TORSION-FREE G_3-CONNECTIONS THAT ARE NOT ANALYTIC

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Abstract. Up to now all the known torsion-free G_3-connections have been analytic. We provide an explicit PDE-solving approach to construct a family of smooth torsion-free G_3-connections that are not equivalent to any analytic ones, although each of these connections is locally homogeneous away from a hypersurface of the base manifold.

1. Introduction

Perhaps one of the ever enlightening remarks that shed much light on the importance of torsion-free linear connections in differential geometry has been that of Hermann Weyl's [19] that points out that the very existence of inertia systems in the Universe warrants that its connection is torsion-free.

The question of finding torsion-free connections, if there are any, on a given principal subbundle of the frame bundle of a manifold immediately becomes one in partial differential equations, in principle. Berger in his remarkable paper [1] classified those irreducibly acting reductive groups of the general linear group that may arise as the holonomy group of a torsion-free connection. His classification is complete in the metric case. However, as Bryant pointed out [2], the classification is incomplete in the non-metric case. These missing holonomy groups from Berger's list were referred to by Bryant as *exotic holonomies*.

We shall not dwell on the recent development of the holonomy classification, for which the reader can consult the survey articles [4], [17] and the papers [2], [6], [9], [14], [18]. Instead, we shall return once more to exotic G_3-structures, which is an anomaly among all exotic holonomies. Bryant first found in [2] exotic H_3 ($= SL(2,\mathbb{R})$) and (analytic) G_3 ($= GL^+(2,\mathbb{R})$) connections on 4-manifolds that do not preserve any metric. It turns out that the H_3-structures fall in the

Date: July 28, 2003.
1991 *Mathematics Subject Classification.* Primary: 53C10.
Key words and phrases. Torsion-free linear connection, holonomy.
The author was partially supported by NSF Grant No. DMS-0103838.

symplectic family [6] arising from symplectic representations of simple Lie groups realized as the holonomy groups, whereas the G_3-structures stand out in a peculiar way. Namely, first off we could replace the simple Lie groups G in the symplectic family by $\mathbb{R}^+G$. However, since these holonomies are reductions of conformally symplectic structures, which in turn can be reduced to symplectic structures when the dimension of the manifold > 4 ([2]), the only reductive group extrapolated out of the symplectic family is G_3, which is $\mathbb{R}^+H_3$.

G_3-structures are salient in another striking manner. Namely, the moduli space of the H_3 family (in fact, of every holonomy in the symplectic family [6]) is analytic and finite-dimensional whereas the moduli space of the generic (nondegenerate) analytic G_3-connections is parametrized by four functions in three variables [2], which Bryant deduced using exterior differential systems (EDS) coupled with large MAPLE-calculations that are indispensable, because a certain nowhere vanishing 8×8 determinant J each of whose entries is a linear function in twenty-eight variables must be reckoned with. The magnitude of the symbolic calculations encountered plus the nonconstructive nature of EDS seems to render it hard to study the geometry of each individual of these generic structures. On the other hand, Bryant developed in [2] the powerful (and conceptual) twistorial approach in the holomorphic category, where each real analytic G_3-structure sits as a real slice of its complexification, and thus he could apply the deformation theory to gain a structure theory for real analytic G_3-structures from its holomorphic counterpart, which the author utilized, with the aid of the notion of infinitesimal neighborhoods of (holomorphic) submanifolds, to analyze degenerate G_3-structures, whose moduli depends at least on four restricted functions (in an appropriate sense) in three variables [7], [8]. However, due to the qualitative nature of this holomorphic approach, it seems in general difficult to answer, e.g., the basic question of writing down torsion-free G_3-connections in terms of, say, their Christoffel symbols Γ^i_{jk} in local coordinates, which can provide much local and often global geometric information. Furthermore, it seems remote that these techniques, being analytic in nature, could handle the fundamental question of the existence of smooth torsion-free G_3-connections. Of course, here by a smooth connection we mean one that is not equivalent to any analytic ones.

In this paper we shall prove yet a further result about the intriguing anomalous torsion-free G_3-structures, which the title of this paper addresses. To motivate the ground work, let us first briefly recall [15] that the space of G_3-bundles over a 4-manifold M can be identified with the space of maps (smooth, analytic, holomorphic, etc.) from M to $GL(4)/GL(2)$; hence locally it depends on twelve functions in four variables. For a G_3-bundle admitting a torsion-free structure, locally it is defined, when we set the intrinsic torsion to be zero, by an underdetermined system of eight 1st-order partial differential equations with twelve functions in four variables not of Cauchy-Kowalewskaya type [2]. It seems daunting to even guess whether there is a smooth solution to the system, much less proving that a smooth solution, if existent, generates a torsion-free G_3-connection that is not equivalent to any analytic ones.

A simple example in the mechanics of nonholonomic systems can shed much light on this question. Consider a circle rolling vertically without skidding on a plane. If the angle made by the plane containing the circle and the x-axis of the plane on which the circle rolls is ϕ, then the (underdetermined) equation of motion of the circle is

$$dy = (\tan\phi)dx.$$

When the circle is confined to move along a line, the solution is just the line itself. However, with one more degree of freedom introduced the solutions can be very complicated; any smoothly parametrized curve on the plane gives rise to a solution. As a result, singular solutions can occur, as we all have experienced when we, for instance, roll a swimming tube.

