Symplectic Connections and Exotic Holonomy

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Abstract

We show that exotic holonomies arise naturally from torsion-free symplectic connections.

0. Introduction. Advances have been made in the past few years since the first appearance of Bryant's examples of H_3- and G_3-holonomy groups of torsion-free connections not preserving any nondegenerate symmetric bilinear forms, referred to by Bryant as exotic holonomies [3], which do not appear on Berger's original classification list of irreducibly acting reductive holonomy groups [1]. For example, it was proved in [5] that in fact there exists an infinite number of exotic holonomies whose holonomy groups are $SL(2,\mathbb{C})SO(n,\mathbb{C})$. (For simplicity and clarity in presentation we only look at representations of complex Lie groups in the sequel.) All these constructions follow from the following recipe in [5], Theorem 3.7. and Theorem 3.10. Namely,

(i) Let G be a connected reductive Lie group and g its Lie algebra which act irreducibly on V. Let $K(g)$ be the curvature tensor space associated with g and V, i.e., $K(g)$ is the subspace of $\wedge^2(V^*)\otimes g$ satisfying the 1st Bianchi identity. If one can find a G-equivariant element

$$\phi \in (S^2(g)\otimes\wedge^2(V^*))\cap(g\otimes K(g)), \tag{1}$$

then via the Poisson geometry one can construct a non-symmetric torsion-free connection, referred to as a Poisson connection, whose holonomy group is G if the induced map

$$\phi' : g^* \longrightarrow K(g) \tag{2}$$

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by ϕ satisfies the condition that there is an open set U in g^* such that every element $\rho \in \phi(U)$ generates $K(g)$ in the sense that the set

$$\{\rho(x, y) : x, y \in V\} \tag{3}$$

generates g.

(ii) Let $K^1(g)$ be the subspace of $V^* \otimes K(g)$ satisfying the 2nd Bianchi identity. If furthermore ϕ in (1) satisfies the property that both ϕ' in (2) and

$$\phi'' : V^* \longrightarrow K^1(g) \tag{4}$$

induced by ϕ are in fact isomorphisms, so that

$$K(g) \simeq g^*, K^1(g) \simeq V^*, \tag{5}$$

then all torsion-free connections whose holonomy group is contained in G are Poisson connections.

In view of the above recipe, one way to find exotic holonomies is to utilize (ii) above. Namely, find a g such that (5) holds, a possible candidate for exotic holonomy. Then one tries to find an equivariant ϕ inducing these two isomorphisms. In fact, this approach has been responsible for the discovery of the aforementioned $SL(2, \mathbb{C})SO(n, \mathbb{C})$ family and their real counterparts, and the recent $E_7(\mathbb{C})$ example and their real counterparts in [6].

However, in addition to the technical task of establishing (5), which is abstract in nature via, for example, the Borel-Weil-Bott sheaf-cohomological machinery ([2]), one needs to find in each case an explicit equivariant map ϕ in (ii), if it exists, through a detailed knowledge of g.

It is therefore desirable then to search for a unifying general principle as simple and geometric as possible for the construction of exotic holonomies, which is the purpose of this paper.

The class of representations under our investigation will be the *symplectic* representations, i.e., the irreducible representations V of semisimple Lie algebras g such that V admit a g-invariant 2-form $< \cdot, \cdot >$. Any manifold carrying a torsion-free connection with a holonomy group which is a symplectic representation on the typical tangent space is automatically a symplectic manifold. Conversely, locally any symplectic manifold admits many torsion-free connections whose holonomy groups are all symplectic representations on the typical tangent space via the symplectic form. Hence investigating the torsion-free connections of a symplectic manifold should lead to interesting holonomies. As in the Riemannian case, a generic torsion-free connection of a symplectic manifold assumes the symplectic group as the holonomy group. Such a generic connection bears an important feature. Namely, the space of 2-forms of a typical tangent space V of the manifold splits into

$$\wedge^2 V = W \oplus \mathbb{C}$$

for some irreducible $Sp(V)$-module W. This follows by considering the natural $Sp(V)$-equivariant map

$$\mathbf{e} : a \wedge b \longmapsto < a, b >$$

from $\wedge^2(V)$ onto $\mathbb{C}$. Weyl's construction of the representations of the classical groups ([7]) asserts that $Ker(\mathbf{e})$, which is W above, is irreducible. It turns out that this decomposition alone warranties the existence of exotic holonomies. More precisely, we prove the following theorems.

Theorem 1 *Let G be a connected simple Lie group with Lie algebra g. Suppose V is an irreducible representation of g satisfying the following condition:*

$$\wedge^2(V) = W \oplus \mathbb{C},$$

as a g-module, where W is irreducible. (Hence in particular V is a symplectic representation.)

Then there is a non-symmetric torsion-free connection whose holonomy group is G.

Notice that $W = 0$ implies that V is 2-dimensional with the standard $SL(2, \mathbb{C})$-action on it, which gives the 2-dimensional $SL(2, \mathbb{C})$-holonomies. We will therefore assume $W \neq 0$ from now on.

The proof of Theorem 1 utilizes only some elementary properties of symplectic representations.

