighborhood U of η. Then there exist C^∞ functions f_{ij} on U such that $[Z^i, d\alpha_s^0 X] = \sum_{j=1}^n f_{ij} Z^j$ holds in U. Let ϵ be a positive number such that $\beta_s(\gamma_t)(\eta) \in U$ for $|t| < \epsilon$, noting that $\beta_s(\gamma_t)$ is a continuous map of t and $\beta_s(\gamma_0)(\eta) = \eta$. Then $d\beta_s(\gamma_t)[Z^i, d\alpha_s^0 X] = \sum_{j=1}^n f_{ij} d\beta_s(\gamma_t) Z^j$ for $|t| < \epsilon$. Set $V^j(t) = d\beta_s(\gamma_t)Z^j$. Then $V^j(t)$, with $|t| < \epsilon$, satisfies the linear differential equation

$$\frac{dV^j(t)}{dt} = \sum_{j=1}^n f_{jk} V^k(t) \;\; j = 1, \ldots, n\,. \tag{48}$$

The solution $V^j(t)$ can be written as $V^j(t) = \sum_{k=1}^n g_{jk}(t) V^k(0)$, where (g_{jk}) is a regular matrix. Also, we have $V^k(0) \in \hat{\mathcal{B}}(\eta)$ and $V^k(t) \in \hat{\mathcal{B}}(\beta_s(\gamma_t))(\eta)$. The map $d\beta_s(\gamma_t) : \hat{\mathcal{B}}(\eta) \to \hat{\mathcal{B}}(\beta_s(\gamma_t))(\eta)$ is bijective because (g_{jk}) is a regular matrix. Moreover, $d\beta_s(\gamma_t)$ retains the structure of the Lie bracket with respect to $d\alpha_s^0 X$. This establishes that $d\beta_s(\gamma_t)$ is an isomorphism between $\hat{\mathcal{B}}(\eta)$ and $\hat{\mathcal{B}}(\beta_s(\gamma_t))(\eta)$ for $|t| < \epsilon$. Since $\gamma_t' \equiv \beta_s(\alpha) \cdot \gamma_t \cdot \beta_s(\alpha)^{-1}$ (with s fixed) is a one-parameter group of transformations generated by $d\beta_s(\alpha)X$ and $d\beta_s(\alpha)X$ belongs to $\tilde{\hat{\mathcal{B}}}$, we know γ_t' (with $t \in R$) belongs to $\mathrm{Exp}\,\tilde{\hat{\mathcal{B}}}$. But $\mathrm{Exp}\,\tilde{\hat{\mathcal{B}}}$ is generated by all such γ_t, so we arrive at the relationship

$$\beta_t(\alpha)(\mathrm{Exp}\,\tilde{\hat{\mathcal{B}}})\beta_t(\alpha)^{-1} \subset \mathrm{Exp}\,\tilde{\hat{\mathcal{B}}}, \qquad \text{for } \alpha \in \tilde{\hat{\mathcal{B}}}\,. \tag{49}$$

Let α be any element of $\widetilde{G_t}$, written as

$$\alpha = \beta_{t_k}(\gamma_k) \cdot \ldots \cdot \beta_{t_1}(\gamma_1), \;\; t_l \geq 0, \max_l t_l = t\,. \tag{50}$$

By induction we can prove that there exist $\tilde{\gamma}_k, \ldots, \tilde{\gamma}_1$ of $\operatorname{Exp} \widetilde{\hat{\mathcal{B}}}$ and $0 \leq s_k \leq \ldots \leq s_1 = t$ such that

$$\beta_{t_k}(\gamma_k) \cdot \ldots \cdot \beta_{t_1}(\gamma_1) = \beta_{s_k}(\tilde{\gamma}_k) \cdot \ldots \cdot \beta_{s_1}(\tilde{\gamma}_1). \tag{51}$$

