Degenerate torsion-free G_3-connections

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Abstract

Using exterior differential systems (EDS), Bryant proved that the moduli space of *nondegenerate* analytic torsion-free G_3-connections depends on four functions in three variables. Here nondegeneracy is a technical condition which imposes the nonvanishing everywhere of a certain determinant pertinent to a torsion-free G_3-connection for EDS to carry through. The finite-dimensional moduli of homogeneous torsion-free G_3-connections are *degenerate* examples, where the determinant vanishes identically. We establish in fact that the moduli space of analytic *inhomogeneous* degenerate torsion-free G_3-connections is infinite-dimensional.

0. Introduction. The notion of holonomy group has played a fundamentally important role in differential geometry since it was first introduced by Cartan [4] to study symmetric spaces. It was not until Berger, who classified in [1] all irreducibly acting reductive groups of the general linear group which may arise as the holonomy group of a torsion-free connection, that systematic studies of holonomy groups began to ensue in the years to follow. (All connections from now on will be understood to be torsion-free.) Berger's classification is complete in the metric case and, as Bryant pointed out [2], is incomplete in the non-metric case; these missing holonomy groups from Berger's list were referred to by Bryant as *exotic holonomies*. Indeed, Bryant found in [2] exotic H_3- and G_3-connections on 4-manifolds that do not preserve any metric, and subsequently gave exotic $Spin(10)$ and E_6 families in [3]. On the other hand, motivated by Bryant's H_3 holonomies, Chi

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et al. [6] , by exploring Poisson geometry and the Borel-Weil-Bott theorem, exhibited an infinite series of exotic $SO(3,\mathbb{C}) \times SO(n,\mathbb{C})$ holonomies and their real counterparts. Pursuing the method further, Chi et al. established exotic E_7 holonomies in [7]. Moreover, utilizing symplectic connections and symplectic representations, the author showed [5] in a unifying way the existence of and classified completely a class of exotic holonomies, which will be referred to as the *symplectic* family henceforth, which includes the aforementioned $H_3(= SL(2))$ and E_7, plus three more, namely, $SL(6,\mathbb{C})$, $Sp(6,\mathbb{C})$, and $Spin(12,\mathbb{C})$ families and their real counterparts. Finally, Merkulov and Schwachhöfer proved in [10] that the above exhaust all the exotic holonomies.

In retrospect, the G_3 family is an anomaly. Notice that we could replace the simple Lie groups G in the symplectic family by $\mathbb{C}^* G$. However, since these holonomies are reductions of conformally symplectic structures, which in turn can be reduced to symplectic structures when the dimension of the manifold > 4 ([2]), the only reductive group extrapolated out of the symplectic family is G_3, which is $\mathbb{C}^* H_3$.

G_3 sticks out in another aspect. Namely, the moduli spaces of all the holonomies in the symplectic family are analytic and finite-dimensional [5] whereas the moduli space of the analytic G_3 class is parametrized by four functions in three variables [2], which Bryant deduced using exterior differential systems. More precisely, let V_n be the space of homogeneous polynomials of degree n in two variables. The Lie algebra $\mathcal{G}_3$ of G_3 can be identified with $V_0 \oplus V_2$, and the curvature space $K(\mathcal{G}_3)$ of the $\mathcal{G}_3$-representation space V_3 satisfying the first Bianchi identity is isomorphic to

$$\mathcal{V} := V_2 \oplus V_4.$$

Therefore a torsion-free connection on a principal G_3-frame bundle P over a 4-manifold naturally gives rise to the $V_0 \oplus V_2$-valued connection form $\lambda \oplus \phi$ and the V_3-valued canonical form ω on P. Furthermore, by the structure of the curvature space mentioned earlier, there is a G_3-equivariant $V_2 \oplus V_4$-valued function $a_2 \oplus a_4$ on P giving the curvature form. In the same vein, the curvature space $K^1(\mathcal{G}_3)$ of the representation V_3 satisfying the second Bianchi identity is isomorphic to

$$\mathcal{W} := V_1 \oplus V_3 \oplus V_5 \oplus V_7,$$

so that the covariant derivative of the curvature tensor of the torsion-free G_3-connection is given in terms of a G_3-equivariant $V_1 \oplus V_3 \oplus V_5 \oplus V_7$-valued function $b_1 \oplus b_3 \oplus b_5 \oplus b_7$. One derives an identity

