Degenerate torsion-free G_3-connections revisited

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0. Introduction. Differential-geometric structures are usually defined in terms of highly nonlinear partial differential equations, whose solution for many occasions can be achieved by powerful geometric reformulations. Take a linear connection for instance. Its Christoffel symbol Γ_{ij}^k satisfies the transition formula

$$\overline{\Gamma}_{ij}^k = \sum_{t,m,n} \Gamma_{mn}^t \frac{\partial \epsilon_m}{\partial \overline{\epsilon}_i} \frac{\partial \epsilon_n}{\partial \overline{\epsilon}_j} \frac{\partial \overline{\epsilon}_k}{\partial \epsilon_t} + \sum_t \frac{\partial^2 \epsilon_t}{\partial \overline{\epsilon}_i \partial \overline{\epsilon}_j} \frac{\partial \overline{\epsilon}_k}{\partial \epsilon_t}$$

between any two coordinates (ϵ_i) and $(\overline{\epsilon}_i)$. Hence any choice of Γ_{ij}^k in one coordinate chart suffices to define the connection in another. Let $T_{ij}^k := \Gamma_{ij}^k - \Gamma_{ji}^k$ be the torsion tensor of the connection. The case when torsion is zero (torsion-free) is of particular interest in geometry. In Hermann Weyl's terms [11], the very existence of inertia systems in the Universe warrants that its linear connection is torsion-free.

Cartan [3] first introduced the holonomy group of a linear connection to study symmetric spaces. By definition, the holonomy group of a manifold M at a point p is the Lie group generated by the parallel translations along all loops starting and ending at p. It is thus a subgroup of the general linear group of the tangent space at p. This group is in general not connected; however, its connected component containing the identity is generated by small loops at p. Therefore, a simply connected manifold always has a connected holonomy group, which we will assume henceforth.

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By parallel translation, one can construct a holonomy bundle over the manifold, where each fiber is the holonomy group at the base point. Conversely it was proved [7] that all subgroups of the general linear group can be realized as the holonomy group of a linear connection of a manifold, if we do not require torsion-freeness.

The question of finding torsion-free connections, if there are any, on a given principal subbundle of the frame bundle of a manifold immediately becomes a PDE one, in principle. It was not until Berger, who classified in [1] all irreducibly acting reductive groups of the general linear group which may arise as the holonomy group of a torsion-free connection, that systematic studies of holonomy groups began to ensue in the years to follow. Berger's classification is complete in the metric case and, as Bryant pointed out [2], is incomplete in the non-metric case; these missing holonomy groups from Berger's list were referred to by Bryant as *exotic* holonomies.

This paper is a sequel to [4]. (See also [4] for further references to the recent holonomy development.) We continue to explore the degenerate (analytic) torsion-free G_3-connections, which are anomalous of all the exotic holonomies.

Bryant found in [2] exotic G_3-connections that do not preserve any metric on 4-manifolds. Indeed, let V_n be the space of homogeneous polynomials of degree n in two variables x and y . $GL(2,\mathbb{R})$ (or $GL(2,\mathbb{C})$ in the holomorphic case) acts on V_3 by

$$(g \cdot p)(x, y) = p((x, y)g),$$

where $g \in GL(2)$ and $p(x, y) \in V_3$. We refer to $GL(2)$ with this representation as G_3. The Lie algebra $\mathcal{G}_3$ of G_3 is identified with $V_0 \oplus V_2$.

It is well-known [9] that the space of nonisomorphic G_3-bundles over a 4-fold M can be identified with the space of maps (smooth, analytic, holomorphic, etc.) from M to $GL(4)/GL(2)$. Hence locally, it depends on twelve functions in four variables. For a G_3-bundle over M, let θ be its canonical form and ω be any G_3-connection form. θ is V_3-valued and ω is $V_0 \oplus V_2$-valued. As is pointed out by Bryant [2], we can always perturb ω by adding to it an appropriate $\mathcal{G}_3$-valued 1-form η so that the torsion 2-form T of the new connection form $\omega_1 := \omega + \eta$, which is

$$T := d\omega_1 + \omega_1 \wedge \theta,$$

assumes the simplest form. In fact, there is a V_7-valued G_3-equivariant map

τ on the G_3-bundle such that

$$T =< \tau, < \theta, \theta >_1>_4 .$$

Here, $< \cdot, \cdot >_s$ stands for the standard pairing in the Clebsch-Gordon formula given by

$$< p, q >_s= \frac{1}{s!} \sum_{n=0}^{s} (-1)^n \binom{s}{n} \frac{\partial^s p}{\partial x^{s-n} \partial y^n} \frac{\partial^s q}{\partial x^n \partial y^{s-n}}.$$

It turns out that T is independent of the connection ω chosen. For this reason T is called the *intrinsic* torsion of the G_3-bundle.

As a consequence, a G_3-bundle admits a (unique) torsion-free connection if and only if $\tau = 0$. Therefore, locally torsion-free G_3-connections are defined by eight first-order PDEs with twelve functions in four variables. Since an isomorphism of the base manifold gives isomorphic G_3-bundles, we know these equations will be invariant under diffeomorphisms (biholomorphisms, etc.) of the manifold. It follows that the equations cannot be of Cauchy-Kowalewskaya type.