In fact we can classify the representations satisfying the condition in Theorem 1.

Theorem 2 *All the representations satisfying the condition in Theorem 1 are*

(a) $SL(2, \mathbb{C})$ *acting on* $\mathbb{C}^4$*;*

(b) $SL(6, \mathbb{C})$ *acting on* $\mathbb{C}^{20}$*;*

(c) $Sp(2n, \mathbb{C})$ *acting on* $\mathbb{C}^{2n}$ *for all* n*;*

(d) $Sp(6, \mathbb{C})$ *acting on* $\mathbb{C}^{14}$*;*

(e) $Spin(12, \mathbb{C})$ *acting on* $\mathbb{C}^{32}$*;*

(f) $E_7(\mathbb{C})$ *acting on* $\mathbb{C}^{52}$*.*

As a consequence, Theorems 1 and 2 constitute a simple and unifying criterion for the construction of exotic holonomies in items (a), (b), (d), (e) and (f) and the standard $Sp(2n)$-holonomy that already appeared on Berger's list, where item (a) is the H_3-connections mentioned above and item (f) was recently found in [6].

In fact we could have replaced the above simple groups G by the reductive $\mathbb{C}^* G$. However, since these holonomies are reductions of $\mathbb{C}^* Sp(V)$-structures, which in turn can be reduced to $Sp(V)$-structures when $\dim V > 4$ ([4]), the only reductive group added to our list is G_3 for Item (a).

The author was informed lately by S. Merkulov and L. Schwachhöfer that they had classified all the remaining groups that can possibly arise as exotic holonomies ([8]): Items (b), (d), (e) are the only ones that have not been previously known.

Lastly we remark that in fact the Lie algebras of all except item (c) in Theorem 2 satisfy (5) above, which can be proved by, for instance, a Borel-Weil-Bott type calculation analogous to the one given in [6] for the E_7-case. Hence all torsion-free connections whose holonomy group is one of these groups are Poisson connections.

The author would like to acknowledge the enthusiastic interest of S. Merkulov and L. Schwachhöfer in a weak version of Theorem 1 shown to them in the summer of 1996 when they began to work on the classification, which enabled them to also come up with a different existence proof, resorting to their classification result (and symplectic representation as well), of the aforementioned holonomies in [8]. In contrast, our existence proof is simple and elementary.

1. Symplectic connections. Let V be an even-dimensional vector space equipped with a nondegenerate 2-form $<\cdot,\cdot>$. Via the 2-form V^*, the dual of V, is naturally identified with V by

$$v \longmapsto v^* =:< v, \cdot > . \tag{6}$$

The symplectic group $Sp(V)$ is the group of automorphisms of V preserving the symplectic form, whose Lie algebra $sp(V)$, the space of matrices in $gl(V)$ equivariant with respect to $<\cdot,\cdot>$, can therefore be canonically identified, via (6), with $S^2(V)$, the symmetric product of V, through the morphism $S^2(V) \longrightarrow sp(V)$ defined by

$$x \odot y : z \longmapsto < x, z > y + < y, z > x. \tag{7}$$

Under this identification the Lie bracket assumes the form

$$[a \odot b, c \odot d] =< a, c > b \odot d + < a, d > b \odot c + < b, c > a \odot d + < b, d > a \odot c,$$

and the Killing form is given by

$$(a \odot b, c \odot d) =< a, c >< b, d > + < a, d >< b, c > . \tag{8}$$

(Reason: the right hand side is symmetric and $sp(V)$-equivariant.) We say that a frame $(e_1, \cdots, e_{2s})$ of V is *symplectic* if $< e_i, e_{i+s} >= 1, 1 \leq i \leq s$, and zero otherwise.

Now let M be a symplectic manifold with the symplectic form ω. By Darboux's theorem locally there is a coordinate system $(z^1, \cdots, z^{2s})$ relative to which $(\partial_{z^1}, \cdots, \partial_{z^{2s}})$ is a symplectic frame. Given a connection on M, via the identification (7) its connection form can be locally written as

$$\sum_k \Gamma_{ijk} dz^k,$$

where Γ is symmetric in i and j. It is not hard to check that the connection is torsion-free if and only if Γ is symmetric in all i, j and k. The curvature tensor of a torsion-free connection is of the form

$$\sum_{ijkl} R_{ijkl} e_i \wedge e_j \otimes e_k \odot e_l,$$

where R_{ijkl} is anti-symmetric in i, j, symmetric in k, l and satisfies the 1st Bianchi identity

$$R_{ijkl} + R_{kijl} + R_{jkil} = 0;$$

here we identify $e_i^* \wedge e_j^*$ with $e_i \wedge e_j$ through (6). We are precisely looking at a specific representation space of $sp(V)$. Namely, starting with

$$a \otimes b \otimes c \otimes d \tag{9}$$

in $\otimes^4 V$, we symmetrize the last three vectors and anti-symmetrize the first two vectors, or rather applying the Young symmetrizers ([7]) of the Young diagram

$$\begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & & \end{array}$$

to the element in (9). The resulting subspace of $\otimes^4 V$, generated by a typical element of the form