$$d(a_2 + a_4) = J(\lambda + \phi + \omega), \tag{1}$$

where J is a certain 8×8 matrix whose entries are linear polynomials in $a_2, a_4, b_1, b_3, b_5, b_7$. To prove the existence of G_3-connections, Bryant starts with the open set $\mathcal{O}$ of $\mathcal{V} \oplus \mathcal{W}$ on which J is nonsingular. Motivated by (1) one then considers the form defined by

$$\lambda + \phi + \omega := J^{-1} d(a_2 + a_4),$$

with respect to which one can define three natural 2-forms Θ, Λ, Φ, which measure the extent to which a G_3-connection fails to be torsion-free. Then one shows that the differential ideal generated by Θ, Λ, Φ on $\mathcal{O}$ is differentially closed, from which one applies Cartan-Kähler theory to arrive at the existence and the moduli.

One special feature in Bryant's existence proof is that for the theory of differential systems to work, one has to assume the nonsingularity of the matrix J. We will call those G_3-connections constructed over the domain $\mathcal{O}$, that is, over where the determinant of J is nonzero, *nondegenerate* G_3-connections.

Are there any *degenerate* G_3-connections, i.e., are there any G_3-connections for which J vanishes identically? Since nondegenerate G_3-connections do not admit any nontrivial infinitesimal symmetries [2], the homogeneous G_3-connections are indeed degenerate examples, which have been classified by Schwachhöfer [12]: they form one 4-dimensional component, seven 1-dimensional components and fourteen points. So the right question to ask is whether there are any *inhomogeneous* degenerate G_3-connections. Clearly, Cartan-Kähler theory no longer applies. This is the question we will address in this paper. In fact, we will prove the following.

Theorem 1 *The moduli space of analytic inhomogeneous degenerate G_3-connections is infinite-dimensional.*

Recall that Bryant [2] pointed out a twistorial interpretation of G_3-connections in the holomorphic category (real-analytic G_3-connections are real slices of them). Namely, let W be a complex contact 3-fold in which there is a smooth rational curve C such that $L|_C$, the restriction of the contact line bundle L of W to C, is $\mathcal{O}(3)$. Then the 4-dimensional complete deformation family of C in W is naturally endowed with a G_3-connection. Conversely, a holomorphic G_3-connection gives rise to a contact 3-fold W and a 4-dimensional deformation family around a smooth rational curve in W, to which the restriction of the contact line bundle of W is $\mathcal{O}(3)$.

Our idea is to deform the 1st-jet bundle of $\mathcal{O}(3)$, which carries a natural contact structure, to an infinite-dimensional family of contact 3-folds each of whose members is equipped with a G_3-structure with a nontrivial infinitesimal symmetry. Then such G_3-structures must be degenerate, and since the homogeneous ones are only finite-dimensional, the theorem follows.

Our result complements in a sense one of LeBrun's [9] about a construction of quaternionic Kähler manifolds via the twistor theory of such manifolds. In his case, the family of rational curves are transversal to the contact distribution of the twistor space whereas in our case the rational curves are contact curves and thus are tangent to the contact distribution. In fact, some fruitful ideas in [9] have had apparent influence on the present paper.

1. Preliminaries. let W be a complex contact manifold. By that we mean there endows on W a holomorphic line bundle L^* of 1-forms such that if θ is a local section of L^* (called a local contact form), then $\theta \wedge d\theta$ is a nondegenerate 3-form. The dual of L^* in TW is the 2-dimensional contact distribution D, with respect to which L, the dual of L^* called the contact line bundle of W, is isomorphic to TW/D.