Here we only consider the case $k = 2$. If $t_2 \leq t_1$, there is no need for proof. Suppose $t_2 > t_1$, and set $t_3 = t_2 - t_1$. Then we may write $\beta_{t_2}(\gamma_2) \cdot \beta_{t_1}(\gamma_1) = \beta_{t_1}(\beta_{t_3}(\gamma_2) \cdot \gamma_1)$. By relationship (49), there exists $\tilde{\gamma}_1$ of $\operatorname{Exp} \widetilde{\mathcal{B}}$ such that $\beta_{t_3}(\gamma_2) \cdot \gamma_1 \cdot \beta_{t_3}(\gamma_2)^{-1} = \tilde{\gamma}_1$, i.e., $\beta_{t_3}(\gamma_2) \cdot \gamma_1 = \tilde{\gamma}_1 \cdot \beta_{t_3}(\gamma_2)$. This implies

$$\beta_{t_2}(\gamma_2) \cdot \beta_{t_1}(\gamma_1) = \beta_{t_1}(\beta_{t_3}(\gamma_2) \cdot \gamma_1) = \beta_{t_1}\left(\tilde{\gamma}_1 \cdot \beta_{t_3}(\gamma_2)\right) = \beta_{t_1}(\tilde{\gamma}_1) \cdot \beta_{t_2}(\gamma_2)\,. \tag{52}$$

More detailed proofs may be found in Refs. [55, 67].

Theorem 4.6 *Suppose that* $\dim \hat{\mathcal{C}}(\eta) = n - 1 = m$ *holds for all* $\eta \in N$, *and suppose that* $[\hat{\mathcal{B}}, \hat{\mathcal{C}}](\eta) \subset \hat{\mathcal{B}}(\eta)$ *holds for all* η. *Let* $I(\eta)$ *be the maximally connected integral manifold containing* η *whose corresponding Lie algebra is* $\hat{\mathcal{C}}$. *Then* $\alpha_t^0(I(\eta)) = \hat{R}_t(\eta)$.

Proof: Clearly we have $\{\alpha\alpha_t^0(\eta) | \alpha \in F_t\} \subset \{\alpha\alpha_t^0(\eta) | \alpha \in \widetilde{F}_t\}$. In fact, the closures of these two sets coincide. Since $\widetilde{F}_t = \widetilde{G_t} \supset G_t$, it is seen that

$$\begin{aligned} \operatorname{cl} \hat{R}_t^0(\eta) &= \operatorname{cl} \{\alpha\alpha_t^0(\eta) | \alpha \in F_t\} \\ &= \operatorname{cl} \{\alpha\alpha_t^0(\eta) | \alpha \in \widetilde{F}_t\} \\ &= \operatorname{cl} \{\alpha\alpha_t^0(\eta) | \alpha \in \widetilde{G_t}\} \quad \text{(by Lemma 4.5)} \\ &= \operatorname{cl} \alpha_t^0(I(\alpha_{-t}^0(\alpha_t^0(\eta)))) \quad \text{(by Lemma 4.4)} \\ &= \operatorname{cl} \alpha_t^0(I(\eta))\,. \end{aligned} \tag{53}$$

But Lemma 4.3 tells us that $\operatorname{cl} \hat{R}_t^0(\eta) = \operatorname{cl} \hat{R}_t(\eta)$, so we obtain $\operatorname{cl} \hat{R}_t(\eta) = \operatorname{cl} \alpha_t^0(I(\eta))$. And from Lemma 16 we know that $\operatorname{int} \hat{R}_t(\eta) = \operatorname{int}(\operatorname{cl} \hat{R}_t(\eta))$, which implies $\operatorname{int} \hat{R}_t(\eta) = \alpha_t^0(I(\eta))$ under the topology of $\alpha_t^0(I(\eta))$. Finally, $\hat{R}_t(\eta) \subset \alpha_t^0(I(\eta))$ by Theorem 4.1, and we arrive at $\hat{R}_t(\eta) = \alpha_t^0(I(\eta))$.

4.3 Strong Analytic Controllability of the Actual System

In subsection 4.2, we investigated the reachable set at time t of the time-independent augmented system formed by enlarging the state space to include an extra dimension corresponding to the variable t. Now we return to the original quantum control system (4) to discover conditions under which it is strongly analytically controllable.