The technique of exterior differential systems (EDS) comes to the rescue now in the ”generic” case. Namely, the Lie algebra $\mathcal{G}_3$ of G_3 can be identified with $V_0 \oplus V_2$ and the curvature space $K(\mathcal{G}_3)$ of the $\mathcal{G}_3$-representation space V_3 satisfying the first Bianchi identity is isomorphic to

$$\mathcal{V} := V_2 \oplus V_4.$$

So for a torsion-free G_3-connection there is a G_3-equivariant $V_2 \oplus V_4$-valued function $a_2 \oplus a_4$ on the G_3-bundle giving the curvature form. In the same vein, the curvature space $K^1(\mathcal{G}_3)$ of the representation V_3 satisfying the second Bianchi identity is isomorphic to

$$\mathcal{W} := V_1 \oplus V_3 \oplus V_5 \oplus V_7,$$

so that the covariant derivative of the curvature tensor of the torsion-free G_3-connection is given in terms of a G_3-equivariant $V_1 \oplus V_3 \oplus V_5 \oplus V_7$-valued function $b_1 \oplus b_3 \oplus b_5 \oplus b_7$. One derives an identity [2]

$$d(a_2 + a_4) = J(\lambda + \phi + \omega), \tag{1}$$

where J is a certain 8×8 matrix whose entries are linear polynomials in $a_2, a_4, b_1, b_3, b_5, b_7$. To prove the existence of G_3-connections, Bryant started with the open set $\mathcal{O}$ of $\mathcal{V} \oplus \mathcal{W}$ on which J is nonsigular. Motivated by (1) he then considered the form defined by

$$\lambda + \phi + \omega := J^{-1} d(a_2 + a_4),$$

with respect to which one can define three natural 2-forms Θ, Λ, Φ, which measure the extent to which a G_3-connection fails to be torsion-free. Then he showed that the differential ideal generated by Θ, Λ, Φ on $\mathcal{O}$ is differentially closed, from which he applied the Cartan-Kähler theory to arrive at the existence of torsion-free G_3-connections and verified that the moduli of such connections depends on four functions in three variables.

One special feature in Bryant's existence proof is that for EDS to work, one has to assume the nonsingularity of the matrix J. We will call those G_3-connections constructed over the domain $\mathcal{O}$, that is, over where the determinant of J is nonzero, *nondegenerate* G_3-connections.

Are there any *degenerate* G_3-connections, i.e., are there any G_3-connections for which J vanishes identically? Homogeneous G_3-connections are degenerate examples [4]. Clearly, the Cartan-Kähler theory no longer applies. The author proved in [4] the following.

The moduli space of inhomogeneous analytic degenerate G_3-connections is infinite-dimensional.

We briefly sketch the idea. Indeed, Bryant [2] pointed out a beautiful twistorial interpretation of G_3-connections in the holomorphic category (real-analytic G_3-connections are real slices of them). Namely, let W be a complex contact 3-fold in which there is a smooth rational curve C such that $L|_C$, the restriction of the contact line bundle L of W to C, is $\mathcal{O}(3)$. Then the 4-dimensional complete deformation family of C in W is naturally endowed with a G_3-connection. Conversely, a holomorphic G_3-connection gives rise to a contact 3-fold W and a 4-dimensional deformation family around a smooth rational curve in W, to which the restriction of the contact line bundle of W is $\mathcal{O}(3)$. Our idea is to deform the 1st-jet bundle of $\mathcal{O}(3)$, which carries a natural contact structure, to an infinite-dimensional family of contact 3-folds each of whose members is equipped with a G_3-structure with a nontrivial infinitesimal symmetry. Then such G_3-structures must be degenerate, and hence the theorem follows.

In clear hind sight, the degenerate torsion-free G_3-connections we constructed in [4] are very restricted, as if out of fluke they had fallen out. In the present paper we will systematically capture a large class sufficiently close to the flat G_3-connection that depends on four restricted functions in three variables. Our starting point is the aforementioned fact that the moduli of torsion-free real-analytic connections depends on four arbitrary functions in three variables. Say, these four functions are $\sum_{ijk} a^s_{ijk} x^i y^j z^k$ for $1 \leq s \leq 4$. Then one can form two holomorphic functions

$$f := \sum_{ijk} (a^1_{ijk} + \sqrt{-1} a^2_{ijk}) u^i v^j w^k$$

and

$$g := \sum_{ijk} (a^3_{ijk} + \sqrt{-1} a^4_{ijk}) u^i v^j w^k,$$

where u, v, w are complex variables whose real parts are respectively x, y, z. This leads us to perturb the 1st-jet bundle by two holomorphic functions to achieve inequivalent torsion-free holomorphic G_3-connections whose real slices depend on four functions in three variables, from which we read off the large class that is degenerate.

We remark that in a subsequent paper [5], we will study those smooth torsion-free G_3-connections that are *not* analytic via an entirely different approach. Apparently the above methods, being analytic in nature, fail to provide any useful information in this case.

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1. Preliminaries. let W be a complex contact manifold. By that we mean there endows on W a holomorphic line bundle L^* of 1-forms such that if θ is a local section of L^* (called a local contact form), then $\theta \wedge d\theta$ is a nondegenerate 3-form. The dual of L^* in TW is the 2-dimensional contact distribution D, with respect to which L, the dual of L^* called the contact line bundle of W, is isomorphic to TW/D.

A holomorphic vector field X on W is called a contact vector field if for any local contact form θ of L^*,

$$\mathcal{L}_X \theta = t\theta \tag{2}$$

for some holomorphic function t. By Darboux's theorem, there is a local coordinate system (x, y, z) with respect to which the contact form θ can be written as

$$\theta = dz - ydx.$$

From this it is easy to see that given a local section s of L and θ of L^*, there is a unique contact vector field X such that

$$\theta(X) = \theta(s).$$

In fact, if we let $f = \theta(s)$, then relative to the above Darboux coordinates,

$$X = -f_y \frac{\partial}{\partial x} + (f_x + y f_z) \frac{\partial}{\partial y} + (f - y f_y) \frac{\partial}{\partial z}. \tag{3}$$

Here subscripts denote partial differentiation. It also follows that the natural projection

$$\pi : TW \longrightarrow L$$

sets up a one-to-one $\mathbb{C}$-morphism between local sections of L and local contact vector fields, so that the sequence

$$0 \longrightarrow D \longrightarrow TW \longrightarrow L \longrightarrow 0$$

$\mathbb{C}$-splits (but does not split as vector bundles).

A 1-dimensional complex submanifold C of W is called a Legendre submanifold if C is everywhere tangent to D. From now on a Legendre submanifold C in W in this paper is always understood to be a rational curve such that $L|_C$, the restriction of L to C, is $\mathcal{O}(3)$.