This helps explain why on the sheer group level H_3 differs from G_3 only by a scaling factor, and yet on the manifold level torsion-free H_3-connections, which are analytic and finite-dimensional in moduli and can be fully accounted for by Poisson geometry, similar to the rolling circle confined to move on a line, explodes into a universe of analytic torsion-free G_3-connections whose moduli depends on four functions in three variables, similar to the general solutions to the motion of the rolling circle on the plane, when we perturb from H_3 to G_3.

With the motivation by this example, we shall prove in this paper the following.

Theorem 1. *There are smooth torsion-free G_3-connections that are not equivalent to any analytic ones, although each of these connections is locally homogeneous (in fact, is left-invariant on a local solvable group) away from a hypersurface of the base manifold.*

We remark that as a byproduct, we can display the Christoffel symbols Γ^i_{jk} of these torsion-free G_3-connections explicitly in coordinates, which is crucial to our proof.

2. Preliminaries

2.1. **The G_3-actions.** Let V be the real vector space of dimension 4 identified with homogeneous polynomials of degree 3 in two variables x and y. The group $G_3 := GL^+(2,\mathbb{R})$ acts on V in a natural way. Namely, for $p(x,y) \in V$ and $A \in G_3$, the action of A on $p(x,y)$ is given by $(A \cdot p)(x,y) = p((x,y)\cdot A)$. This action induces an action on V^*, the dual of V, given by $(A \cdot f)(v) = f(A^{-1}\cdot v)$ for $f \in V^*$ and $v \in V$.

$H_3 := SL(2,\mathbb{R})$, the semisimple part of G_3, acts on V with x^3 the maximal weight, whose G_3-orbit is a cone in V. As can be easily seen, this cone comprises of all polynomials of the form $(ax+by)^3$. When projectivized, the cone turns into the twisted cubic $[a^3 : 3a^2b : 3ab^2 : b^3]$ in the homogeneous coordinates relative to the basis x^3, x^2y, xy^2, y^3. Following [2], we will call the cone the intrinsic cone and its projectivization the intrinsic twisted cubic.

Likewise, let $f_1, \cdots, f_4$ be the dual basis to x^3, x^2y, xy^2, y^3. Then f_4 is the maximal weight of the dual G_3-action, whose G_3-orbit is the cone $a^3f_1 + a^2bf_2 + ab^2f_3 + b^3f_4$, of which the projectivization is the twisted cubic $[a^3 : a^2b : ab^2 : b^3]$ in the homogeneous coordinates relative to $f_1, \cdots, f_4$. We will call the cone and its projectivization the dual intrinsic cone and the dual intrinsic twisted cubic.

2.2. **G_3-structures.** We will review some results in [2] in this subsection. Let M be a smooth 4-dimensional manifold equipped with a torsion-free G_3-connection. The holonomy group G_3 acts on the typical tangent space of M, which induces an action on the cotangent space as above. The collection of the dual intrinsic cones with the zero vector removed is denoted by $\mathcal{V} \subset T^*M$, whose projectivization, denoted by $\mathcal{C}$, is the circle bundle of dual twisted cubics over M.

Now let the canonical symplectic form of T^*M be α, and let Δ be the distribution over $\mathcal{V}$ defined, for $y \in \mathcal{V}$, by

$$\Delta(y) = \{X \in T_y\mathcal{V} : \alpha(X, T_y\mathcal{V}) \equiv 0\}.$$

Δ is an integrable 2-dimensional distribution whose leaf space $\mathcal{F}$, when restricted to a sufficiently small open set U of M, inherits from α a symplectic structure. $\mathbb{P}(\mathcal{F})$, the projectivization of $\mathcal{F}$ over U, is a contact 3-fold W of the G_3-structure on M. Therefore over U there is a double fibration $\Pi : \mathcal{C} \longrightarrow U$ and $\Psi : \mathcal{C} \longrightarrow W$ such that for $p \in U$, Ψ sends $\mathcal{C}_p := \Pi^{-1}(p)$ to a contact circle in W.

We remark that in a standard local coordinate system $(q_s, p^s), 1 \leq s \leq 4$, for T^*M, we have $\alpha = \sum_s dp^s \wedge dq_s$ and a straightforward calculation gives that Δ is spanned by, for $i = 1, 2$,

$$X_i = -2 \sum_{jk} a^{(i)}_{jk} p^j \frac{\partial}{\partial q_k} + \sum_{jks} p^j p^k \frac{\partial a^{(i)}_{jk}}{\partial q_s} \frac{\partial}{\partial p^s},$$

where

$$F_i := \sum_{jk} a^{(i)}_{jk}(q) p^j p^k = 0,$$

$1 \leq i \leq 2$, locally carve out the dual intrinsic cone at q. It follows that if $\pi : T^*M \longrightarrow M$ is the projection, then for $i = 1, 2$,

$$\pi_*(X_i) = -2 \sum_{jk} a^{(i)}_{jk} p^j \frac{\partial}{\partial q^k}.$$

As an immediate consequence, at q any vector f in $T^*_q M$ tangent to the dual intrinsic cone at (p^s), so that $dF_i(f) = 0$, must nullify $\pi_*(X_i)$ for $i = 1, 2$. Therefore, we conclude that the plane spanned by $\pi_*(X_i)$, $i = 1, 2$, in T_qM is tangent to the intrinsic cone. In other words, each leaf in Δ projects to a surface in U that is tangent to intrinsic cones everywhere.

The important fact is that the converse is also true [2], which characterizes a torsion-free G_3-connection.