A holomorphic vector field X on W is called a contact vector field if for any local contact form θ of L^*,

$$\mathcal{L}_X \theta = t\theta \tag{2}$$

for some holomorphic function t. By Darboux's theorem, there is a local coordinate system (x, y, z) with respect to which the contact form θ can be written as

$$\theta = dz - ydx.$$

From this it is easy to see that given a local section s of L and θ of L^*, there is a unique contact vector field X such that

$$\theta(X) = \theta(s).$$

In fact, if we let $f = \theta(s)$, then relative to the above Darboux coordinates,

$$X = -f_y \frac{\partial}{\partial x} + (f_x + y f_z)\frac{\partial}{\partial y} + (f - y f_y)\frac{\partial}{\partial z}. \tag{3}$$

Here subscripts denote partial differentiation. It also follows that the natural projection

$$\pi : TW \longrightarrow L$$

sets up a one-to-one $\mathbb{C}$-morphism between local sections of L and local contact vector fields, so that the sequence

$$0 \longrightarrow D \longrightarrow TW \longrightarrow L \longrightarrow 0$$

$\mathbb{C}$-splits (but does not split as vector bundles).

A 1-dimensional complex submanifold C of W is called a Legendre submanifold if C is everywhere tangent to D. From now on a Legendre submanifold C in W in this paper is always understood to be a rational curve such that $L|_C$, the restriction of L to C, is $\mathcal{O}(3)$.

By the ith infinitesimal neighborhood of C we mean it is the ringed space $(C, \mathcal{O}_{(i)})$, where $\mathcal{O}_{(i)} = \mathcal{O}_W/\mathcal{I}^{(i+1)}$, $\mathcal{O}_W$ is the structural sheaf of W and $\mathcal{I}$ is the ideal sheaf of C in W. More generally, we define $\mathcal{O}_{(i)}(L) = \mathcal{O}(L)/\mathcal{I}^{(i+1)}\mathcal{O}(L)$, the ith infinitesimal neighborhood of L around C. Let N be the normal bundle of C in W. We have the exact sequence

$$0 \longrightarrow L|_C \otimes S^i(N^*) \longrightarrow \mathcal{O}_{(i)}(L) \longrightarrow \mathcal{O}_{(i-1)}(L) \longrightarrow 0. \tag{4}$$

Let $\mathcal{D}_{(i)}$ be the sheaf of derivations of $\mathcal{O}_{(i)}$, where each stalk is the Lie algebra of derivations of the stalk of $\mathcal{O}_{(i)}$. By the sheaf of automorphisms $\mathcal{AUT}_{(i)}$ of $\mathcal{O}_{(i)}$ we mean it is the sheaf $\exp(\mathcal{D}_{(i)})$. Locally, each stalk element of $\mathcal{O}_{(i)}$ is represented by a Taylor series

$$a_0(x) + a_1(x)y + a_2(x)z + \cdots \mod (i+1),$$

where $(x, y = 0, z = 0)$ parametrizes C and $dz - ydx$ is the contact form. (We use mod $(i+1)$ to denote terms of order $\geq i+1$). An element in the stalk of $\mathcal{D}_{(i)}$ is represented by a local vector field

$$f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} + h\frac{\partial}{\partial z}, \tag{5}$$

where $f, g, h \in \mathcal{O}_{(i)}$.

By a *contact* derivation we mean a vector field X in (5) such that (2) is satisfied mod $(i+1)$. Let $\mathcal{CD}_{(i)}$ be the subsheaf of $\mathcal{D}_{(i)}$ consisting of contact derivations D such that $D : \mathcal{I}/\mathcal{I}^{(i+1)} \longrightarrow \mathcal{I}/\mathcal{I}^{(i+1)}$. Let $\mathcal{CAUT}_{(i)}$ be the subsheaf $\exp(\mathcal{CD}_{(i)})$ of $\mathcal{AUT}_{(i)}$. Geometrically, an element in $\mathcal{CAUT}_{(i)}$ represents a contact transformation that leaves C invariant up to the ith order. Consider the sequence

$$1 \longrightarrow \mathcal{K}_{(i)} \longrightarrow \mathcal{CAUT}_{(i)} \longrightarrow \mathcal{CAUT}_{(i-1)} \longrightarrow 1, \tag{6}$$

where $\mathcal{K}_{(i)}$ is the kernel. We wish to understand $\mathcal{K}_{(i)}$.