Theorem 4.7 *For the control system defined by Eq. (4), let*

$$\begin{aligned}
&\mathcal{B}(t) = \mathcal{L}(H_1(t), \ldots, H_r(t)) \\
&B_1 = -[H_0, \mathcal{B}] + \tfrac{\partial}{\partial t}\mathcal{B} \\
&\vdots \\
&B_n = -[H_0, B_{n-1}] + \tfrac{\partial}{\partial t} B_{n-1} \\
&\vdots \\
&\mathcal{C} = \mathcal{L}\{\mathcal{B}, B_1, \ldots, B_n, \ldots\}\,.
\end{aligned} \tag{54}$$

Suppose $\dim \mathcal{C}(t)\psi(t) = m$ *holds for all* $\psi \in M \cap \mathcal{D}_A$*, and* $[\mathcal{B}, \mathcal{C}](t) \subset \mathcal{B}(t)$ *is the case for all* t*. Then the time-dependent quantum control system (4) is strongly analytically controllable.*

Proof: We apply Theorem 4.6 to the augmented control system (12). To do so, we need to examine the Lie algebras $\mathcal{B}$ and $\mathcal{C}$ for this problem. For $\mathcal{B}$ we readily find

$$\begin{aligned}
\mathcal{B} =& \mathcal{L}\{W_1, \ldots, W_r\} \\
=& \mathcal{L}\left\{\begin{pmatrix} 0 \\ H_1(t) \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ H_r(t) \end{pmatrix}\right\}\psi(t) = \begin{pmatrix} 0 \\ \mathcal{L}\{H_1(t), \ldots, H_r(t)\} \end{pmatrix}\psi(t) = \begin{pmatrix} 0 \\ \mathcal{B}(t)\psi(t) \end{pmatrix}.
\end{aligned} \tag{55}$$

Next let us construct $\mathcal{C}$. For any

$$W(\eta) = W(t, \psi) = \begin{pmatrix} 0 \\ H(t)\psi(t) \end{pmatrix} \in \mathcal{B}\,, \tag{56}$$

where $\eta \in N$, we have

$$\begin{aligned}
\mathrm{ad}_{W_0} W = [W_0, W] =& \left[\begin{pmatrix} 1 \\ H_0(t)\psi(t) \end{pmatrix}, \begin{pmatrix} 0 \\ H(t)\psi(t) \end{pmatrix}\right] \\
=& \frac{\partial \begin{pmatrix} 0 \\ H(t)\psi(t) \end{pmatrix}}{\partial(t, \psi)} \begin{pmatrix} 1 \\ H_0(t)\psi(t) \end{pmatrix} - \frac{\partial \begin{pmatrix} 1 \\ H_0(t)\psi(t) \end{pmatrix}}{\partial(t, \psi)} \begin{pmatrix} 0 \\ H(t)\psi(t) \end{pmatrix} \\
=& \begin{Bmatrix} 0 \\ -[H_0, H] + \partial H/\partial t \end{Bmatrix}\psi(t)\,.
\end{aligned} \tag{57}$$

Similarly,

$$\mathrm{ad}_{W_0}\mathcal{B} = \begin{pmatrix} 0 \\ -[H_0, \mathcal{B}] + \partial\mathcal{B}/\partial t \end{pmatrix}\psi(t)\,. \tag{58}$$

Setting $B_1 = -[H_0, \mathcal{B}] + \partial\mathcal{B}/\partial t$, we may then derive

$$\mathrm{ad}^2_{W_0}\mathcal{B} = \mathrm{ad}_{W_0}\mathrm{ad}_{W_0}\mathcal{B} = \mathrm{ad}_{W_0}\begin{pmatrix} 0 \\ B_1\psi(t) \end{pmatrix} = \begin{pmatrix} 0 \\ -[H_0, B_1] + \partial B_1/\partial t \end{pmatrix}\psi(t)\,. \tag{59}$$