2.3. **Three lemmas.** From now on we assume M, parametrized by $\epsilon := (\epsilon_0, \cdots, \epsilon_3)$, is small enough to allow the aforementioned double fibration. Let $p \in M$ be the origin in the coordinates. Since the fibration $\Psi : \mathcal{C} \longrightarrow M$ is trivial when M is shrunken even more, we can select maps

$$\iota_q : \mathcal{C}_p \longrightarrow \mathcal{C}_q$$

for all $q \in M$ such that ι_q gives rise to a smooth parametrization $\Gamma(t, \epsilon)$, analytic in t, from $\mathcal{C}_p \times M$ to $\mathcal{C}$ via its trivialization. Γ maps a vector $v \in T_qM$ to a vector field along C_q given by $\Gamma_{t*}(v)$, where $\Gamma_t(\cdot) := \Gamma(t, \cdot)$ for a fixed t, and q has coordinates ϵ.

Now let $\mathcal{D}$ be the contact distribution of W and let $\mathbf{L} := TW/\mathcal{D}$ be the contact line bundle of W. For a contact circle $\mathbf{S}$ in W we let $\mathbf{L}_{|\mathbf{S}}$ be the restriction of $\mathbf{L}$ onto $\mathbf{S}$. For the contact circle $\mathbf{S} := \Psi(C_q)$ we can set up the map

$$\rho : T_qM \longrightarrow H^0(\mathbf{L}_{|\mathbf{S}})$$

whose image is given by $\Psi_*(\Gamma_{t*}(v))$ in TW followed by the projection onto $\mathbf{L}_{|\mathbf{S}}$. Here, $H^0(\mathbf{L}_{|\mathbf{S}})$ denotes the space of smooth sections of $\mathbf{L}_{|\mathbf{S}}$, which is infinite-dimensional. ρ is independent of the parametrization Γ of $\mathcal{C}$ chosen, since any other parametrization amounts to only introducing an additional vector tangent to the fibers of $\mathcal{C}$.

Lemma 2. *The map ρ is injective.*

Proof. Suppose $\rho(v) = 0$ for some $v \neq 0$. Then $\Psi_*(\Gamma_{t*}(v)) \in \mathcal{D}$ for all $t \in C_q$. Let $v = \epsilon'(0)$ for a curve $\epsilon(s) \in M$. Then we have a 2-parameter family $\gamma(t,s) := \Gamma(t, \epsilon(s))$. Let $c(t,s) = \Psi(\gamma(t,s))$. Then

$$\frac{\partial c(t,s)}{\partial s}|_{s=0} = \Psi_*(\Gamma_{t*}(v)) \in \mathcal{D}. \tag{1}$$

Now $c(t,s)$ is a contact circle in W for each s. So the deformation of the first order at $s = 0$ of the contact curves $c(t,s)$ belongs to $\mathcal{D}$ for each t.

We may cover the contact circle $c(t,0)$ by contact charts (x_i, y_i, z_i) with contact form $\theta_i = dy_i - z_i dx_i$ of W in such a way that $(x_i, 0, 0)$ parametrizes $c(t,0)$. By shrinking M even further, we may assume that the 2-parameter family $c(t,s)$ are graphs over $(x := x_i, 0, 0)$ over each contact chart, with respect to which $c(t,s)$ is $(x_i(x,s), y_i(x,s), z_i(x,s))$. With this the preceding equation is equivalent to

$$\frac{\partial y_i}{\partial s}(x, 0) = 0. \tag{2}$$

However, the fact that $c(t,s)$ is a contact curve for each s gives $\partial y_i/\partial x = z_i \partial x_i/\partial x$. Differentiating this equation with respect to s at $s = 0$ and invoking (2), $z_i(x,0) = 0$ and $\partial x_i/\partial x = 1$ at $s = 0$, we obtain $\partial z_i/\partial s(x,0) = 0$, so that $\partial c/\partial s$ is tangent to the contact circle at $s = 0$. In view of (1), this implies that $\Gamma_{t*}(v)$ belongs to the span of Δ', the image of Δ via the projection from $\mathcal{V}$ to $\mathcal{C}$, and the tangent space of C_q at $\Gamma(t, \epsilon)$ in $\mathcal{C}$. It follows that the projection of $\Gamma_{t*}(v)$ onto T_qM is tangent to the intrinsic cone at all t. However, clearly this projection is v. Hence v lies in all the tangent planes to the intrinsic cones, which is absurd. □

Corollary 3. *$\rho(v)$ vanishes only at isolated points.*

Proof. This follows from the last three sentences of the preceding lemma. □

Lemma 4. *For any $z \in \mathcal{C}_q$, the subspace V_1 of T_qM consisting of all vectors v such that $\rho(v)$ vanishes at z is 3-dimensional. The subspace V_2 consisting of all vectors v such that $\rho(v)$ vanishes to the 1st order at z is 2-dimensional.*

Proof. As before we let $\Gamma_t(\epsilon)$ be z. Consider the linear map $T_1 : v \longmapsto \rho(v)'(t)$ from $\mathbb{R}^4$ to $\mathbb{R}$. If $\dim V_1 = 4$, then since the kernel of T_1 is of dimension at least 3, we have $\dim V_2 \geq 3$. However, the arguments of the preceding lemma shows that all $v \in V_2$ would lie in a 2-plane tangent to the intrinsic cone, which is impossible. Consequently, $\dim V_1 \leq 3$. On the other hand the map $T_2 : v \longmapsto \rho(v)(t)$ implies that $\dim V_1 \geq 3$. So $\dim V_1 = 3$.