Proposition 1

$$\mathcal{K}_{(i)} \simeq \mathcal{O}((L^*|_C)^{(i-1)}).$$

Proof. Anything A in $\mathcal{K}_{(i)}$ is a ring automorphism of $\mathcal{O}_{(i)}$ such that it acts trivially up to the $(i-1)$th order. Such an object in $\mathcal{AUT}_{(i)}$ can easily be seen to be of the form $A = 1 + V$ with V a (local) vector field vanishing to the $(i-1)$-th order at C (see also [8] for an intrinsic proof). It then follows that, in our setting, V is a contact vector field (up to the ith order), so that by (3) V gives rise to a (local) section of L vanishing to the $(i-1)$th order, and vice versa.

Now let us write a local section of L as

$$f = \sum_{k=0}^{i} \sum_{\alpha+\beta=k} f_{\alpha\beta}(x) y^\alpha z^\beta \bmod (i+1)$$

with respect to the above chosen coordinates (x, y, z). A straightforward substitution of f into (3) gives that the generated contact vector field X vanishes to the $(i-1)$th order if and only if $f_{\alpha\beta}(x) = 0$ except for f_{0i}, so that the local section of L is of the form

$$f_{0i}(x) z^i \bmod (i+1).$$

Hence the local section, when viewed as a section in $L|_C \otimes S^i(N^*)$ via (4), is $f_{0i}(x)(dz)^i$. On the other hand, dz is just the contact form $dz - y dx$ restricted to C, which generates $L^*|_C$. Hence $f_{0i}(dz)^i$ is a section of $L|_C \otimes (L^*|_C)^i = (L^*|_C)^{(i-1)}$. $\square$

Remark 1 *Strictly speaking, the isomorphism established in the proposition is only a* $\mathbb{C}$*-isomorphism in view of the* $\mathbb{C}$*-isomorphism between contact vector fields and sections of* L*. However, it makes no difference when we descend to the associated cohomology spaces of (6), which are* $\mathbb{C}$*-modules.*

2. The 1st-jet of L**.** Let L be a line bundle over any Riemann surface C. Let $\{U_\alpha\}$ be a collection of open covering of C with coordinate x_α. Let e_α be a generator of L over U_α. Consider the 1st-jet bundle J^1L of L over C. Locally J^1L is isomorphic to $U_\alpha \times \mathbb{C}^2$ given explicitly by $(x_\alpha; y_\alpha, z_\alpha) \in U_\alpha \times \mathbb{C}^2 \longmapsto$ the equivalence class of sections of L at x_α with value $z_\alpha e_\alpha$ and slope $y_\alpha = dz_\alpha/dx_\alpha$. Hence there is associated with J^1L the canonical contact structure $dz_\alpha - y_\alpha dx_\alpha$, where the contact curves are just the sections of L sitting in J^1L.

From now on, $\mathcal{CAUT}_{(i)}$ will always be referred to as the sheaf of ith infinitesimal contact automorphisms of the contact manifold J^1L.

Returning to our specialized setup, for any Legendre submanifold C of a contact 3-fold W with contact line bundle $L \simeq \mathcal{O}(3)$, locally we can always cover C by coordinates $(x_\alpha, y_\alpha, z_\alpha)$ such that $\theta_\alpha := dz_\alpha - y_\alpha dx_\alpha$ is the contact form with $(x_\alpha, 0, 0)$ parametrizing C. Moreover, since the coordinate transformations between U_α and U_β are contact transformations leaving C invariant, we see that the neighborhood $\cup_\alpha U_\alpha$ of C in W defines an element $(\tau_{\alpha\beta})$ of $H^1(\mathcal{CAUT})$, and vice versa. (Here, $\mathcal{CAUT}$ without subscript denotes the genuine sheaf of contact automorphisms of J^1L leaving C invariant.) Therefore, the geometric interpretation of $H^1(\mathcal{CAUT}_{(i)})$ is the space of all contact structures defined in a neighborhood of C up to the ith order.