$$a \wedge b \otimes c \odot d + a \wedge c \otimes b \odot d + a \wedge d \otimes b \odot c$$

is precisely $K(sp(V))$, the curvature space of $sp(V)$ acting canonically on V. In fact there is a natural $sp(V)$-equivariant surjective contraction map

$$\pi : K(sp(V)) \longrightarrow sp(V) \simeq S^2(V)$$

given by

$$\pi : a \otimes b \otimes c \otimes d \longmapsto < a, b > c \otimes d,$$

or in coordinate terms

$$\pi : \sum_{ijkl} R_{ijkl} e_i \wedge e_j \otimes e_k \odot e_l \longmapsto \sum_{ijkl} R_{ijkl} < e_i, e_j > e_k \odot e_l,$$

with respect to which one has the split exact sequence

$$0 \longrightarrow Ker(\pi) \longrightarrow K(sp(V)) \longrightarrow sp(V) \longrightarrow 0,$$

where the split morphism

$$f : sp(V) \longrightarrow K(sp(V))$$

is given, up to a constant factor, by

$$f : a \otimes b \longmapsto \sum_{i,j} \Psi_{i,j}(a \otimes b),$$

where for each unordered pair of distinct numbers i, j between 1 and 4, $\Psi_{i,j}$ is the map that inserts the dual of $< \cdot, \cdot >$, i.e., $\sum_{i=1}^{s} e_i \wedge e_{i+s}$ with respect to a symplectic frame, into the i and j slots to form a 4-vector followed by the appropriate symmetrization. Explicitly,

$$\begin{aligned} f \quad &: \quad a \odot b \longmapsto 2 \sum_i e_i \wedge e_{i+s} \otimes a \odot b \\ &- \quad \sum_i a \wedge e_i \otimes e_{i+s} \odot b - \sum_i e_{i+s} \wedge a \otimes e_i \odot b \\ &- \quad \sum_i b \wedge e_i \otimes e_{i+s} \odot a - \sum_i e_{i+s} \wedge b \otimes e_i \odot a, \end{aligned} \tag{10}$$

or in more intrinsic forms

$$f : A \in S^2(V) \simeq sp(V) \longmapsto R(u, v) =: 2 < u, v > A + Au \odot v - Av \odot u \tag{11}$$

through the identification in (7). In particular,

$$K(sp(V)) = sp(V) + Ker(\pi).$$

In fact, by Weyl's construction of the representations of the classical groups ([7]), one sees that $Ker(\pi)$ is an irreducible representation space of $sp(V)$ associated with the above Young diagram.

Notice that V as a $sp(V)$-module has the property that

$$\wedge^2(V) = W \oplus \mathbb{C},$$

where W is irreducible as a $sp(V)$-module, a property pertinent to our discussions in the sequel. Again this follows from Weyl's construction by applying only the column Young anti-symmetrizer to $a \otimes b$ where the $sp(V)$-equivariant contraction map sends $a \wedge b$ to $< a, b >$, whose kernel is exactly W.

We point out that the identification (7) is equivalent to the following more intrinsic identity

$$(x \odot y, A) =< Ax, y > \tag{12}$$

for A in $sp(V)$ in view of (8), which will be important for later developments.

2. The existence. We prove Theorem 1 in this section. We begin with the observation in the following lemma, remarking that $< \cdot, \cdot >$ is the symplectic form.

Lemma 1 *Let V be an irreducible representation of a simple Lie algebra such that $V = W \oplus \mathbb{C}$ with W irreducible. Then the equivariant subspace of $S^2(\wedge^2(V))$ is 2-dimensional. Moreover, this subspace is generated by the two morphisms*

$$\begin{aligned} (u \wedge s) \odot (v \wedge t) &\longmapsto\ < u, s >< v, t >, && (13)\\ (u \wedge s) \odot (v \wedge t) &\longmapsto\ < u, v >< s, t > - < u, t >< s, v > && (14) \end{aligned}$$

when we identify $\wedge^2(V)$ with $\wedge^2(V^*)$ via the symplectic form.

Proof. In general we have the identity

$$S^2(\wedge^2(V)) = S^2(W) \oplus W \oplus \mathbb{C}.$$

Now by the fact that W is irreducible we must have that $S^2(W)$ has at most one $\mathbb{C}$-summand since all invariant symmetric forms on W are multiples of one another ([10]). Hence $S^2(\wedge^2(V))$ has at most two $\mathbb{C}$-summand. It must therefore have exactly two $\mathbb{C}$-summands, because it is easy to see that the two morphisms in (13) and (14) are independent equivariant elements in $S^2(\wedge^2(V))$ as long as $\dim V > 2$. $\square$

Recall the Killing form of g is denoted by $(\cdot, \cdot)$. Motivated by (12), one can also define a symmetric product "$\circ$" on V with values in g by

$$(x \circ y, A) =< Ax, y > \tag{15}$$

for all A in g and x, y in V.