By the ith infinitesimal neighborhood of C we mean it is the ringed space $(C, \mathcal{O}_{(i)})$, where $\mathcal{O}_{(i)} = \mathcal{O}_W / \mathcal{I}^{(i+1)}$, $\mathcal{O}_W$ is the structural sheaf of W and $\mathcal{I}$ is the ideal sheaf of C in W. More generally, we define $\mathcal{O}_{(i)}(L) = \mathcal{O}(L)/\mathcal{I}^{(i+1)}\mathcal{O}(L)$, the ith infinitesimal neighborhood of L around C. Let N be the normal bundle of C in W. We have the exact sequence

$$0 \longrightarrow L|_C \otimes S^i(N^*) \longrightarrow \mathcal{O}_{(i)}(L) \longrightarrow \mathcal{O}_{(i-1)}(L) \longrightarrow 0. \tag{4}$$

Let $\mathcal{D}_{(i)}$ be the sheaf of derivations of $\mathcal{O}_{(i)}$, where each stalk is the Lie algebra of derivations of the stalk of $\mathcal{O}_{(i)}$. Locally, each stalk element of $\mathcal{O}_{(i)}$ is represented by a Taylor series

$$a_0(x) + a_1(x)y + a_2(x)z + \cdots \ (mod\ i+1)\,,$$

where $(x, y = 0, z = 0)$ parametrizes C and $dz - ydx$ is the contact form. (We use $(\mathrm{mod}\ i+1)$ to denote terms of order $\geq i+1$). An element in the stalk of $\mathcal{D}_{(i)}$ is represented by a local vector field

$$f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} + h\frac{\partial}{\partial z}, \tag{5}$$

where $f, g, h \in \mathcal{O}_{(i)}$, and g and h have no zero order term.

By a *contact* derivation we mean a vector field X in (5) such that (2) is satisfied $(\mathrm{mod}\ i+1)$. Let $\mathcal{CD}_{(i)}$ be the subsheaf of $\mathcal{D}_{(i)}$ consisting of contact derivations D such that $D : \mathcal{I}/\mathcal{I}^{(i+1)} \longrightarrow \mathcal{I}/\mathcal{I}^{(i+1)}$. Let $\mathcal{CAUT}_{(i)}$ be the sheaf of contact automorphisms of $\mathcal{O}_{(i)}$ that preserve the form $dz - ydx$ up to a factor. Geometrically, an element in $\mathcal{CAUT}_{(i)}$ represents a contact transformation that leaves C invariant up to the ith order. Consider the sequence

$$1 \longrightarrow \mathcal{K}_{(i)} \longrightarrow \mathcal{CAUT}_{(i)} \longrightarrow \mathcal{CAUT}_{(i-1)} \longrightarrow 1, \tag{6}$$

where $\mathcal{K}_{(i)}$ is the kernel. It was established in [4] that

$$\mathcal{K}_{(i)} \simeq \mathcal{O}((L^*|_C)^{(i-1)}). \tag{7}$$

Let L be a line bundle over any Riemann surface C. Let $\{U_\alpha\}$ be a collection of open covering of C with coordinate x_α. Let e_α be a generator of L over U_α. Consider the 1st-jet bundle J^1L of L over C. Locally J^1L is isomorphic to $U_\alpha \times \mathbb{C}^2$ given explicitly by $(x_\alpha; y_\alpha, z_\alpha) \in U_\alpha \times \mathbb{C}^2 \longmapsto$ the equivalence class of sections of L at x_α with value $z_\alpha e_\alpha$ and slope $y_\alpha = dz_\alpha/dx_\alpha$. Hence there is associated with J^1L the canonical contact structure $dz_\alpha - y_\alpha dx_\alpha$, where the contact curves are just the sections of L sitting in J^1L.

From now on, $\mathcal{CAUT}_{(i)}$ will always be referred to as the sheaf of ith infinitesimal contact automorphisms of the contact manifold J^1L.

Returning to our specialized setup, for any Legendre submanifold C of a contact 3-fold W with contact line bundle $L \simeq \mathcal{O}(3)$, locally we can always cover C by coordinates $(x_\alpha, y_\alpha, z_\alpha)$ such that $\theta_\alpha := dz_\alpha - y_\alpha dx_\alpha$ is the contact form with $(x_\alpha, 0, 0)$ parametrizing C. Moreover, since the coordinate transformations between U_α and U_β are contact transformations leaving C invariant, we see that the neighborhood $U := \cup_\alpha U_\alpha$ of C in W defines an element $(\tau_{\alpha\beta})$ of $H^1(U, \mathcal{CAUT}|_{\mathcal{U}})$. Here, $\mathcal{CAUT}$ without subscript denotes the genuine sheaf of contact automorphisms of J^1L leaving C invariant. Passing to the limit as U approaches C one obtains the sheaf $\mathcal{CAUT}|_C := \iota^*(\mathcal{CAUT})$ with $\iota : C \to J^1L$ the embedding, the sheaf of contact automorphisms on the germs of neighborhoods of C. Therefore, the geometric interpretation of $H^1(\mathcal{CAUT}_{(i)})$ is the space of all contact structures defined on the germs of neighborhoods of C up to the ith order.