From the exact sequence (6), Proposition 1 and $L \simeq \mathcal{O}(3)$, we see that $H^0(\mathcal{CAUT}_{(i)})$ always embeds in $H^0(\mathcal{CAUT}_{(i-1)})$ as long as $i \geq 2$. For $i = 1$ we have

$$1 \longrightarrow \mathbb{C} \longrightarrow H^0(\mathcal{CAUT}_{(1)}) \longrightarrow H^0(\mathcal{CAUT}_{(0)}) \longrightarrow 1.$$

Proposition 2 $H^0(\mathcal{CAUT}_{(1)})$ *is the Lie group of bundle automorphisms of* L *over* C*.*

Proof. Any contact automorphism of J^1L leaving C invariant must be a bundle automorphism of L over C. The $\mathbb{C}$-factor in the sequence corresponds to the natural dilation along the fibers of L over C. $H^0(\mathcal{CAUT}_{(0)})$ corresponds to the automorphism group of C, which is $PSL(2, \mathbb{C})$. □

Proposition 3 *$H^0(\mathcal{CAUT}_{(i)}) \simeq H^0(\mathcal{CAUT}_{(i-1)})$ for all $i \geq 2$. In particular we have the exact sequence*

$$1 \longrightarrow H^1((L^*|_C)^{(i-1)}) \longrightarrow H^1(\mathcal{CAUT}_{(i)}) \longrightarrow H^1(\mathcal{CAUT}_{(i-1)}) \longrightarrow 1. \quad (7)$$

Proof. As noted above $H^0(\mathcal{CAUT}_{(i)})$ always embeds into $H^0(\mathcal{CAUT}_{(i-1)})$ for $i \geq 2$. Choose the coordinates x and x' covering C such that $x' = 1/x$. In view of (3), we choose to solve

$$\begin{aligned} dx/dt &= kx, \\ dy/dt &= -ky + lz, \\ dz/dt &= lz, \end{aligned}$$

for some constants k, l. (If $f = a(x) + b(x)y + c(x)z$ satisfies (3), then $a(x) = 0, b(x) = -kx, c(x) = l$ for some constants k, l.)

The solution defines a 1-parameter of automorphisms

$$\Phi_t : (x, y, z) \longmapsto (e^{kt}x, e^{-kt}y + \frac{l}{k+l}(e^{lt} - e^{-kt})z, e^{lt}z),$$

which leaves C invariant. Φ_t defines a bundle automorphism of J^1L covering the automorphism $x \longmapsto kx$ of C. Now since any automorphism of C is of the form $x \longmapsto kx$ with respect to an appropriate affine coordinate, we see that Φ_t generates all the 4-dimensional bundle automorphisms of L over C. Since Φ_t is of 1st order, it is an automorphism for all $\mathcal{CAUT}_{(i)}, i \geq 1$. Thus we see $H^0(\mathcal{CAUT}_{(i)}) = H^0(\mathcal{CAUT}_{(i-1)})$ for $i \geq 2$, from which the exact sequence (7) follows by inspecting the associated long exact sequence of (6). □

We now utilize (7). From the fact that $H^1(\mathcal{CAUT}_{(0)}) = 1$ (the 0th order neighborhood of C in any contact 3-fold W is always C), we have $H^1(\mathcal{CAUT}_{(1)}) = 1$ since $H^1(\mathcal{O}_C) = 0$, which is also clear since the 1st order neighborhood of C in any W is always J^1L. Hence $H^1(\mathcal{CAUT}_{(2)}) = H^1(L^*) \simeq \mathbb{C}^2$ $(L \simeq \mathcal{O}(3))$, which says already that there are 2-parameter worth of contact structures whose 2nd order neighborhoods are different.

The important case starts at $i = 3$, where $H^1((L^*|_C)^2) \simeq \mathbb{C}^5$ injects into $H^1(\mathcal{CAUT}_{(3)})$ in view of (7), which gives a 5-parameter worth of contact structures whose 3rd order neighborhoods are different. Our task next is to construct these contact structures explicitly.