The map T_1 restricted to V_1 gives that $\dim V_2 \geq 2$. For the same reason that any vector $v \in V_2$ lies in a tangent plane to the intrinsic cone, we see $\dim V_2 \leq 2$. So $\dim V_2 = 2$. □

Set $V := T_qM$. With Lemma 4 at hand, we can, as in projective geometry, define a map

$$\begin{aligned}\Lambda : \mathcal{C}_q &\longrightarrow \mathbb{P}(V^*)\\ : z &\longmapsto \text{the dual of } V_1,\end{aligned}$$

where V_1 is as in the preceding lemma.

Lemma 5. *$\Lambda(\mathcal{C}_q)$ is the dual twisted cubic at q.*

Proof. The dual of V_2 in V^* defines a projective line $l(z)$ containing the point $\Lambda(z)$. In fact, $l(z)$ is tangent to the curve $\Lambda(\mathcal{C}_q)$ at $\Lambda(z)$. Hence the $l(z)$ envelopes $\Lambda(\mathcal{C}_q)$ as z varies in C_q. On the other hand, as explained in Lemma 2, V_2 is a tangent plane to the intrinsic cone. Therefore, V_2 envelopes the intrinsic cone and its dual envelopes the dual intrinsic cone. Putting these together we see that the curve $\Lambda(\mathcal{C}_q)$ is the dual twisted cubic at q. □

3. The governing partial differential equations of a class of torsion-free G_3-connections

3.1. **The primitive PDEs defining the connections.** Suppose now the 4-parameter family of contact circles described in Section 2.2 lies in a contact 3-fold W (the leaf space), which is covered by contact charts $U_\alpha = \{(x_\alpha, y_\alpha, z_\alpha)\}$ with the contact forms $dy_\alpha - z_\alpha dx_\alpha$ in such a way that the contact circle for $\epsilon = 0$ is given by $(x_\alpha, 0, 0)$. Clearly, we may assume that the transition between $(x_\alpha, 0, 0)$ and $(x_\beta, 0, 0)$ is analytic. As we mentioned, by shrinking M enough we may assume that all these contact circles are graphs over $(x_\alpha, 0, 0)$. Then they are parametrized as $(x_\alpha, y_\alpha(x_\alpha, \epsilon), \partial y_\alpha/\partial x_\alpha(x_\alpha, \epsilon))$. The map ρ encountered in the preceding section now assumes the form

$$\rho : \frac{\partial}{\partial \epsilon_s} \longmapsto (\frac{\partial y_\alpha}{\partial \epsilon_s}),$$

and $[\partial y_\alpha/\partial\epsilon_0 : \cdots : \partial y_\alpha/\partial\epsilon_3]$ parametrized by x_α is exactly the dual intrinsic twisted cubic relative to the tangent basis $(\partial y_\alpha/\partial\epsilon_0), \cdots, (\partial y_\alpha/\partial\epsilon_3)$ at ϵ.

Motivated by this observation and Lemma 5, we consider a primitive set of partial differential equations as follows. For $0 \le i \le 3$,

$$\frac{\partial y_\alpha}{\partial \epsilon_i} = r_\alpha(x_\alpha, \epsilon) \sum_{j=0}^{3} a_i^j(\epsilon)\zeta_j(x_\alpha), \tag{3}$$

subject to the condition $y_\alpha(x_\alpha, 0) = 0$, for some positive smooth function r_α. Here $\zeta_j(x_\alpha)$ are the standard sections x^3, x^2y, xy^2, y^3 of $\mathbb{P}^3$ relative to a coordinate system x_α of $\mathbb{P}^1 := \{[x : y]\}$. Note that, with (3), $[\partial y_\alpha/\partial\epsilon_0 : \cdots : \partial y_\alpha/\partial\epsilon_3]$ automatically is a twisted cubic at ϵ.

What is remarkable is that the integrability conditions to (3) turn out to produce explicit torsion-free G_3-connections.

3.2. **Integrability conditions to the primitive PDEs.** For notational convenience, we let

$$\eta_i := \sum_{j=0}^{3} a_i^j(\epsilon)\zeta_j(x_\alpha)$$

in (3). We will drop the subscript α since we now work only in a single coordinate chart . We may assume $\eta_0 = \zeta_0$. For by the chain rule, (3) is transformed to

$$\frac{\partial y}{\partial \overline{\epsilon}_i} = r \sum_l \overline{a}_i^l \zeta_l,$$

where

$$\overline{a}_i^l = \sum_s a_s^l \frac{\partial \epsilon_s}{\partial \overline{\epsilon}_i}.$$

Hence $\overline{a}_0^0 = 1$ and $\overline{a}_0^i = 0$ for $i > 0$ if and only if

$$\frac{\partial \epsilon}{\partial \overline{\epsilon}_0} = A^{-1} \cdot (1,0,0,0)^{tr}$$

with $A := (a_s^l)$. This is always solvable, when we shrink the domain of ϵ if necessary, since $\overline{\epsilon}_0$ is the flow parameter to the vector field $A^{-1} \cdot (1,0,0,0)^{tr}$.

We are now solving

$$\frac{\partial y}{\partial \epsilon_i} = r\eta_i \tag{4}$$

with $\eta_0 = \zeta_0$.