From the exact sequence (6), (7) and $L \simeq \mathcal{O}(3)$, we see that $H^0(\mathcal{CAUT}_{(i)})$ always embeds in $H^0(\mathcal{CAUT}_{(i-1)})$ as long as $i \geq 2$. For $i = 1$ we have

$$1 \longrightarrow \mathbb{C} \longrightarrow H^0(\mathcal{CAUT}_{(1)}) \longrightarrow H^0(\mathcal{CAUT}_{(0)}) \longrightarrow 1.$$

We observed in [4] that $H^0(\mathcal{CAUT}_{(1)})$ is the Lie group of bundle automorphisms of L over C. Moreover, $H^0(\mathcal{CAUT}_{(i)}) \simeq H^0(\mathcal{CAUT}_{(i-1)})$ for all $i \geq 2$, so that in particular we have the exact sequence

$$1 \longrightarrow H^1((L^*|_C)^{(i-1)}) \longrightarrow H^1(\mathcal{CAUT}_{(i)}) \longrightarrow H^1(\mathcal{CAUT}_{(i-1)}) \longrightarrow 1. \quad (8)$$

2. Constructing the contact 3-folds. Let us recall first the twistorial construction of a holomorphic G_3-structure over a complex 4-fold M [2]. The G_3-holonomy representation at each cotangent space T^*_pM generates the orbit of the highest weight, which is a cone $\mathcal{V}_p$ in T^*_pM. Remove the zero of $\mathcal{V}_p$ and denote the resulting bundle by $\mathcal{V}$, whose projectivization $\mathcal{P}$ is a $\mathbb{P}^1$-bundle over M. Let the canonical symplectic form of T^*M be α, and let Λ be the distribution over $\mathcal{V}$ defined, for $y \in \mathcal{V}$ by

$$\Lambda(y) = \{X \in T_y\mathcal{V} : \alpha(X, T_y\mathcal{V}) \equiv 0\}.$$

Λ is an integrable distribution whose local leaf space $\mathcal{F}$ inherits from α a symplectic structure. $\mathbb{P}(\mathcal{F})$, the projectivization of $\mathcal{F}$, is the contact 3-fold W of the G_3-structure and each $\mathbb{P}(\mathcal{V}_p)$ sits as a Legendre submanifold C_p in W. The contact line bundle L of W is the one naturally associated with the $\mathbb{C}^*$-action of $\mathcal{F}$ over W. When restricted to each Legendre submanifold C_p, $L|_{C_p}$ is isomorphic to the hyperplane bundle of $\mathbb{P}(T^*_pM)$ restricted to C_p; thus $L|_{C_p} \simeq \mathcal{O}(3)$. One can now interpret M as the complete deformation family of rational curves around a rational curve C in W for which $L|_C \simeq \mathcal{O}(3)$, and the G_3-bundle as the bundle of automorphisms of L restricted to each curve in the deformation family.

We state here the result [4] that for a holomorphic G_3-structure, the infinitesimal symmetries of the connection are in one-to-one correspondence with the holomorphic sections of the contact line bundle L of the associated contact 3-fold W. Therefore, we will construct contact 3-folds with nontrivial contact line bundle sections.

To construct the contact 3-folds, fix, as in [4], two numbers r_1 and r_2 such that $1 \ll r_1 \ll r_2$. Cover $\mathbb{P}^1$ by three open sets

$$\begin{aligned} U_1 &= \{z \in \mathbb{C} : |z| < r_1\}, \\ U_2 &= \{z : r_1 - 1 < |z| < r_2 + 1\}, \\ U_3 &= \{z : |z| > r_2\} \cup \{\infty\}. \end{aligned}$$

Let $L = \mathcal{O}(3)$ and let J^1L be the 1st-jet bundle of L. Over each U_i, let J^1L be parametrized by a holomorphic chart $D_i := \{(x_i, y_i, z_i)\}$ with $(x_i, 0, 0)$ parametrizing U_i. Note that U_1 is disjoint from U_3. Moreover, clearly we can assume that D_1 and D_2 share the same coordinates, so that whenever without confusion we will denote both (x_1, y_1, z_1) and (x_2, y_2, z_2) by (x, y, z) and (x_3, y_3, z_3) by (x', y', z') for notational simplicity.

We now make the convention that for a holomorphic function H in three variables x, y, z, we denote by $H^{(m)}$ the Taylor expansion of H around $y = z = 0$ up to the mth order (the coefficients are functions of x), and by $H_{|m+1}$ the (m+1)th term in the Taylor expansion of H. We wish to deform the germs of neighborhoods of $\mathbb{P}^1$ in J^1L via contact transformations to produce inequivalent contact 3-folds around C.

We first find $H^1(\mathcal{CAUT}_{(i)})$ explicitly. Note that U_1, U_2, U_3 suffice for the purpose since $H^s((L|_C)^{i-1}) = 0$ for all $s \geq 1$ over an annuli or a disc, which are Stein. Hence (6) implies that $H^1(\mathcal{CAUT}_{(i)}) = 0$ for all i over an annuli or a disc. The standard sheaf theory then asserts that the three Stein sets are sufficient to calculate $H^1(\mathcal{CAUT}_{(i)})$ with.

We start with the observation that $H^1(\mathcal{CAUT}_{(1)}) = 1$, which follows from the fact that (6) and (7) for $i = 1$ imply $H^1(\mathcal{CAUT}_{(1)}) = H^1(\mathcal{CAUT}_{(0)})$. On the other hand, $H^1(\mathcal{CAUT}_{(0)}) = 1$ since it is the space of inequivalent $\mathbb{P}^1$. Therefore up to a 1st-order change of coordinates we can assume that the 1st-order contact transition $\Phi_{ij}^{(1)}$ between D_1 and D_2 and between D_2 and D_3 are, respectively,

$$\begin{aligned} x_2 &= x_1, \\ y_2 &= y_1, \end{aligned} \tag{9}$$

$$z_2 = z_1$$

and

$$\begin{aligned} x_3 &= 1/x_2, \\ y_3 &= -y_2/x_2 + 3z_2/(x_2)^2, \\ z_3 &= z_2/(x_2)^3. \end{aligned} \tag{10}$$

This 1st-order transition glues D_1, D_2, D_3 into J^1L, which is our 1st-order deformation of the contact 3-fold to be constructed. The corresponding 0th order transition $g_{\alpha\beta}$ of the contact line bundle L of the contact 3-fold over $\mathbb{P}^1$ is

$$g_{21}^{(0)} = 1, g_{32}^{(0)} = 1/x^3, \tag{11}$$

with the 0th order 4-dimensional sections

$$s_1^{(0)} = a + bx + cx^2 + dx^3. \tag{12}$$

Here the subscript for s_1 indicates that it is expressed over D_1.