3. Constructing the contact 3-folds. To see $H^1((L^*|_C)^2) \simeq \mathbb{C}^5$ more closely, let $C \simeq \mathbb{P}^1$ be covered by the affine charts U_1, U_2 with coordinates x, x' where $x \in \mathbb{C}$ and $x' = 1/x$ over $\mathbb{C}^*$. We interpret $\mathcal{O}(-6)$ as $\mathcal{O}(-6\infty)$, where ∞ is the point $x' = 0$. Since $(x')^6$ is a holomorphic section of $\mathcal{O}(-6)$ over U_2 nowhere vanishing on $U_1 \cap U_2$, an elment λ of $H^1(\mathcal{O}(-6))$ is a multiple of $(x')^6$ by a function holomorphic over $\mathbb{C}^*$ modulo the holomorphic parts contributed from U_1 and U_2. In other words,

$$\lambda = \frac{a_1}{x} + \frac{a_2}{x^2} + \frac{a_3}{x^3} + \frac{a_4}{x^4} + \frac{a_5}{x^5}.$$

Therefore, the 5-parameter family of contact manifolds is obtained by setting

$$f_{03} = \lambda$$

and, for convenience, all other $f_{ij} \equiv 0$, where f_{ij} are given in Proposition 1. We then form the contact vector field X_λ corresponding to λ and the local flow Φ_t generated by X_λ on J^1L for sufficiently small t. The idea is to deform J^1L via Φ_t so that the resulting contact 3-folds $W_{t\lambda}$ will be the 5-parameter family.

To carry this out precisely, let

$$\pi : J^1L \longrightarrow C$$

be the natural projection map and let W_1, W_2 be respectively the preimage of U_1, U_2 under π. By (3), the x-component of X_λ is zero, which implies that the contact flow Φ_t only translates in the $y - z$ planes. Naively, one would tempt to define the contact 3-fold $W_{t\lambda}$ to be the disjoint union of W_1, W_2 modulo the identification $\sim$ given by $p \in W_1 \sim q \in W_2$ if $p, q \in W_1 \cap W_2$ and $q = \Phi_t(p)$. $W_{t\lambda}$ so constructed would constitute the desired 5-parameter family in view of Proposition 1 if there was a fixed small range of t over which the local flow Φ_t is defined around every point. This is certainly not true in general. On the other hand, We could try to shrink the size of W_1 and W_2 so that they only cover a small neighborhood of C in J^1L. However, this would not take care of the size of the neighborhoods in all directions; the radius of convergence of t still varies with the initial position because points in say, U_1, can still reach infinity.

To remedy this, fix two numbers r_1 and r_2 such that $1 \ll r_1 \ll r_2$. This time cover C by three open sets

$$\begin{aligned} U_1 &= \{z \in \mathbb{C} : |z| < r_1\}, \\ U_2 &= \{z : r_1 - 1 < |z| < r_2 + 1\}, \\ U_3 &= \{z : |z| > r_2\} \cup \{\infty\}. \end{aligned}$$

Note that U_1 is disjoint from U_3. The function $t\lambda$ restricted to $U_1 \cap U_2$ and $U_2 \cap U_3$ respectively now defines a 1-cocycle $\Omega_{t\lambda}$ in $H^1(\mathcal{O}(-6))$. To find $\Omega_{t\lambda}$ explicitly, we recall the Serre duality

$$H^0(\mathcal{O}(4)) \otimes H^1(\mathcal{O}(-6)) \longrightarrow H^1(\mathcal{O}(-2)) \simeq \mathbb{C}$$

defined explicitly with respect to the covering U_1, U_2, U_3 by

$$G \otimes H \longmapsto \int_{\gamma_1} GH dz + \int_{\gamma_2} GH dz,$$

where γ_1 and γ_2 are two homologically nontrivial circles in $U_1 \cap U_2$ and $U_2 \cap U_3$, respectively, and

$$\begin{aligned} G &= A_4 z^4 + A_3 z^3 + \cdots + A_0, \\ H &= B_1/z + B_2/z^2 + \cdots + B_5/z^5. \end{aligned}$$

Via the duality, it is easy to see that, up to a constant factor,

$$\Omega_{t\lambda} \longmapsto (ta_1, ta_2, \cdots, ta_5),$$

so that the contact 3-folds to be constructed will all be distinct. Now pick three appropriate open sets W_1, W_2, W_3 in $\pi^{-1}(U_1), \pi^{-1}(U_2), \pi^{-1}(U_3)$ respectively such that they cover C and are finite in size. Then glue them together by the contact flow $\Phi_{t\lambda}$ to form the contact 3-folds $W_{t\lambda}$ as before. The range of t can then be chosen to be independent of the initial position as long as t remains sufficiently small.