The Lie algebra $(sp(V), < \cdot, \cdot >)$ contains g as a Lie subalgebra in a natural way and the Killing form $(\cdot, \cdot)_{sp}$ of $sp(V)$, when restricted to g, is a constant multiple of the Killing form $(\cdot, \cdot)$ of g; we may thus assume they are equal on g without loss of generality. With this understood, we have on V two symmetric products $x \odot y$ and $x \circ y$ relative to $(\cdot, \cdot)_{sp}$ and $(\cdot, \cdot)$, respectively.

Lemma 2 *g is spanned by elements of the form $x \circ y$.*

Proof. With the assumption that $(\cdot,\cdot)_{sp} = (\cdot,\cdot)$ on g , it follows from (15) that $(x \odot y - x \circ y, A)_{sp} = 0$ for all A in g. In other words, $x \circ y$ is the orthogonal projection of $x \odot y$ under $(\cdot,\cdot)_{sp}$. The conclusion follows. □

We are now ready to prove Theorem 1. Consider the following equivariant element in $S^2(\wedge^2(V^*))$:

$$(u \circ v, s \circ t) - (u \circ t, s \circ v), \tag{16}$$

where $(\cdot,\cdot)$ is the Killing form. Under the condition of Theorem 1, the quantity in (16) is a linear combination of (13) and (14) thanks to Lemma 1. Hence

$$\begin{aligned} & (u \circ v, s \circ t) - (u \circ t, s \circ v) \\ = \ & c_1 < u, s >< v, t > + c_2(< u, v >< s, t > - < u, t >< s, v >) \end{aligned} \tag{17}$$

for some constant c_1 and c_2.

Motivated by (11), for each $A \in g$ consider the tensor

$$R(A) : x \wedge y \longrightarrow -c_1 < x, y > A + Ax \circ y - Ay \circ x.$$

It is elementary to check by (17) that $R(A)(x, y) \in g$ satisfies the 1st Bianchi identity. Furthermore,

$$\phi(x, y, A, B) =: (R(A)(x, y), B)$$

is an equivariant element given in (1) and ϕ' defined in (2) is exactly R when we identify g^* with g via the Killing form $(\cdot,\cdot)$.

To complete the proof of Theorem 1, let us apply criterion (i) stated in Section 0. More precisely, we want to show that (3) is satisfied for an element of g. Then it will be true in a neighborhood of this element by continuity, and hence the conclusion of the theorem.

To this end, let us pick an element H in the Cartan subalgebra such that $(\mu - \nu)(H) \neq 0$ for all weights μ and ν and $H \neq t_\mu$, the dual of μ, for all weights μ. Consider $R(H)$ defined above. Now

$$Hx_\mu \circ x_\nu - Hx_\nu \circ x_\mu = (\mu - \nu)(H) x_\mu \circ x_\nu,$$

so that for all μ and ν, $x_\mu \circ x_\nu$ plus a multiple of H is in the span of $\{R(H)(x, y) : x, y \in V\}$, which therefore generates g by Lemma 2. Theorem 1 is thus proved. □

3. The classification. Since symplectic representations have been well understood, we will leave the reader to consult, e.g., [10], for details. In what follows, we will denote by $\lambda_i, 1 \leq i \leq n$, the fundamental weights of a simple Lie algebra g of rank n whose simple roots are $\alpha_i, 1 \leq i \leq n$. For a dominant weight μ, we will denote by $V(\mu)$ the irreducible representation space of g whose highest weight is μ.

Lemma 3 *Let V be an irreducible representation of a simple Lie algebra g with highest weight λ. Let*

$$\lambda = \sum_{i=1}^{n} a_i \lambda_i \tag{18}$$

for some nonnegative integers a_i. If $a_s > 0$ for some s, then $2\lambda - \alpha_s$ is a dominant weight such that $V(2\lambda - \alpha_s)$ is a summand of $\wedge^2(V)$.

Proof. It is easy to see $2\lambda - \alpha_s$ is a dominant weight by a look at the Cartan matrices and to express α_s as a linear combination of the fundamental weights. Now the weights in the weight chain $\lambda - k\alpha_s, 1 \leq k \leq a_s$, are all weights for V. In particular, taking the wedge product of a weight vector with weight λ and one with weight $\lambda - \alpha_s$ we see that $2\lambda - \alpha_s$ is a dominant weight for $\wedge^2(V)$. The lemma follows by observing that $2\lambda - \alpha_s$ must be the highest weight in the summand of $\wedge^2(V)$ where it belongs. $\Box$

Lemma 4 *Let V be an irreducible representation of a simple Lie algebra g satisfying*

$$\wedge^2(V) = W \oplus \mathbb{C}$$

for some irreducible W. Then V can only possibly be one of the following:

(i) *$g = sl(4n+2, \mathbb{C})$ acting on $V(m\lambda_{2n+1})$;*

(ii) *$g = so(2n+1, \mathbb{C})$ acing on $V(m\lambda_n)$;*

(iii) *$g = sp(2n, \mathbb{C})$ acting on $V(m\lambda_{2i+1})$ for all $i \geq 0$;*

(iv) *$g = so(2n, \mathbb{C})$ acting on $V(m\lambda_{n-1})$ or $V(m\lambda_n)$;*

(v) *$g = e_7(\mathbb{C})$ acting on $V(m\lambda_1)$, $V(m\lambda_3)$ or $V(m\lambda_7)$*

for some positive odd integer m. (It suffices to consider $V(m\lambda_n)$ in item (iv) by symmetry.)