Write

$$\mu_i =: \frac{\partial \log r}{\partial \epsilon_i}$$

and

$$\eta_{ji} =: \frac{\partial \eta_j}{\partial \epsilon_i}.$$

The integrability conditions are

(5) $$\mu_i\eta_j - \mu_j\eta_i = \eta_{ij} - \eta_{ji}.$$

In particular, setting $j = 0$ we obtain

(6) $$\eta_{k0} = \mu_k\zeta_0 - \mu_0\eta_k$$

since $\eta_{0k} = 0$. Substituting μ_k solved from the preceding equation into (5) we obtain

$$(\mu_k\eta_i - \mu_i\eta_k)\zeta_0 = \eta_{k0}\eta_i - \eta_{i0}\eta_k,$$

so that

$$(\eta_{ki} - \eta_{ik})\zeta_0 = \eta_{k0}\eta_i - \eta_{i0}\eta_k,$$

or

(7) $$\sum_{s,t} a_i^t \frac{\partial a_k^s}{\partial \epsilon_0}\zeta_s\zeta_t - \sum_{s,t} a_k^t \frac{\partial a_i^s}{\partial \epsilon_0}\zeta_s\zeta_t = \sum_{\alpha}\left(\frac{\partial a_k^\alpha}{\partial \epsilon_i} - \frac{\partial a_i^\alpha}{\partial \epsilon_k}\right)\zeta_\alpha\zeta_0.$$

Dividing through (7) by $(\zeta_0)^2$ and employing the fact that $(\zeta_0/\zeta_0, \cdots, \zeta_3/\zeta_0)$ is the twisted cubic $(1, x, x^2, x^3)$, we can compare the coefficients of $x^\alpha, 0 \le \alpha \le 3$, in (7) to yield

(8) $$\sum_{s+t=\alpha\le 3}\left(a_i^t \frac{\partial a_k^s}{\partial \epsilon_0} - a_k^t \frac{\partial a_i^s}{\partial \epsilon_0}\right) = \frac{\partial a_k^\alpha}{\partial \epsilon_i} - \frac{\partial a_i^\alpha}{\partial \epsilon_k},$$

(9) $$\sum_{s+t=\alpha\ge 4}\left(a_i^t \frac{\partial a_k^s}{\partial \epsilon_0} - a_k^t \frac{\partial a_i^s}{\partial \epsilon_0}\right) = 0.$$

3.3. A canonical non-torsion-free G_3-connection.

Proposition 6. $\Gamma_{ti}^s = \sum_\mu \overline{a}_\mu^s \frac{\partial a_i^\mu}{\partial \epsilon_t}$, *where* $A^{-1} = (\overline{a}_\mu^s)$, *is a* G_3-*connection that is not torsion-free in general.*

Proof. Consider the principal G_3-bundle for which the vector fields $X_i, 0 \le i \le 3$, such that

$$\frac{\partial}{\partial \epsilon_i} = \sum_{j=0}^{3} a_i^j X_j,$$

form an adapted G_3-frame of the bundle. We define the "canonical" connection ∇ such that

$$\nabla X_i = 0$$

for all i. A straightforward calculation shows that the Christoffel symbols are of the desired form with the torsion

$$T^s_{ti} = \sum_\mu \overline{a}^s_\mu \left(\frac{\partial a^\mu_i}{\partial \epsilon_t} - \frac{\partial a^\mu_t}{\partial \epsilon_i}\right). \tag{10}$$

□

Let the dual forms of X_j be denoted by $\theta^j, j = 0, ..., 3$. In view of (10) and (8), the torsion 2-form T is given in matrix form by $T = -\Omega \wedge \theta$ so that the structural equation reads

$$d\theta = -\Omega \wedge \theta, \tag{11}$$

where

$$\Omega = \begin{pmatrix} \Omega^0_0 & 0 & 0 & 0 \\ \Omega^1_0 & \Omega^0_0 & 0 & 0 \\ \Omega^2_0 & \Omega^1_0 & \Omega^0_0 & 0 \\ \Omega^3_0 & \Omega^2_0 & \Omega^1_0 & \Omega^0_0 \end{pmatrix}$$

with

$$\begin{aligned} \Omega^\alpha_\beta &= \sum_{k=0}^{3} \frac{\partial a^\alpha_k}{\partial \epsilon_\beta} d\epsilon^k, \\ \theta^\beta &= \sum_{k=0}^{3} a^\beta_k d\epsilon^k. \end{aligned} \tag{12}$$

3.4. **The PDEs of the torsion-free G_3-connections.** An element in the Lie algebra $\mathcal{G}_3$ of G_3 is of the form

$$\begin{pmatrix} 3A & B & 0 & 0 \\ 3C & 2A+D & 2B & 0 \\ 0 & 2C & A+2D & 3B \\ 0 & 0 & C & 3D \end{pmatrix} \tag{13}$$

To obtain a torsion-free G_3-connection, we perturb (11) to produce a connection form ω with values in $\mathcal{G}_3$ such that $d\theta = -\omega \wedge \theta$. To this end, set

$$\overline{\omega} =: \begin{pmatrix} \Omega^0_0 & 0 & 0 & 0 \\ \Omega^1_0 & \Omega^0_0 & 0 & 0 \\ 0 & 2/3\Omega^1_0 & \Omega^0_0 & 0 \\ 0 & 0 & 1/3\Omega^1_0 & \Omega^0_0 \end{pmatrix},$$

which is $\mathcal{G}_3$-valued. Then (11) is nothing but

$$d\theta = -\overline{\omega} \wedge \theta + \overline{T},$$

where $\overline{T} = \overline{\Omega} \wedge \theta$ with

$$\overline{\Omega} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\Omega^2_0 & -1/3\Omega^1_0 & 0 & 0 \\ -\Omega^3_0 & -\Omega^2_0 & -2/3\Omega^1_0 & 0 \end{pmatrix}$$