Continuing in this fashion with

$$B_n = -[H_0, B_{n-1}] + \partial B_{n-1}/\partial t \tag{60}$$

for $n = 2, 3, \ldots$, we find

$$\mathrm{ad}^n_{W_0}\mathcal{B} \;\; = \begin{pmatrix} 0 \\ -[H_0, B_{n-1}] + \frac{\partial B_{n-1}}{\partial t} \end{pmatrix} \psi(t) = \begin{pmatrix} 0 \\ B_n \psi(t) \end{pmatrix} . \tag{61}$$

Thus

$$\begin{aligned} \mathcal{C} &= \mathcal{L}\{\mathcal{B}, \mathrm{ad}_{W_0}\mathcal{B}, \ldots, \mathrm{ad}^n_{W_0}\mathcal{B}, \ldots\} \\ &= \mathcal{L}\left\{ \begin{pmatrix} 0 \\ \mathcal{B}(t)\psi(t) \end{pmatrix}, \begin{pmatrix} 0 \\ B_1(t)\psi(t) \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ B_n(t)\psi(t) \end{pmatrix}, \ldots \right\} \\ &= \begin{pmatrix} 0 \\ \mathcal{L}\{\mathcal{B}(t), B_1(t), \ldots, B_n(t), \ldots\}\psi(t) \end{pmatrix} = \begin{pmatrix} 0 \\ \mathcal{C}(t)\psi(t) \end{pmatrix} . \end{aligned} \tag{62}$$

From the assumption that $[\mathcal{B}, \mathcal{C}](t) \subset \mathcal{B}(t)$, $\forall (t)$, we have

$$[\mathcal{B}, \mathcal{C}](t)\psi(t) \subset \mathcal{B}(t)\psi(t), \; \forall (t) \, . \tag{63}$$

Hence

$$\left[\begin{pmatrix} 0 \\ \mathcal{B}\psi \end{pmatrix}, \begin{pmatrix} 0 \\ \mathcal{C}\psi \end{pmatrix} \right] \subset \begin{pmatrix} 0 \\ \mathcal{B}\psi \end{pmatrix} , \tag{64}$$

so that $[\mathcal{B}, \mathcal{C}](\eta) \subset \mathcal{B}(\eta)$, $\forall \eta \in N$.

By assumption, $\dim \mathcal{C}(t)\psi(t) = m$, $\forall \psi \in M \cap \mathcal{D}_A$, which implies that $\dim \mathcal{C}(\eta) = m = n - 1$ holds for all $\eta \in N$. According to Theorem 4.6, $\alpha^0_t(I(\eta)) = \hat{R}_t(\eta)$, $\forall t > 0$, and since $\alpha^0_t(I(\alpha^0_{-t}(\eta))) = \begin{pmatrix} t_0 \\ M \cap \mathcal{D}_A \end{pmatrix}$, we obtain $\alpha^0_t(I(\eta)) = \begin{pmatrix} t + t_0 \\ M \cap \mathcal{D}_A \end{pmatrix}$.

Let $\pi : N \to M \cap \mathcal{D}_A$ be the projection map that in effect annihilates the time-dimension of the augmented problem corresponding to the variable t, and brings us back to the original control system. In fact, the extension and projection maps mediate a one-to-one correspondence between the states of the augmented system and those of the original system. The simplicity of this relationship stems from the fact that t is a strictly increasing variable.

To reiterate our strategy: We have dealt with the explicit time-dependence of the original control problem by adding an extra dimension to its state space, such that, as viewed in the augmented space, the augmented control problem is time-independent. After analyzing controllability within this extension, the results are projected to the original space by removing the extra time dimension, recovering the exact states of the original system.

Accordingly, $\pi(\alpha^0_t(I(\eta))) = M \cap \mathcal{D}_A$, while $\pi \hat{R}_t(\eta) = R_{t+t_0}(\psi)$, $\forall \psi \in M \cap \mathcal{D}_A$. Hence $R_t(\psi) = M \cap \mathcal{D}_A, \forall t > t_0$, and the system (4) is strongly analytically controllable on M.

We may note that upon introducing the Lie algebra $\mathcal{A}(t) = \mathcal{L}\{H_0(t), H_1(t), \ldots, H_r(t)\}$, it is readily established from property (13) that $\mathcal{B} \subset \mathcal{C} \subset \mathcal{A}$ for all t.