Proof. Let λ be the highest weight of V. We know the complete classification of all symplectic representations ([10]), which are (notation as in (18))

(1) $g = sl(n+1, \mathbb{C}), n = 1 \; mod(4)$, acting on $V(\lambda)$ with $a_1 = a_n, a_2 = a_{n-1}, a_3 = a_{n-2}$, etc, and $a_{(n+1)/2}$ being odd;

(2) $g = so(2n+1, \mathbb{C}), n = 1, 2 \; mod(4)$, acting on $V(\lambda)$ with a_n being odd;

(3) $g = sp(2n, \mathbb{C})$ acting on $V(\lambda)$ with $a_1 + a_3 + a_5 + \cdots$ being odd;

(4) $g = so(2n, \mathbb{C}), n = 2 \; mod(4)$, acting on $V(\lambda)$ with $a_{n-1} + a_n$ being odd;

(5) $g = e_7(\mathbb{C})$ with $a_1 + a_3 + a_7$ being odd.

Now it follows immediately from Lemma 3 and the assumption that W is irreducible that there is one and only one simple root α_s such that $a_s \neq 0$; hence the lemma is proved. $\square$

In particular, under the condition of Lemma 4, we have

$$W = V(2\lambda_s - \alpha_s) \tag{19}$$

for the appropriate s in Lemma 4.

To complete the classification, it suffices now to show that, in addition to $g = sp(2n, \mathbb{C})$ acting on $V(\lambda_1) = \mathbb{C}^{2n}$, the pairs (n, m) in Lemma 4 are $(0, 3)$ and $(1, 1)$ for item (i), and $(2, 1), (3, 1)(i = 1), (3, 1)$ for items (ii) through (iv), respectively, and $m = 1$ with e_7 acting on $V(\lambda_1)$ in item (v). Note that since $Spin(5, \mathbb{C}) \simeq Sp(4, \mathbb{C})$, item (ii) is not included in Theorem 2.

The case when $n = 0$ in item (i) readily follows from the Clebsch-Gordan formula

$$\wedge^2(m\lambda_1) = \oplus_{i \geq 0} V((2m - 4i - 2)\lambda_1)$$

for $SL(2, \mathbb{C})$.

We next settle the cases when the highest weight is $m\lambda_n$ in items (ii) through (iv) and all the ones in item (v) by showing, via Weyl's dimension formula

$$\dim V(\lambda) = \prod_{\alpha} (\lambda + \rho, \alpha)/(\rho, \alpha),$$

where $(\cdot,\cdot)$ is the Killing form, $\rho = \lambda_1 + \cdots + \lambda_n$ and α ranges through all positive roots, that

$$\dim \wedge^2 V(m\lambda_s) = \dim V(2\lambda_s - \alpha_s) + 1 \tag{20}$$

if and only if $m = 1$ and $n = 2, 3, 3$ for items (ii) through (iv), respectively, while $m = 1$ with e_7 acting on λ_1 for item (v).

Since the explicit dimension formulas for the classical groups are available ([7]), we will only work out the $sp(2n, \mathbb{C})$ case in some details, the remaining classical cases being similar. The simple observation which is crucial in the estimates is the inequality

$$(m + k)^2 \geq (k + l)(2m + k - l) \tag{21}$$

for any real numbers m, k, l. Let $\lambda = a_1\lambda_1 + \cdots + a_n\lambda_n$ be the highest weight of an irreducible representation. Let

$$p_i = a_i + \cdots + a_n + n - i.$$

Then for $sp(2n, \mathbb{C})$

$$\dim V(\lambda) = \prod_{i<j}(p_i - p_j)(p_i + p_j + 2)\prod_i (p_i + 1)/(2n - 1)!(2n - 3)! \cdots 1!.$$

In the case we are interested in we have two weights $m\lambda_n$ and $2m\lambda_n - \alpha_n$ to compare with in view of (19). Now it suffices to handle the ratio

$$(\dim V(\lambda_n))^2 / \dim V(2\lambda_n - \alpha_n), \tag{22}$$

which comes down to

$$\frac{\frac{(m+1)^2}{2m-1}\prod_{i<j<n}\frac{(j-i)(2m+2n+2-i-j)^2}{4m+2n+2-i-j}\prod_{i<n}\frac{(n-i)^2(2m+n+2-i)^2(m+n-i+1)^2}{(n-i+2)(4m+n-i)(2m+n-i+1)}}{(2n-1)!(2n-3)!\cdots 1!}.$$

One then employs (21) to replace the ratios with m in the expression by one without m. The upshot is that the ratio in (22) is

$$\geq (n + 3)(n + 2)(n - 1)!/12(n + 1)n > 3.7$$

for $n \geq 5$. Hence (20) does not hold if $n \geq 5$. For $n = 3, 4$, one calculates directly the left side of (20) subtracted by its right side. In fact, shifting m

to $m+2$ one finds that the difference is a polynomial of degree 12 and 20, respectively, in m with positive coefficients whose constant term is 1028 and 86031, respectively (it is 36 for $n=4$ and $m=1$), so that (20) holds only when $n=3$ and $m=1$.