The question is now reduced to asking whether there is a $\mathcal{G}_3$-valued 1-form Λ such that

$$\overline{T} = -\Lambda \wedge \theta; \tag{14}$$

if yes, then $\omega = \overline{\omega} + \Lambda \in \mathcal{G}_3$ will satisfy $d\theta = -\omega \wedge \theta$, which gives rise to a torsion-free G_3-connection.

Proposition 7. *The existence of* Λ *is a consequence of* (9).

Proof. Let us set

$$\Lambda := \begin{pmatrix} 3\omega^0_0 & \omega^0_1 & 0 & 0 \\ 3\omega^1_0 & 2\omega^0_0 + \omega^3_3 & 2\omega^0_1 & 0 \\ 0 & 2\omega^1_0 & \omega^0_0 + 2\omega^3_3 & 3\omega^0_1 \\ 0 & 0 & \omega^1_0 & 3\omega^3_3 \end{pmatrix}$$

with $\omega^\alpha_\beta := \sum_\gamma f^\alpha_{\beta\gamma}\theta^\gamma$. Then (14) establishes several identities as follows.

I: From $3\omega^0_0 \wedge \theta^0 + \omega^0_1 \wedge \theta^1 = 0$ we obtain

$$0 = f^0_{10} - 3f^0_{01}, \tag{15}$$

$$0 = f^0_{02} = f^0_{03} = f^0_{12} = f^0_{13}. \tag{16}$$

II: From $3\omega^1_0 \wedge \theta^0 + 2\omega^0_0 \wedge \theta^1 + \omega^3_3 \wedge \theta^1 + 2\omega^0_1 \wedge \theta^2 = 0$ and (16) we derive

$$\begin{aligned} 0 &= 2f^0_{00} - 3f^1_{01} + f^3_{30}, \\ 0 &= 2f^0_{10} - 3f^1_{02}, \\ 0 &= 2f^0_{11} - f^3_{32} - 2f^0_{02} = 2f^0_{11} - f^3_{32}, \\ 0 &= f^1_{03}, \\ 0 &= f^3_{33} + 2f^0_{03} = f^3_{33}. \end{aligned} \tag{17}$$

III: Let $\Omega^\alpha_\beta = \sum_\gamma \Omega^\alpha_{\beta\gamma}\theta^\gamma$. From

$$2\omega^1_0 \wedge \theta^1 + \omega^0_0 \wedge \theta^2 + 2\omega^3_3 \wedge \theta^2 + 3\omega^0_1 \wedge \theta^3 = \Omega^2_0 \wedge \theta^0 + 1/3\Omega^1_0 \wedge \theta^1$$

we find

$$
\begin{aligned}
-\Omega^2_{01}+1/3\Omega^1_{00}&=2f^1_{00},\\
-\Omega^2_{02}&=f^0_{00}+2f^3_{30},\\
-\Omega^2_{03}&=3f^0_{10},\\
-1/3\Omega^1_{02}&=-2f^1_{02}+f^0_{01}+2f^3_{31},\\
-1/3\Omega^1_{03}&=-2f^1_{03}+3f^0_{11}=3f^0_{11}.
\end{aligned}\tag{18}
$$

IV: From $\Omega^3_0\wedge\theta^0+\Omega^2_0\wedge\theta^1+2/3\Omega^1_0\wedge\theta^2=\omega^1_0\wedge\theta^2+3\omega^3_3\wedge\theta^3$ we deduce

$$
\begin{aligned}
-\Omega^3_{01}+\Omega^2_{00}&=0,\\
-\Omega^3_{02}+2/3\Omega^1_{00}&=f^1_{00},\\
-\Omega^3_{03}&=3f^3_{30},\\
-\Omega^2_{02}+2/3\Omega^1_{01}&=f^1_{01},\\
-\Omega^2_{03}&=3f^3_{31},\\
-2/3\Omega^1_{03}&=-f^1_{03}+3f^3_{32}=3f^3_{32}.
\end{aligned}\tag{19}
$$

Therefore, from these identities we immediately arrive at

$$
\begin{aligned}
\omega^0_0&=f^0_{00}\theta^0+f^0_{01}\theta^1\\
&=(-\Omega^2_{02}+2/3\Omega^3_{03})\theta^0-1/9\Omega^2_{03}\theta^1,\\
\omega^0_1&=f^0_{10}\theta^0+f^0_{11}\theta^1\\
&=-1/3\Omega^2_{03}\theta^0-1/9\Omega^1_{03}\theta^1,\\
\omega^1_0&=f^1_{00}\theta^0+f^1_{01}\theta^1+f^1_{02}\theta^2\\
&=(-\Omega^3_{02}+2/3\Omega^1_{00})\theta^0+(2/3\Omega^1_{01}-\Omega^2_{02})\theta^1-2/9\Omega^2_{03}\theta^2,\\
\omega^3_3&=f^3_{30}\theta^0+f^3_{31}\theta^1+f^3_{32}\theta^2\\
&=-1/3\Omega^3_{03}\theta^0-1/3\Omega^2_{03}\theta^1-2/9\Omega^1_{03}\theta^2.
\end{aligned}
$$

Here, we use the third, fifth and sixth identities in (19) and the fifth identity in (17) to get ω^3_3. The second and the fourth identities in (19) and the second and the fourth identities in (17) are employed to obtain ω^1_0. Next, (16), the third and fifth identities in (18) come in to fix ω^0_1. Lastly, (15), (16) and the second identity of (18) define ω^0_0.