To conclude this section, we prove the following crucial lemma.

Lemma 1 $H^0(W_{t\lambda}) \neq 0$.

Proof. In general, let $(x_\alpha, y_\alpha, z_\alpha)$ be a coordinate cover of a contact 3-fold with the contact form $\theta_\alpha = dz_\alpha - y_\alpha dx_\alpha$ such that $(x, 0, 0)$ parametrizes a

Legendre submanifold C. Then the transition function $(g_{\alpha\beta})$ of the contact line bundle L is given by

$$g_{\alpha\beta} = \frac{\partial z_\alpha}{\partial z_\beta} - y_\alpha \frac{\partial x_\alpha}{\partial z_\beta}. \tag{8}$$

Now the contact vector field $X_{t\lambda}$, by (3), is

$$X_{t\lambda} = (t\lambda'(x)z^3 + 3t\lambda(x)yz^2)\frac{\partial}{\partial y} + t\lambda(x)z^3\frac{\partial}{\partial z}.$$

$\Phi_{t\lambda}$ is thus the local flow of the differential equation

$$\begin{aligned} dx/dt &= 0, \\ dy/dt &= \lambda'(x)z^3 + 3\lambda(x)yz^3, \\ dz/dt &= \lambda(x)z^3. \end{aligned}$$

Let (x_0, y_0, z_0) be the initial point. Then $x = x_0$ and z has the solution

$$z = \sum_{k=0}^{\infty}(-1)^k \binom{\frac{1}{2}}{k} 2^k(\lambda(x_0))^k(z_0)^{2k+1}t^k. \tag{9}$$

In particular, we have

$$\frac{\partial z}{\partial z_0}(z_0)^3 = z^3. \tag{10}$$

On the other hand, since $\partial x/\partial z_0 = 0$, (8) gives

$$g_{\alpha\beta} = \frac{\partial z}{\partial z_0}. \tag{11}$$

(Here α is for z and β is for z_0.) Thus (10) and (11) say we are looking at a nontrivial section of L. □

4. Infinitesimal symmetries and $H^0(W_{t\lambda}, L)$. We will prove in this section that for a G_3-structure, the infinitesimal symmetries of the connection are in one-to-one correspondence with the holomorphic sections of the contact line bundle L of the associated contact 3-fold W.

Let us recall first the twistorial construction of a G_3-structure over a 4-fold M. The G_3-holonomy representation at each cotangent space T_p^*M generates the orbit of the highest weight, which is a cone $\mathcal{V}_p$ in T_p^*M. Remove the zero of $\mathcal{V}_p$ and denote the resulting bundle by $\mathcal{V}$, whose projectivization $\mathcal{P}$ is a $\mathbb{P}^1$-bundle over M. Let the canonical symplectic form of T^*M be α, and let Λ be the distribution over $\mathcal{V}$ defined, for $y \in \mathcal{V}$, by

$$\Lambda(y) = \{X \in T_y\mathcal{V} : \alpha(X, T_y\mathcal{V}) \equiv 0\}.$$

Λ is an integrable distribution and the local leaf space $\mathcal{F}$ of it inherits from α a symplectic structure. $\mathbb{P}(\mathcal{F})$, the projectivization of $\mathcal{F}$, is the contact 3-fold W of the G_3-structure and each $\mathbb{P}(\mathcal{V}_p \backslash \{0\})$ sits as a Legendre submanifold C_p in W. The contact line bundle L of W is the one naturally associated with the $\mathbb{C}^*$-action of $\mathcal{F}$ over W. When restricted to each Legendre submanifold C_p, $L|_{C_p}$ is isomorphic to the hyperplane bundle of $\mathbb{P}(T_p^*M)$ restricted to C_p; thus $L|_{C_p} \simeq \mathcal{O}(3)$. One can now interpret M as the complete deformation family of rational curves around a rational curve C in W for which $L|_C \simeq \mathcal{O}(3)$, and the G_3-bundle as the bundle of automorphisms of L restricted to each curve in the deformation family.