To complete the formal analysis, we state two corollaries that devolve immediately from Theorem 4.7:

Corollary 4.8 *From the operators H_i entering control system (4), form the Lie algebras $\mathcal{B} = \mathcal{L}\{H_1, \ldots, H_r\}$ and $\mathcal{C} = \mathcal{L}\{\mathcal{B}, \mathrm{ad}_{H_0}\mathcal{B}, \ldots, \mathrm{ad}^n_{H_0}\mathcal{B}, \ldots\}$. Suppose that the H_i do not possess explicit dependence on the time t, that* dim $\mathcal{C}\psi(t) = m$ *holds for all $\psi \in M \cap \mathcal{D}_A$, and that $[\mathcal{B}, \mathcal{C}] \subset \mathcal{B}$ is satisfied. Then the time-invariant system (4) is strongly analytically controllable.*

Corollary 4.9 *For the control system (4), form the Lie algebra $\mathcal{B}(t) = \mathcal{L}(H_1(t), \ldots, H_r(t))$, and suppose that* dim $\mathcal{B}(t)\psi(t) = m$ *holds for all $\psi \in M \cap \mathcal{D}_A$. Then system 4 is strongly analytically controllable.*

The latter corollary follows because $[\mathcal{B}, \mathcal{C}](t) \subset \mathcal{B}(t)$ must hold, once dim $\mathcal{B}(t)\psi(t) = m$.

5 Examples of Strong Analytic Controllability

In this section, we present three examples that meet the criteria for analytic controllability enunciated in Theorem 4.7. The examples selected are relevant to problems of interest in mathematical physics or engineering applications of quantum mechanics.

Example 1 The strong analytic controllability theorem can be applied to the simple degenerate parametric oscillator, a problem of importance in physics and engineering. Introducing an appropriate effective Hamiltonian allows the corresponding control system to be written in the form [68]

$$i\frac{\partial}{\partial t}\psi = \left\{\omega(t)a^\dagger a + \frac{1}{2}\chi(t)\left[e^{-2i\omega t}(a^\dagger)^2 + e^{2i\omega t}a^2\right]\right\}\psi\,. \tag{65}$$

Here $a^\dagger$ and a represent, in turn, the creation and annihilation operators of the pump mode of frequency $\omega(t)$, while $\chi(t)$ is the time-dependent coupling function related to the second-order nonlinear susceptibility of the pumped medium. We may consider $\omega(t)$ and $\chi(t)$ as control functions playing the role of the u_l in Eq. (4), since they are real and can be adjusted to piecewise-constant functions of time t, outside the system itself.

Following precedent [69, 70, 71, 72], we define the operators

$$K_+ = \frac{1}{2}(a^\dagger)^2\,, \quad K_- = \frac{1}{2}a^2\,, \quad K_0 = \frac{1}{2}(a^\dagger a + aa^\dagger)\,, \tag{66}$$

which satisfy the commutation relations of $SU(1,1)$, thus

$$[K_0, K_\pm] = \pm K_\pm\,, \quad [K_+, K_-] = -2K_0\,. \tag{67}$$

Setting

$$
\begin{aligned}
H_0 &= -iK_0\,, & (68)\\
H_1 &= -\frac{i}{2}[e^{-2i\omega t}K_+ + e^{2i\omega t}K_-]\,, & (69)\\
H_2 &= \frac{1}{2}[e^{-2i\omega t}K_+ - e^{2i\omega t}K_-]/2\,, & (70)
\end{aligned}
$$

the control system (65) may be written in the more familiar form

$$\frac{\partial}{\partial t}\psi = [\omega(t)H_0 + \chi(t)H_1(t)]\,\psi\,. \tag{71}$$

The skew-Hermitian operators H_0, H_1, and H_2 satisfy the commutation relations

$$[H_0, H_1] = -H_2\,, \quad [H_0, H_2] = H_1\,, \quad [H_1, H_2] = H_0\,. \tag{72}$$