Likewise, for $so(2n+1,\mathbb{C})$ it comes down to the estimate

$$2^{n-3}\cdot 3(n+1)(n-1)!/n > 4$$

for $n \geq 3$. For $n=2$, again a calculation shows that, shifting m to $m+2$, the difference in (20) is a polynomial of degree 6 in m with positive coefficients whose constant term is 9, so that (20) is violated except when $m=1$.

For $so(2n,\mathbb{C})$ the estimate yields

$$(n-2)!(n+1)/12(n-1) > 4$$

for $n \geq 7$, while for $n=6$, a calculation, shifting m to $m+2$, shows that the difference in (20) is a polynomial of degree 30 in m with positive coefficients whose constant term is 21548, so that again only $m=1$ for (20) to hold.

For $e_7(\mathbb{C})$, the 63 positive roots are

$$\begin{aligned} e_i - e_j \quad &, \quad 1 \leq i < j \leq 7, \\ e_8 - e_i \quad &, \quad 1 \leq i \leq 7, \\ e_i + e_j + e_k + e_8 \quad &, \quad 1 \leq i < j < k \leq 7, \end{aligned}$$

where $e_1, \cdots, e_8$ are vectors in $\mathbb{R}^8$, with the 7 simple roots $e_i - e_{i+1}, 1 \leq i \leq 6$, and $e_5+e_6+e_7+e_8$. It is then straightforward to express all positive roots in terms of the simple ones and calculate by Weyl's dimension formula; indeed writing $\alpha = \sum_i n_i \alpha_i$, the typical term in the dimension formula is simply

$$\sum_i (a_i+1)n_i / \sum_i n_i$$

for $\lambda = a_1\lambda_1 + \cdots + a_n\lambda_n$. It then follows that (20) is true only when $m=1$ and the representation is λ_1. For interested readers, MATHEMATICA can handle this in a matter of minutes. In fact, in the e_7 case, shifting m to $m+2$ for λ_1 and m to $m+1$ for λ_3 and λ_7, one finds that the difference in (20) is a polynomial in m of degree 54, 100 and 84 with positive coefficients whose constant term is 153747, 172765802 and 49665 for λ_1, λ_3 and λ_7, respectively.

The above estimate for the dimension count becomes more and more disabled if we shift more and more to the left side of the Dynkin diagram. To take care of the two remaining cases, namely, $sl(4n+2,\mathbb{C})$ acting on $V(m\lambda_{2n+1}), n \geq 1$, and $sp(2n,\mathbb{C})$ acting on $V(m\lambda_{2i+1})$ for $1 < 2i+1 < n$, we first observe the following:

(•) If μ and τ are two different weights of $V(\lambda)$, then $\mu+\tau$ is a weight of the space $\wedge^2 V(\lambda)$ with multiplicity at least $(\dim V_\mu)(\dim V_\tau)$, where V_μ, etc., stands for the weight space of μ.

In general, the weights of a $sl(k+1,\mathbb{C})$-representation $V(\lambda)$ can be explicitly described ([7]). Namely, for a dominant weight $\lambda = a_1\lambda_1 + \cdots + a_k\lambda_k$, let

$$d = a_1 + \cdots + a_k$$

and consider the collection of all the partitions of d into at most k ordered parts $(q_1, \cdots, q_k), q_1 \geq q_2 \geq \cdots \geq q_k \geq 0$, and identify each of theses partitions with a dominant weight

$$\sum_{i=1}^{k} (q_i - q_{i+1})\lambda_i \tag{23}$$

with $q_{k+1} = 0$. Alternatively, we can think of the partition in terms of Young diagram where the i-th row in the diagram has q_i boxes. Say two dominant weights τ_1 and τ_2 satisfy $\tau_1 < \tau_2$ if the Young diagram of τ_2 has more boxes than the diagram of τ_1 at the first row where the number of boxes differ for the two diagrams. Then the dominant weights of $V(\lambda)$ are all the weights $\mu < \lambda$ such that the dimension of the weight space of μ in $V(\lambda)$ is the Kostka number $K_{\lambda,\mu} \neq 0$ ([7]). Here, with μ given as in (23), $K_{\lambda,\mu}$ is the number of ways to fill Young's diagram of λ with q_1 1's, q_2 2's,$\cdots$ and q_k k's in such a way that the filling numbers in each row are non-decreasing and those in each column are strictly increasing. All other weights are obtained by permuting arbitrarily the weight components of dominant weights.