Suppose we have found $\Phi_{ij}^{(m)}$, we will construct $\Phi_{ij}|_{m+1}$. For notational ease, we will write

$$\begin{aligned} \Phi_{ij}^{(m)} &= (F(x,y,z), G(x,y,z), H(x,y,z)), \\ \Phi_{ij}|_{m+1} &= (f(x,y,z), g(x,y,z), h(x,y,z)), \\ \Phi_{ij}^{(m+1)} &= (F'(x,y,z), G'(x,y,z), H'(x,y,z)). \end{aligned}$$

$dH' - G'dF'$ and $dH - GdF$ are multiples up to degree $m+1$ and m, say, $\lambda^{(m+1)}$ and $\mu^{(m)}$, respectively, of $dz - ydx$. We know $\mu^{(1)}$ depends only on x. Set

$$(dH - GdF)|_{m+1} = A|_{m+1}dx + B|_{m+1}dy + C|_{m+1}dz.$$

Then expansion gives, mod $(m+2)$,

$$\mu^{(m)}(dz - ydx) + dh - Gdf - gdF - gdf = \lambda^{(m+1)}(dz - ydx).$$

Set $r^{(m+1)} = \lambda^{(m+1)} - \mu^{(m)}$. Now gdf can be ignored since gdf is of order greater than $m+1$. This amounts to, comparing the terms with degree $m+1$,

$$h_x - G^{(0)}f_x - g(F_x)^{(0)} = -yr|_m - A|_{m+1}, \tag{13}$$

$$-G|_1 f_y - g(F_y)|_0 = -B|_{m+1}, \tag{14}$$

$$-G|_1 f_z - g(F_z)|_0 = r|_{m+1} - C|_{m+1}, \tag{15}$$

where subscripts denote partial differentiation. Now for degree m a comparison establishes

$$h_y = G^{(0)} f_y = 0$$

because $G^{(0)} = 0$ by (9) and (10). Now we impose the condition that

$$B|_{m+1} = 0.$$

We see

$$G|_1 f_y = -B|_{m+1} = 0$$

because a glance at (9) and (10) gives that $F_y^{(0)} = 0$ since the 1st-order transition of F has no linear terms in y and z. Thus with $G|_1 \neq 0$ we arrive at

$$f_y = h_y = 0,$$

so that

$$f = p_{m+1}(x) z^{m+1}, \tag{16}$$

$$h = q_{m+1}(x) z^{m+1}, \tag{17}$$

for some functions $p_{m+1}(x)$ and $q_{m+1}(x)$. Comparing degrees and noting that $G^{(0)} = 0$, we get from (13)

$$h_x - g(F_x)^{(0)} = -yr|_m - A|_{m+1}$$

homogeneous of degree $m+1$. On the other hand by the degree m comparison we have

$$r|_m = h_z,$$

so that

$$yh_z + h_x + A|_{m+1} = g(F_x)^{(0)}. \tag{18}$$

Incorporating (16), (17) and (18) we derive

$$g = ((m+1)q_{m+1}(x) y z^m + q'_{m+1}(x) z^{m+1} + A|_{m+1})/(F_x)^{(0)}. \tag{19}$$

Now in view of (16), F' is a function of x and z alone. It follows that $dH' - G'dF'|_{m+2} = -G'dF'|_{m+2}$ is a form in dx and dz only, so that again

$B|_{m+2} = 0$, which completes the induction on m. In fact, a straightforward calculation gives, for $m \geq 1$,

$$\begin{aligned} A|_{m+2} &= p'_{m+1}(x)G|_1 z^{m+1} \\ &+ \sum_{k=2}^{m+1}(kq_k(x)p'_{m+2-k}(x)yz^{m+1} + q'_k(x)p'_{m+2-k}(x)z^{m+2} + p'_{m+2-k}(x)A|_k z^{m+2-k}) \end{aligned} \tag{20}$$

with $A|_2 = 0$. Note that $A|_m$ only has the terms in yz^{m-1} and z^m.

Although it seems that $B|_{m+1} = 0$ is a strong condition to impose, it turns out $B|_{m+1} = 0$ is always true for all m. This is a simple consequence of the fact that for $B|_2 = 0$ for $m = 1$, and hence inductively $B|_{m+1} = 0$ for all m.

In summary, we have derived the following.

Lemma 1 (9), (10), (16), (17), (19) *give rise to the recursion of the Taylor expansions of all contact transitions in* $H^1(\mathcal{CAUT}_{(m)})$ *for all* m.

Proof. The procedure leading to (16), (17) and (19) lends its way to constructing $H^1(\mathcal{CAUT}_{(m+1)})$ explicitly. Specifically, since (16) and (17) live in $H^1(\mathcal{K}_{(m+1)})$ [4] in view of (8), a dimension count of (8) thus exhausts all the elements in $H^1(\mathcal{CAUT}_{(m+1)})$ when we restrict p_{m+1} and q_{m+1} to be polynomials in x of degree $3m - 5$. □

Clearly, appropriate small choices of y, z and the coefficients in the recursion will yield convergent infinite Taylor series. In other words, the data in Lemma 1 yields contact structures in the germs of neighborhoods around $C \simeq \mathbb{P}^1$.

Recall we have the relation

$$g_{\alpha\beta} = \frac{\partial z_\alpha}{\partial z_\beta} - y_\alpha \frac{\partial x_\alpha}{\partial z_\beta},$$

which, when applied to the preceding lemma, gives

$$g_{\alpha\beta} = g^{(0)}_{\alpha\beta} + \sum_{m=0}^{\infty}(m+2)q_{m+2}(x_\beta)z_\beta^{m+1} - y_\alpha \sum_{m=0}^{\infty}(m+1)p_{m+1}(x_\beta)z_\beta^m, \tag{21}$$

where

$$y_\alpha = G|_1+\sum_{m=1}^{\infty}((m+1)q_{m+1}(x_\beta)y_\beta z_\beta^m+q'_{m+1}(x_\beta)z_\beta^{m+1}+A|_{m+1}(x_\beta,y_\beta,z_\beta))/(F_{x_\beta})^{(0)}.$$