We observe that the system (71) does not have a drift term in the usual sense, because the factor $\omega(t)$ can be manipulated externally. We also see immediately that $\mathcal{A} = \mathcal{B} = \mathcal{C} = \mathcal{L}\{H_0, H_1, H_2\}$, and the second condition of Theorem 4.7 is obviated. In addition, H_0 has eigenvectors $|mk\rangle$, with $m = 0, 1, \ldots$ and $k = 1/4,\ 3/4$, which span an analytic domain $\mathcal{D}_A$ [70, 72]. Consequently, we can choose a manifold M such that $\dim \mathcal{C}\psi = \dim M\ \forall\psi \in \mathcal{D}_A \cap M$. All conditions of Theorem 4.7 being met, the system (65) is strongly analytically controllable on M.

Example 2 Defining $Q = i\partial_t + \partial_{x_1x_1} + \partial_{x_2x_2}$, the Schrödinger equation for a free particle moving in two spatial dimensions may be expressed simply as $Qu = 0$. Determination of the maximal symmetry algebra of this equation leads to the following set of nine operators, which form the basis of a nine-dimensional complex Lie algebra: [73]

$$
\begin{aligned}
&K_2 = -t^2\partial_t - t(x_1\partial_{x_1} + x_2\partial_{x_2}) - t + (i/4)(x_1^2 + x_2^2)\,, K_{-2} = \partial_t\,, \quad P_j = \partial_{x_j}\,,\\
&B_j = -t\partial_{x_j} + ix_j/2\,, J = x_1\partial_{x_2} - x_2\partial_{x_1}\,, \quad E = i\,, D = x_1\partial_{x_1} + x_2\partial_{x_2} + 2t\partial_t + 1\,,
\end{aligned}
\tag{73}
$$

with $j = 1, 2$. Of immediate concern is the real Lie algebra spanned by this basis, i.e., the Schrödinger algebra, which has, as alternative basis, the operators B_j, P_j, and E (yielding the five-dimensional Weyl algebra), plus the operator J and the three operators defined by $L_1 = D$, $L_2 = K_2 + K_{-2}$, and $L_3 = K_{-2} - K_2$. The pertinent nonvanishing commutators are specified by [73]:

$$
\begin{aligned}
&[L_1, L_2] = -2L_3\,,\ [L_3, L_1] = 2L_2\,,\ [L_2, L_3] = 2L_1\,, [L_1, B_j] = B_j\,,\ [L_1, P_j] = -P_j\,,\\
&[P_j, J] = (-1)^{j+1}P_l\,, [B_j, J] = (-1)^{j+1}B_l\,,\ [L_2, B_j] = -P_j\,,\ [L_3, B_j] = -P_j\,,\ [L_2, P_j] = B_j\,,\\
&[L_3, P_j] = -B_j,\ [P_j, B_j] = E/2\,,
\end{aligned}
\tag{74}
$$

where $j, l = 1, 2, j \neq l$.

Now we consider the controllability of the system

$$\frac{\partial}{\partial t}\psi = [L_2 + u_1(t)L_1 + u_2(t)L_3 + u_3(t)P_1 + u_4(t)J]\psi \,. \tag{75}$$

In this case there is a time-dependent drift term in the vector field driving ψ. The relations (74) imply the equalities $\mathcal{B} = \mathcal{C} = \mathcal{L}\{L_1, L_2, L_3, P_1, P_2, B_1, B_2, J, E\}$, while the required analytic domain $\mathcal{D}_{\mathcal{A}}$ is furnished by the span of the eigenfunctions $\psi_{n,m}$ of L_3. These take the explicit, time-dependent form [73]

$$\begin{aligned}\psi_{n,m} &= (2^{m+n+1}\pi n!m!)^{-1/2}\exp[i\pi(m+n-1)/2] \\ &\times \exp\left[\frac{(v_1^2+v_2^2)(1-iv_3)}{4}\right]\left(\frac{v_3+i}{v_3-i}\right)^{(m+n)/2} \times \frac{H_m(v_1/\sqrt{2})H_n(v_2/\sqrt{2})}{v_3-i}\,,\end{aligned} \tag{76}$$

where $x_1 = v_1(1+v_3^2)^{1/2}$, $x_2 = v_2(1+v_3^2)^{1/2}$, and $t = v_3$. It follows as before that the system (75) is strongly analytically controllable.