This leaves the first and the third identities in (17), the first and the fourth identities in (18) and the first identity in (19) as compatibility conditions. The first identity in (17) gives

$$
\Omega^2_{02}+\Omega^3_{03}=2\Omega^1_{01}.\tag{20}
$$

The third identity in (17) is automatically true ($\Omega^1_{03} = \Omega^1_{03}$). The first identity in (18) gives

$$\Omega^1_{00} + \Omega^2_{01} = 2\Omega^3_{02}, \tag{21}$$

while the fourth one yields

$$\Omega^2_{03} = \Omega^1_{02}. \tag{22}$$

Lastly, the first identity of (19) clearly results in

$$\Omega^2_{00} = \Omega^3_{01}. \tag{23}$$

Let $(\overline{a}^k_s)$ be the inverse of (a^k_s). Now (12) says

$$\Omega^\alpha_{\beta s} = \sum_k \overline{a}^k_s \frac{\partial a^\alpha_k}{\partial \epsilon_\beta}, \tag{24}$$

with respect to which (9) is equivalent to

$$\begin{aligned} \Omega^3_{0s}\delta^1_t + \Omega^2_{0s}\delta^2_t + \Omega^1_{0s}\delta^3_t &= \Omega^3_{0t}\delta^1_s + \Omega^2_{0t}\delta^2_s + \Omega^1_{0t}\delta^3_s, \\ \Omega^3_{0s}\delta^2_t + \Omega^2_{0s}\delta^3_t &= \Omega^3_{0t}\delta^2_s + \Omega^2_{0t}\delta^3_s, \\ \Omega^3_{0s}\delta^3_t &= \Omega^3_{0t}\delta^3_s. \end{aligned} \tag{25}$$

Setting $s = 1, t = 3$ in the first identity of (25) results in $\Omega^1_{01} = \Omega^3_{03}$, whereas setting $s = 2, t = 3$ in the second identity of (25) gives $\Omega^2_{02} = \Omega^3_{03}$. Hence (20) holds. Setting $s = 0, t = 3$ in the first identity, $s = 1, t = 3$ in the 2nd identity and $s = 2, t = 3$ in the third identity of (25) yields $\Omega^1_{00} = \Omega^2_{01} = \Omega^3_{02} = 0$, so that (21) is true. Setting $s = 3, t = 2$ in the first identity of (25) comes down to exactly (22). Lastly, setting $s = 0, t = 3$ and $s = 1, t = 2$ in the 2nd identity of (25) gives $\Omega^2_{00} = \Omega^3_{01} = 0$, which verifies (23). □

As a consequence we have established the following.

Theorem 8. (8) *and* (9) *are sufficient for the existence of a torsion-free G_3-connection. A given solution to these two sets of integrability equations gives the torsion-free G_3-connection in the local chart $\epsilon_0, \cdots, \epsilon_3$ as*

$$\Gamma^l_{ki} = \sum_s \overline{a}^l_s \frac{\partial a^s_i}{\partial \epsilon_k} + \sum_{s,t} \overline{a}^l_s \omega^s_t (\frac{\partial}{\partial \epsilon_k}) a^t_i,$$

where $(\overline{a}^l_s)$ is the inverse of (a^l_s), and

$$\omega = \begin{pmatrix} 3\lambda^0_0 & \lambda^0_1 & 0 & 0 \\ 3\lambda^1_0 & 2\lambda^0_0 + \lambda^3_3 & 2\lambda^0_1 & 0 \\ 0 & 2\lambda^1_0 & \lambda^0_0 + 2\lambda^3_3 & 3\lambda^0_1 \\ 0 & 0 & \lambda^1_0 & 3\lambda^3_3 \end{pmatrix}$$

with

$$\begin{aligned}\lambda_0^0 &= 1/3\Omega_0^0 - 1/9\Omega_{03}^2\theta^1 - 1/3\Omega_{03}^3\theta^0,\\ \lambda_1^0 &= -1/3\Omega_{03}^2\theta^0 - 1/9\Omega_{03}^1\theta^1,\\ \lambda_0^1 &= 1/3\Omega_0^1 - 2/9\Omega_{03}^2\theta^2 - 1/3\Omega_{03}^3\theta^1,\\ \lambda_3^3 &= 1/3\Omega_0^0 - 1/3\Omega_{03}^2\theta^1 - 2/9\Omega_{03}^1\theta^2 - 1/3\Omega_{03}^3\theta^0\end{aligned}$$

in which (12) *and* (24) *are employed.*

(8) and (9) are the governing partial differential equations for the class of torsion-free G_3-connections that we shall construct next.