Now for $sl(4n+2,\mathbb{C})$ with $n \geq 2$, the weight

$$\mu =: m\lambda_{2n-2} + m\lambda_{2n+4}$$

has the Young diagram

$$(2m, \cdots, 2m, m, \cdots, m, 0, 0, \cdots), \tag{24}$$

where $2m$ appears $2n-2$ times (here the assumption $n \geq 2$ comes in) and m appears 6 times, is a weight of $V(2m\lambda_{2n+1} - \alpha_{2n+1})$, whose Young diagram for its highest weight

$$\lambda =: 2m\lambda_{2n+1} - \alpha_{2n+1}$$

is

$$(2m, \cdots, 2m, 2m-1, 1, 0, 0, \cdots), \tag{25}$$

where $2m$ appears $2n$ times. Now a combinatorial count gives

$$K_{\lambda,\mu} \leq 2m^2.$$

However, μ can be obtained by taking the sum of two weights of $V(m\lambda_{2n+1})$ in view of ($\bullet$) as follows:

($\bullet\bullet$) Let τ be a dominant weight of $V(m\lambda_3)$ for $sl(7, \mathbb{C})$ and let the Young diagram of τ be $(q_1, \cdots, q_6)$. One has the complementary Young diagram $(m - q_6, \cdots, m - q_1)$ for a dominant weight τ^c. Add the diagrams of τ and τ^c, respectively, to the rectangular Young diagram of depth $2n-2$ and length m to form weights σ and σ^c, respectively; σ and σ^c are dominant weights of $V(m\lambda_{2n+1})$. Shuffle the diagram of τ in σ around, by an appropriate Weyl group action, to get a weight σ_1 and lock it up with the weight σ_1^c, obtained by the corresponding shuffling of τ^c in σ^c, to form the diagram $(2m, \cdots, 2m, m, m, m, m, m, m)$. $\sigma_1 + \sigma_1^c$ is precisely the weight μ in (24). $\sigma_1 \neq \sigma_1^c$, because they are equal only when $q_1, \cdots, q_6$ are identical, i.e., only when m is even, which is excluded in view of Lemma 4.

From ($\bullet$) and ($\bullet\bullet$) it follows that there are at least $\dim V(m\lambda_3)/2$ (for $sl(7, \mathbb{C})$) worth of weight vectors with weight μ in $\wedge^2 V(m\lambda_{2n+1})$, which is a polynomial of degree 12 in m with positive coefficients whose m^2-term is $3301m^2/400$, larger than $2m^2$, the maximal possible dimension of the weight space with the same weight in $V(2m\lambda_{2n+1} - \alpha_{2n+1})$.

What we have established so far is that (20) cannot hold if $n \geq 2$. It is now a straightforward dimension count to see for $n = 1$, shifting m to $m + 2$, the difference in (20) is a polynomial of degree 18 in m with positive coefficients whose constant coefficient is 3884, so that once more only $m = 1$ for (20) to be true when $n = 1$.

Now consider the $sp(2n, \mathbb{C})$ case with $1 < 2i + 1 < n$. For a dominant weight $\lambda = a_1\lambda_1 + \cdots + a_n\lambda_n$ let

$$q_i = a_i + \cdots + a_n$$

and identify λ with the vector, or the Young diagram

$$(q_1, \cdots, q_n). \tag{26}$$

Then the action of the Weyl group on λ in the symplectic case is to permute all the q_i's while assign independently plus or minus to each of them. The dominant weight $m\lambda_{2i+1}$ has the Young diagram

$$(m, \cdots, m, 0, 0, \cdots),$$

where m appears $2i + 1$ times. We see by the described Weyl group action that

$$2m\lambda_{2i} - m\lambda_{2i+1}$$

is a weight of $V(m\lambda_{2i+1})$ since its Young diagram is nothing but

$$(m, \cdots, m, -m, 0, 0, \cdots)$$

with m appearing $m - 1$ times. Hence by ($\bullet$)

$$\mu = 2m\lambda_{2i}$$

is a nontrivial (dominant) weight of $\wedge^2 V(m\lambda_{2i+1})$.

Let us look at Kostant's multiplicity formula for the dimension m_μ of a weight space with weight μ appearing in the representation space $V(\lambda)$:

$$m_\mu = \sum_{S \in \mathcal{W}} (-1)^S P(S(\lambda + \rho) - (\mu + \rho)),$$

where $\mathcal{W}$ is the Weyl group, and P is the number of ways a weight can be a sum of positive roots. Moreover, the sign $(-1)^S$ in the multiplicity formula is the multiplication of the assigned pluses and minuses and the permutation sign of S. In our case, set

$$s = 2i + 1$$

for simplicity. The weights pertinent to our consideration are

$$\lambda = 2m\lambda_s - \alpha_s$$

and

$$\mu = 2m\lambda_{s-1}.$$

When we identify them with the diagram in (26), we see that

$$\begin{aligned}\lambda + \rho &= (2m+n, \cdots, 2m+n-s+2, 2m+n-s, n-s+1, n-s-1, \cdots, 1), \\ \mu + \rho &= (2m+n, \cdots, 2m+n-s+2, n-s+1, n-s, n-s-1, \cdots, 1),\end{aligned}$$