Example 3 A quantum control system with position-dependent effective mass $m = (2Ax)^{-1}$ has been described by the time-dependent Schrödinger equation [74]

$$i\frac{\partial}{\partial t}\psi = [iBI_0 + u_1(t)A(t)I_0I_- + iu_2(t)C]\psi \,, \tag{77}$$

where B, $C \in \mathbb{R}$ and $A(t)$ is a real function of time t but in general not piecewise-constant. The operators I_0 and $I_\pm$, which are independent of time, provide a basis for an $su(1,1)$ algebra, and have the concrete realization

$$I_- = -\partial_x \,, \quad I_0 = x\partial_x + 1 \,, \quad I_+ = x^2\partial_x + 2x \,, \tag{78}$$

which satisfies the commutative relations

$$[I_0, I_\pm] = \pm I_\pm, \quad [I_-, I_+] = -2I_0 \,. \tag{79}$$

This effective-mass problem arises in the study of semiconductor heterostructures and, more generally, of inhomogeneous crystals [75]. In the semiconductor application, the effective mass of a carrier depends spatially on the graded composition of the semiconductor alloys used in the barrier and well regions of the microstructures [76].

The wave functions of the stationary states of Eq. (77) can be written as

$$\begin{aligned}\psi_E(t,x) &= \frac{1}{\sqrt{2\pi}}\exp\left\{-iE\int_0^t B(\sigma)d\sigma + \int_0^t [-C(\sigma) - \frac{1}{2}B(\sigma)]d\sigma\right\} \\ &\quad \times \exp\{-a_1(t)\,(x\partial_{xx} + \partial_x)\}\, x^{-iE-1/2} \\ &= \frac{1}{\sqrt{2\pi}}\exp\left\{-iE\int_0^t B(\sigma)d\sigma + \int_0^t [-C(\sigma) - \frac{1}{2}B(\sigma)]d\sigma\right\} \\ &\quad \sum_{n=0}^{\infty}\prod_{l=0}^{n}(iB(t)E + \frac{1}{2} + l)^2[-a_1(t)]^n \times \frac{x^{-iE-n-1/2}}{n!}\,.\end{aligned} \tag{80}$$

These eigenfunctions span the analytic domain relevant to Theorem 4.7.

Let us define

$$H_0 = BI_0 + u_2(t)C, , \quad H_1 = -iA(t)I_0 I_-, \tag{81}$$

where we take $u_2(t) = -B/2C$. Eq. (77) can be recast as the control system

$$\frac{\partial}{\partial t}\psi = [H_0 + u_1(t)H_1]\psi \,. \tag{82}$$

Here the drift term is time-independent. Using the commutation relations (79), we obtain $[H_0, H_1] = -BH_1$. Obviously, $\mathcal{B} = \mathcal{C} \subset \mathcal{A}$, so $[\mathcal{B}, \mathcal{C}] = \mathcal{B}$. Choosing a manifold M such that $\dim M = \dim \mathcal{C}\psi$ for all $\psi \in M$, we are assured that system (77) is strongly analytically controllable.

6 Conclusions

In this paper, we have formulated the time-dependent quantum control problem and studied its controllability. Acknowledging the unbounded nature of operators commonly involved in quantum control systems, our analysis has been predicated on the existence of an analytic domain [32] on which exponentiations of such operators are guaranteed to converge. Within this framework, we have extended the established treatment of time-independent quantum control problems by introducing an augmented system described in a state space that is enlarged by one dimension, yet embodies the true dynamics of the original system. With the aid of techniques and results developed by Kunita [55, 59], we are able to explicate the one-dimension-reduced controllability of the augmented system. Projection onto the original state space then yields a proof of the analytic controllability of the original time-dependent quantum control system, under conditions similar to those required in the time-independent case. The theorem so established has been illustrated with examples drawn from mathematical physics and systems engineering.

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