Observe that $g_{\alpha\beta}$ has terms in $y_\beta z_\beta^m$ and z_β^{m+1} only. Next we find a necessary and sufficient condition for the existence of nontrivial contact line bundle sections for the contact transition. First of all, (11) is $g_{\alpha\beta}^{(0)}$. Suppose we have constructed $s_\beta^{(m)}$ such that

$$s_\alpha^{(m)} = g_{\alpha\beta}^{(m)} s_\beta^{(m)}.$$

We wish to construct $s_\alpha^{(m+1)}$ such that

$$s_\alpha^{(m+1)} = g_{\alpha\beta}^{(m+1)} s_\beta^{(m+1)},$$

which is the case if and only if

$$s_\alpha|_{m+1} = g_{\alpha\beta}^{(m)} s_\beta|_{m+1} + g_{\alpha\beta}|_{m+1}s_\beta^{(m)} + g_{\alpha\beta}|_{m+1}s_\beta|_{m+1},$$

if and only if

$$s_\alpha|_{m+1} = g_{\alpha\beta}^{(0)} s_\beta|_{m+1}. + g_{\alpha\beta}|_{m+1}s_\beta^{(0)}$$

Incorporating these into account for the domains D_1, D_2, D_3, we see that a global section s up to the $(m+1)$th order must be given by

$$\begin{aligned} s_3|_{m+1} &= x^{-3}s_1|_{m+1} + x^{-3}(a+bx+cx^2+dx^3)g_{21}|_{m+1} \\ &+ (a+bx+cx^2+dx^3)g_{32}|_{m+1}, \end{aligned} \tag{22}$$

where $s_3|_{m+1}$ is holomorphic in x', y', z' . Set

$$\begin{aligned} s_1|_{m+1} &= \sum_{i+j=m+1}\sum_{n=-\infty}^{0} a_n^{ij}x^{-n}y^iz^j, \\ g_{21}|_{m+1} &= \sum_{i+j=m+1}\sum_{n=-\infty}^{\infty} b_n^{ij}x^{-n}y^iz^j, \\ g_{32}|_{m+1} &= \sum_{i+j=m+1}\sum_{-\infty}^{\infty} c_n^{ij}x^{-n}y^iz^j \end{aligned} \tag{23}$$

with

$$y = -y'/x' + 3z'/(x')^2,$$
$$z = z'/(x')^3.$$

Note that (21) has terms in $y_\beta z_\beta^m$ and z_β^{m+1} only; hence we have

$$b_n^{ij} = c_n^{ij} = 0 \tag{24}$$

when $i \geq 2$. Substituting (23) into (22) one ends up with

$$\sum_{k=-\infty,s=0}^{\infty,m+1} \sum_{i=s}^{m+1} (-1)^s \binom{i}{s} 3^{i-s} [a_{k-i}^{i,m+1-i} + (b_{k-i}^{i,m+1-i} + c_{k-i+3}^{i,m+1-i})a + (b_{k-i+1}^{i,m+1-i} + c_{k-i+4}^{i,m+1-i})b$$
$$+(b_{k-i+2}^{i,m+1-i} + c_{k-i+5}^{i,m+1-i})c + (b_{k-i+3}^{i,m+1-i} + c_{k-i+6}^{i,m+1-i})d](x')^{k-3m+s}(y')^s(z')^{m+1-s},$$

which must be holomorphic in x', y', z', so that the coefficients of this series are zero when $k < 3m - s$. In other words, for fixed integers $s, 0 \leq s \leq m+1$, and $k, k < 3m - s$ (k may be negative), we have

$$\begin{aligned} 0 \;&=\; \sum_{i=s}^{m+1} (-1)^s \binom{i}{s} 3^{i-s} [a_{k-i}^{i,m+1-i} + (b_{k-i}^{i,m+1-i} + c_{k-i+3}^{i,m+1-i})a + (b_{k-i+1}^{i,m+1-i} + c_{k-i+4}^{i,m+1-i})b \\ &+\; (b_{k-i+2}^{i,m+1-i} + c_{k-i+5}^{i,m+1-i})c + (b_{k-i+3}^{i,m+1-i} + c_{k-i+6}^{i,m+1-i})d]. \end{aligned} \tag{25}$$

Note that $a_n^{ij} = 0$ whenever $n > 0$.

Lemma 2 *All $a_n^{ij}, n \leq 0$, can be determined by a, b, c, d and b_t^{ij} and c_t^{ij}, $-\infty < t < \infty$.*

Proof. Set $s = m + 1$ and $k \leq m + 1$. (25) gives that $a_p^{m+1,0}, p \leq 0$, is in terms of a, b, c, d and various $b_t^{i,m+1-i}$ and $c_t^{i,m+1-i}$. Next set $s = m$ and $k = m$. (25) asserts that $a_0^{m,1}$ is a combination of $a_{-1}^{m+1,0}$, $a, b, , c, d$ and $b_t^{i,m+1-i}$ and $c_t^{i,m+1-i}$. So we can determine $a_0^{m,1}$. Now successively set $k = m - 1, m - 2 \cdots$, etc., to determine $a_p^{m,1}$ for all $p \leq 0$. Next we set $s = m - 1$ and $k = m - 1, m - 2 \cdots$, etc., to find $a_p^{m-1,2}, p \leq 0$. Continuing in this fashion we are done.□

When $m + 2 \leq k \leq 3m - 1$ all $a_{k-i}^{m+1-i} = 0$, so that in this case (25) gives constraints on b_t^{ij} and c_t^{ij}. We next solve these constraint equations. We remark that only $s = 0, 1$ appear in (25) when $k \geq m+2$ in view of (24).