To prove the assertion, given an infinitesimal symmetry of a G_3-connection on M, let ϕ_t be the corresponding 1-parameter symmetries. We claim that ϕ_t must leave the distribution Λ invariant. Indeed, ϕ_t^* induces a natural bundle automorphism of T^*M, and it is elementary to check that ϕ_t^* leaves α invariant. (This is true for any automorphism of M.) The claim will follow if $\mathcal{V}$ is left invariant by ϕ_t^*. But this is true because ϕ_t is also a G_3-bundle automorphism of which $\mathcal{V}$ is an associated bundle. It follows from the claim that ϕ_t induces a 1-parameter automorphisms on the leaf space $\mathcal{F}$ of the distribution which preserve the symplectic structure of $\mathcal{F}$ since α is left invariant by ϕ_t. Therefore ϕ_t induces a 1-parameter contact automorphisms on W, whose associated contact vector field then projects to a section on L.

Conversely, given a section of L and hence a contact vector field of W, let ψ_t be the associated 1-parameter group of contact automorphisms. Regarding M as a deformation family of rational curves and the principal G_3-bundle as the bundle of automorphisms of L restricted to each curve of the deformation family. Then ψ_t induces a 1-parameter G_3-bundle automorphisms because ψ_t leaves L invariant and maps Legendre curves to Legendre curves. ψ_t automatically leaves the canonical form θ of the G_3-bundle invariant [11]. Hence θ and $\psi_t^*(\omega)$, where ω is the connection form, together form another torsion-free structure on the same G_3-bundle. By the uniqueness of torsion-free G_3-structure on a G_3-bundle [2], we conclude that $\psi_t^*(\omega) = \omega$. In other words, ψ_t induces a 1-parameter symmetry of the G_3-connection [11].

That the correspondence is one-to-one follows from the proof and the discussion immediately preceding it.

5. Proof of Theorem 1. We are ready to draw the conclusion that the moduli space of inhomogeneous degenerate G_3-connections are infinite-dimensional. For $i = 3$, Lemma 1 and Section 4 imply that all the 5-parameter contact 3-folds constructed in Section 3 have a nontrivial infinitesimal symmetry. Hence all these 3-folds are degenerate.

On the other hand, the preceding analysis carries through verbatim for any i, in which case we have a $(3i - 4)$-family of distinct contact 3-folds each of which has a nontrivial infinitesimal symmetry. Letting i go to infinity, we see that the moduli space of degenerate G_3-connections must be infinite-dimensional, and so must be that of the inhomogeneous degenerate G_3-connections in view of the finite-dimensionality of the homogeneous G_3-structures.

References

[1] M. Berger, *Sur les groupes d'holonomie des variétés à connexion affine et des variétés Riemanniennes*, Bull. Soc. Math. **83**(1955), 279-330.

[2] R. Bryant, *Two exotic holonomies in dimension four, path geometries, and twistor theory*, Proc. Symp. in Pure Math. **53** (1991), 33-88.

[3] R. Bryant, *Classical, exceptional, and exotic holonomies: a status report*, preprint.

[4] É. Cartan, *les groupes d'holonomie des espaces généralisés*, Acta Math. **48**(1926), 1-40.

[5] Q.-S. Chi, *Symplectic connections and exotic holonomy*, Int. J. Math. **9**(1998), 47-61.

[6] Q.-S. Chi, S. Merkulov and L. Schwachhöfer, *On the existence of infinite series of exotic holonomies*, Inven. Math. **126**(1996), 391-411.

[7] Q.-S. Chi, S. Merkulov and L. Schwachhöfer, *Exotic holonomies E_7^a*, Int. J. Math. **8**(1997), 583-594.

[8] M. Eastwood and C. LeBrun, *Fattening complex manifolds: curvature, and Kodaira-Spencer maps*, J. Geom. Phys. **8**(1992), 123-146.

[9] C. LeBrun, *On complete quaternionic-Kähler manifolds*, Duke Math. J. **63**(1991), 723-743.

[10] S. Merkulov and L. Schwachhöfer, *Classification of irreducible holonomies of torsion-free affine connections*, preprint.

[11] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Vol. 1, Interscience Publishers, New York, London, 1963.

[12] L. Schwachhöfer, *On homogeneous connections with exotic holonomy*, Geom. Dedicata **62**(1996), 193-208.

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