where the appearance of $\cdots$ indicates that the integers decrease by 1, and $2m+n-s+2$ occurs at the $(s-1)$-th slot for both diagrams. A moment's thinking of these two diagrams and the multiplicity formula asserts that the components in the first $s-1$ slots in both diagrams have to be left fixed by any shuffling of the Weyl group, because the counting $P \neq 0$ in the multiplicity formula only when $S(\lambda+\rho) - (\tau+\rho)$ is a combination of simple roots with non-negative integral coefficients. This means that we can now ignore the first $s-1$ slots in the counting as if they were not there while observe that the remaining components of the two diagrams are identified, in order, with the weights $2m\lambda_1 - \alpha_1$ and 0 for $sp(n-s+1, \mathbb{C})$. In other words, the multiplicity of μ in $V(\lambda)$ is equal to the multiplicity of the 0 weight in $V(2m\lambda_1 - \alpha_1)$ for $sp(n-s+1, \mathbb{C})$, which in turn equals, by the character formula ([7]), the constant coefficient of

$$J_{2m-1}J_1 - J_{2m} - J_{2m-2}, \tag{27}$$

where $J_d(x_1, \cdots, x_k) = H_d(x_1, \cdots, x_k, x_1^{-1}, \cdots, x_k^{-1})$, H_d is the complete symmetric polynomial in $2k$ variables and

$$k =: n - s + 1 \geq 2,$$

which equals

$$A \quad = \quad ((2k-2)m - k + 1)(k+m-2)!/m!(k-1)!.$$

On the other hand, the weight μ in $V(\lambda)$ can be obtained in view of ($\bullet$) as follows:

(•••) Let τ be a dominant weight of $V(m\lambda_1)$ in $sp(k, \mathbb{C})$. Add the Young diagram of τ to the rectangular diagram with depth $s-1$ and length m to form a dominant weight σ of $V(m\lambda_s)$. Shuffle τ around, by an appropriate Weyl group action, in σ to get a weight ν. There comes with ν the weight ν' such that the shuffled part of ν' is the negative of that of ν. Then $\nu + \nu'$ is precisely μ. $\nu \neq \nu'$, because they are equal only when $\tau = 0$ and 0 is not a weight for $V(m\lambda_1)$, whose character is J_m, when m is odd.

Therefore, the multiplicity of μ in $\wedge^2 V(\lambda_s)$ equals $\dim V(m\lambda_1)/2$ for $sp(k, \mathbb{C})$, which, by the dimension formula, is

$$B = (2k + m - 1)!/2(2k-1)!m!.$$

Now for a fixed $k \geq 2$, B is a polynomial of degree $2k-1$ in m and A is a polynomial of degree $k-1$ in m, so that B will be larger than A when m gets big enough. To carry this out, we differentiate $\log(B/A)$ with respect to m and find it equal to

$$\begin{aligned} & - \frac{2k-2}{(2k-2)m-k+1} + \sum_{i=k}^{2k} \frac{1}{m-1+i} \\ & \geq \frac{k+1}{2k-1+m} - \frac{2k-2}{(2k-2)m-k+1} \\ & > 0 \end{aligned}$$

when $m \geq 3$. Moreover, it is straightforward to check that

$$B - A = (20k + 29k^2 + 10k^3 + k^4)/24$$

for $m = 4$. Hence $B > A$ for all $m \geq 4$. For $m = 1$, one can easily calculate to see $A = k-1$ while $B = k$. We are only left with $m = 3$ since m is an odd number. (In fact, $B < A$ in this case.) Now it is not hard to see that the weights of $V(3\lambda_1)$ for $sl(k, \mathbb{C})$ are of multiplicity 1 except for $2k$ weights of the form x_i and $x_i^{-1}, 1 \leq i \leq k$, in the character H_3 of $V(3\lambda_1)$, each of which has multiplicity k as a result of the expression $x_i = x_i x_j x_j^{-1}$. Let n_1 be the number of half of the weights of multiplicity 1. Then

$$n_1 + k^2 = B.$$

On the other hand, in view of (•) and (•••), the number of weights in $V(\lambda)$, being contributed by $x_\sigma \wedge x_{-\sigma}$ for x_σ in $V(m\lambda_1)$, is

$$\begin{aligned} & n_1 + k^3 \\ = \; & B - k^2 + k^3 \\ = \; & A + \frac{7k + 5k^3}{6}. \end{aligned}$$

Hence, (20) is violated.

In any event what we have shown is that there is at least one weight vector with weight $2m\lambda_{s-1}$ in $\wedge^2 V(m\lambda_s)$ which does not belong to $V(2m\lambda_s - \alpha_s)$, so that (20) cannot hold true in this case at all.

In summary, we have arrived at the classification in Theorem 2. To conclude we point out the interesting property that all the weight spaces of W in the classification are 1-dimensional.

We also remark that the method in this paper also yields the real holonomies $SL(6,\mathbb{R}), SU(1,5)$ and $SU(3,3)$ acting on $\mathbb{R}^{20}$, $Sp(6,\mathbb{R})$ acting on $\mathbb{R}^{14}$, $Spin(2,10)$ and $Spin(6,6)$ acting on $\mathbb{R}^{32}$, and E_7^5 and E_7^7 acting on $\mathbb{R}^{56}$ corresponding to their complex counterparts in Theorem 2 by checking the self-conjugacy and calculating the indexes of the real forms of these complex Lie algebras ([9]), which we will not dwell upon.

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