Set $s = 0$ so that $m + 2 \le k \le 3m - 1$. We obtain from (25), for $i = 1, 2$,

$$
\begin{aligned}
0 &= (b_k^{0,m+1} + c_{k+3}^{0,m+1} + 3b_{k-1}^{1,m} + 3c_{k+2}^{1,m})a + (b_{k+1}^{0,m+1} + c_{k+4}^{0,m+1} + 3b_k^{1,m} + 3c_{k+3}^{1,m})b \\
&+ (b_{k+2}^{0,m+1} + c_{k+5}^{0,m+1} + 3b_{k+1}^{1,m} + 3c_{k+4}^{1,m})c + (b_{k+3}^{0,m+1} + c_{k+6}^{0,m+1} + 3b_{k+2}^{1,m} + 3c_{k+5}^{1,m})d.
\end{aligned} \tag{26}
$$

Set $s = 1$ so that $m + 2 \le k \le 3m - 2$. We obtain from (25), for $i = 1$,

$$
\begin{aligned}
0 &= (b_{k-1}^{1,m} + c_{k+2}^{1,m})a + (b_k^{1,m} + c_{k+3}^{1,m})b \\
&+ (b_{k+1}^{1,m} + c_{k+4}^{1,m})c + (b_{k+2}^{1,m} + c_{k+5}^{1,m})d
\end{aligned} \tag{27}
$$

Hence there are three sets of constraint equations. Namely, (27) and

$$
\begin{aligned}
0 &= (b_k^{0,m+1} + c_{k+3}^{0,m+1})a + (b_{k+1}^{0,m+1} + c_{k+4}^{0,m+1})b \\
&+ (b_{k+2}^{0,m+1} + c_{k+5}^{0,m+1})c + (b_{k+3}^{0,m+1} + c_{k+6}^{0,m+1})d
\end{aligned} \tag{28}
$$

for $m + 2 \le k \le 3m - 2$ obtained by subtracting (27) from (26), and

$$
\begin{aligned}
0 &= (b_{3m-1}^{0,m+1} + c_{3m+2}^{0,m+1} + 3b_{3m-2}^{1,m} + 3c_{3m+1}^{1,m})a + (b_{3m}^{0,m+1} + c_{3m+3}^{0,m+1} + 3b_{3m-1}^{1,m} + 3c_{3m+2}^{1,m})b \\
&+ (b_{3m+1}^{0,m+1} + c_{3m+4}^{0,m+1} + 3b_{3m}^{1,m} + 3c_{3m+3}^{1,m})c + (b_{3m+2}^{0,m+1} + c_{3m+5}^{0,m+1} + 3b_{3m+1}^{1,m} + 3c_{3m+4}^{1,m})d
\end{aligned} \tag{29}
$$

for $k = 3m - 1$ in (26). The upshot is the following.

Lemma 3 *Say $a \neq 0$ in (12). Then $b_{3m-1+t}^{1,m} + c_{3m+2+t}^{1,m}, 0 \le t \le 2$, and $b_{3m+t}^{0,m+1} + c_{3m+3+t}^{0,m+1}, 0 \le t \le 2$, linearly generate all other similar terms in (27) and (28), and generate $b_{3m-1}^{0,m+1} + c_{3m+2}^{0,m+1} + 3b_{3m-2}^{1,m} + 3c_{3m+1}^{1,m}$ in (29). The constraint space is thus of dimension $4m + 9$.*

Now for D_1 and D_2, $q_1 = 1, p_1 = 0$ and $G|_1 = y$. Write, for $m \ge 0$,

$$
p_{m+1} := \sum_{n=-\infty}^{\infty} \alpha_n^{1,m} x^{-n},
$$

$$
q_{m+2} := \sum_{n=-\infty}^{\infty} \beta_n^{0,m+1} x^{-n}.
$$

Then (21) and (23) give, for $m \geq 1$,

$$\begin{aligned} b_n^{0,m+1} &= (m+2)\beta_n^{0,m+1} \mod(2), \\ b_n^{1,m} &= -(m+1)\alpha_n^{1,m} \mod(2), \end{aligned} \tag{30}$$

where mod(2) denotes terms involving products of $\alpha_i^{1,m}$ and $\beta_j^{0,m+1}$. Likewise, for D_2 and D_3, We have $q_1 = 1/x^3, p_1 = 0$ and $G|_1 = -y/x + 3z/x^2$. Write, for $m \geq 0$,

$$p_{m+1} = \sum_{n=-\infty}^{\infty} \gamma_n^{1,m} x^{-n},$$

$$q_{m+2} = \sum_{n=-\infty}^{\infty} \delta_n^{0,m+1} x^{-n}.$$

Then we see

$$\begin{aligned} c_n^{1,m} &= (m+1)\gamma_{n-1}^{1,m} \mod(2), \\ c_n^{0,m+1} &= (m+2)\delta_n^{0,m+1} - 3(m+1)\gamma_{n-2}^{1,m} \mod(2). \end{aligned} \tag{31}$$

Now let us say that the contact transition and the resulting contact 3-fold are of *pole type* if for each $m \geq 0$, there is an integer N_m (N_m may be negative) such that $\alpha_i^{1,m}, \beta_i^{0,m+1}, \gamma_i^{1,m}, \delta_i^{0,m} = 0$ for all $i \leq N_m$.

Therefore near zero, by the implicit function theorem we can solve $\alpha_n^{1,m}, \beta_n^{0,m+1}, \gamma_n^{1,m}, \delta_n^{0,m+1}$ in terms of $b_n^{1,m}, b_n^{0,m+1}, c_n^{1,m}, c_n^{0,m+1}$ and finitely many other $\alpha_i^{1,m}, \beta_j^{0,m+1}, \gamma_k^{1,m}, \delta_l^{0,m+1}$ for the pole type. Therefore the $(4m+9)$-dimensional linear constraint space in the preceding lemma generates a solution space of dimension at least $4m+9$ for the involved coefficients of p_{m+1} and q_{m+2} over $D_1 \cap D_2$ and $D_2 \cap D_3$; all other coefficients are arbitrary.

In conclusion we have constructed a large class of contact 3-folds with $L|_C = \mathcal{O}_3$ for its contact line bundle L and its 4-parameter Legendre rational curves C, referred to as the class of pole type, each of whose members has nontrivial L-sections.

3. Generic contact 3-folds of pole type near the 1st-jet are non-isomorphic. Let us now consider a 1-parameter family (with parameter t) of the contact transition $\Phi_{ij}(t)$ as specified by the data for the pole type such that $\Phi_{ij}(0)$ is given in (9) and (10), the transition of the 1st-jet of $\mathcal{O}(3)$.

This is equivalent to specifying $\alpha_n^{1,m}(t,x,z), \beta_n^{0,m+1}(t,x,z),\ \gamma_n^{1,m}(t,x,z)$ and $\delta_n^{0,m+1}(t,x,z)$ so that they are identically zero at $t=0$.

Now it is well-known [10] that the vector filed $d/dt|_{t=0}\Phi_{ij}(t)$ evaluated by the contact form results in an element Ω in H^1L, where L is the contact line bundle of the 1st-jet of $\mathcal{O}(3)$. Explicitly, $\Omega=(h_{ij})$, where

$$h_{12} = \sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}(\alpha_n^{1,m})'(0)x^{-n}yz^m + (\beta_n^{0,m+1})'(0)x^{-n}z^{m+1} \tag{32}$$

$$h_{23} = \sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}(\gamma_n^{1,m})'(0)x^{-n}yz^m + (\delta_n^{0,m})'(0)x^{-n}z^{m+1}. \tag{33}$$

(Prime denotes differentiation with respect to t.) Ω naturally projects to an element $\Omega^{(m)}$ in $H^1(\mathcal{O}_{(m)})$ for all m by ignoring the terms of the expansion of Ω of degree $\geq m+1$.

Suppose we have another family $\Psi_{ij}(t)$ whose derivative at $t=0$, denoted by Θ, projects to $\Theta^{(m-1)}$ so that $\Omega^{(m-1)}=\Theta^{(m-1)}$. Then by (4) we see that $\Phi^{(m)}-\Theta^{(m)}$ lives in $H^1(L|_C\otimes S^m(N^*))$. Since $N^*=\mathcal{O}(2)\oplus\mathcal{O}(2)$ [2], we know

$$L|_C\otimes S^m(N^*)=\mathcal{O}(3-2m)\otimes\mathbb{C}^{m+1},$$

so that

$$H^1(L|_C\otimes S^m(N^*))=H^0(\mathcal{O}(2m-5))\oplus\cdots\oplus H^0(\mathcal{O}(2m-5)), \tag{34}$$

where $H^0(\mathcal{O}(2m-5))$ appears $m+1$ times.

Accordingly, let us represent $\Phi^{(m)}-\Theta^{(m)}$ by the cocycle

$$\Phi^{(m)}-\Theta^{(m)} := (g_{12}(x)yz^{m-1}+h_{12}(x)z^m, g_{23}(x)yz^{m-1}+h_{23}(x)z^m).$$

Serre duality then identifies the cohomology class of this cocycle with one in (34) explicitly by identifying (g_{12},g_{23}) (and likewise for (h_{12},h_{23})) with a polynomial A in x of degree $3m-5$ given by the pairing

$$A\otimes(g_{12},g_{23})\longmapsto\int_{\Gamma_1}Ag_{12}dx+\int_{\Gamma_2}Ag_{23}dx\in\mathbb{C}.$$

where Γ_1 and Γ_2 are homologically nontrivial circles in $D_1\cap D_2$ and $D_2\cap D_3$, respectively, with opposite orientations. This leads to the identification of

$\Phi^{(m)} - \Theta^{(m)}$ with

$$yz^{m-1}\sum_{i=0}^{3m-5}(g_{12,i+1} - g_{23,i+1})x^i + z^m\sum_{i=0}^{3m-5}(h_{12,i+1} - h_{23,i+1})x^i, \qquad (35)$$

in $H^1(L|_C \otimes S^m(N^*))$, where $g_{12} = \sum_{n=-\infty}^{\infty} g_{12,n}x^{-n}$, etc. In terms of $\Phi^{(m)}$ and $\Theta^{(m)}$, (35) is

$$\begin{aligned} &yz^{m-1}(\textstyle\sum_{i=0}^{3m-5}(\alpha_{\Phi,i+1}^{1,m-1})'(0) - (\gamma_{\Phi,i+1}^{1,m-1})'(0) - (\alpha_{\Theta,i+1}^{1,m-1})'(0) + (\gamma_{\Theta,i+1}^{1,m-1})'(0)) \\ &\quad + z^m(\textstyle\sum_{i=0}^{3m-5}(\beta_{\Phi,i+1}^{0,m})'(0) - (\delta_{\Phi,i+1}^{0,m})'(0) - (\beta_{\Theta,i+1}^{0,m})'(0) + (\delta_{\Theta,i+1}^{0,m})'(0)), \end{aligned} \qquad (36)$$

which is generically nonzero as $m \to \infty$ in view of the defining data for the pole type. The nonvanishing of (36) then asserts that generic such contact 3-folds constructed are not isomorphic.

4. Concluding remarks. In light of Section 3, Lemma 1 clearly recovers generically nonisomorphic contact 3-folds that depend on two holomorphic functions in three variables (take $N_m = 0$ for the pole type); therefore their real slices depend on four functions in three variables, as was proved for the moduli of generic G_3-connections by Bryant using EDS. Suggested by this comparison, it seems that Lemma 1 has already exhausted all the possible such contact 3-folds. Of course, Lemma 1 captures indeed all the *formal* extensions of the natural contact structure of the 1st-jet of $\mathcal{O}(3)$. What we must do to classify all such contact 3-folds is to study the uniqueness part of the extension problem. Namely, will $H^1(\mathcal{CAUT}_{(s)}), 1 \leq s < \infty$ suffice to determine $H^1(\mathcal{CAUT}|_C)$?

What we have shown is that we are able to construct real *degenerate* torsion-free G_3-connections that generically depend on four (restricted) functions in three variables by studying the formal extension problem. The examples we found in [4] belong to the subclass of the pole type when $\alpha_n^{1,m} = \gamma_n^{1,m} \equiv 0$ for all n ($N_m = 0$). There are no higher order terms in (30) and (31) in this case. This subclass generically depends on two (restricted) functions in three variables.

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