4. Reduction of the PDEs

4.1. **A first reduction.** Recall from Section 3.2 that $a_0^i = \delta_0^i$. We choose to set

$$\begin{pmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \\ a_1^3 & a_2^3 \end{pmatrix} := \begin{pmatrix} Fa_3^1 + Ea_3^2 + Ca_3^3 & F^*a_3^1 + E^*a_3^2 + C^*a_3^3 \\ Fa_3^2 + Ea_3^3 & F^*a_3^2 + E^*a_3^3 \\ Fa_3^3 & F^*a_3^3 \end{pmatrix},$$

where C, E, F, C^*, E^*, F^* are functions depending only on $\epsilon_1, \epsilon_2, \epsilon_3$, while $a_1^0, a_2^0, a_3^0, a_3^1, a_3^2, a_3^3$ depend on $\epsilon_0, \epsilon_1, \epsilon_3, \epsilon_4$.

Proposition 9. *The above specialization of* $\left(a_j^i\right)$ *solves* (9)*.*

Proof. (9) consists of three sets of equations

$$\begin{aligned} a_i^3\frac{\partial a_k^3}{\partial\epsilon_0} &= a_k^3\frac{\partial a_i^3}{\partial\epsilon_0},\\ a_i^2\frac{\partial a_k^3}{\partial\epsilon_0} + a_i^3\frac{\partial a_k^2}{\partial\epsilon_0} &= a_k^2\frac{\partial a_i^3}{\partial\epsilon_0} + a_k^3\frac{\partial a_i^2}{\partial\epsilon_0},\\ a_i^1\frac{\partial a_k^3}{\partial\epsilon_0} + a_i^2\frac{\partial a_k^2}{\partial\epsilon_0} + a_i^3\frac{\partial a_k^1}{\partial\epsilon_0} &= a_k^1\frac{\partial a_i^3}{\partial\epsilon_0} + a_k^2\frac{\partial a_i^2}{\partial\epsilon_0} + a_k^3\frac{\partial a_i^1}{\partial\epsilon_0}, \end{aligned} \tag{26}$$

with $(i,k) = (1,2),(1,3),(2,3)$. Set $k=3$ in the first equation of (26). We first assume a_3^3 is nowhere vanishing. Then it is readily seen that $\partial(a_i^3/a_3^3)/\partial\epsilon_0 = 0$. That is,

$$\begin{aligned} a_1^3 &= Fa_3^3,\\ a_2^3 &= F^*a_3^3, \end{aligned} \tag{27}$$

for some F and F^* depending only on $\epsilon_1, \epsilon_2, \epsilon_3$. Conversely, it is straightforward to check that (27) solves the first equation of (26).

With (27), we obtain in the second equation of (26), for $(k,i) = (1,2)$,

$$Fa_2^2\frac{\partial a_3^3}{\partial\epsilon_0} + F^*a_3^3\frac{\partial a_1^2}{\partial\epsilon_0} = F^*a_1^2\frac{\partial a_3^3}{\partial\epsilon_0} + Fa_3^3\frac{\partial a_2^2}{\partial\epsilon_0},$$

so that

$$F\frac{\partial(a_2^2/a_3^3)}{\partial\epsilon_0} = F^*\frac{\partial(a_1^2/a_3^3)}{\partial\epsilon_0}$$

if a_3^3 is nowhere vanishing. It follows that

$$Fa_2^2 = F^*a_1^2 + Ia_3^3$$

with I independent of ϵ_0. Likewise, for $(k,i) = (1,3)$ and $(2,3)$, we have

$$Fa_3^2 = a_1^2 - Ea_3^3 \tag{28}$$

and

$$F^*a_3^2 = a_2^2 - E^*a_3^3 \tag{29}$$

for some E and E^* not depending on ϵ_0. We solve (28) and (29) to get the desired a_1^2 and a_2^2, which in turn yields

$$I = E^*F - EF^*.$$

For $(k,i) = (1,2)$, the third equation of (26) is

$$a_2^1\frac{\partial a_1^3}{\partial\epsilon_0} + a_2^2\frac{\partial a_1^2}{\partial\epsilon_0} + a_2^3\frac{\partial a_1^1}{\partial\epsilon_0} = a_1^1\frac{\partial a_2^3}{\partial\epsilon_0} + a_1^2\frac{\partial a_2^2}{\partial\epsilon_0} + a_1^3\frac{\partial a_2^1}{\partial\epsilon_0},$$

which is

$$F\frac{\partial(a_2^1/a_3^3)}{\partial\epsilon_0} = F^*\frac{\partial(a_1^1/a_3^3)}{\partial\epsilon_0} - (EF^* - E^*F)\frac{\partial(a_3^2/a_3^3)}{\partial\epsilon_0},$$

so that

$$Fa_2^1 = F^*a_1^1 - (EF^* - E^*F)a_3^2 + Ja_3^3 \tag{30}$$

for some J independent of ϵ_0.

Likewise, for $(k,i) = (2,3)$ we obtain

$$a_3^1\frac{\partial a_2^3}{\partial\epsilon_0} + a_3^2\frac{\partial a_2^2}{\partial\epsilon_0} + a_3^3\frac{\partial a_2^1}{\partial\epsilon_0} = a_2^1\frac{\partial a_3^3}{\partial\epsilon_0} + a_2^2\frac{\partial a_3^2}{\partial\epsilon_0} + a_2^3\frac{\partial a_3^1}{\partial\epsilon_0},$$

which is

$$F^*\frac{\partial(a_3^1/a_3^3)}{\partial\epsilon_0} = \frac{\partial(a_2^1/a_3^3)}{\partial\epsilon_0} - E^*\frac{\partial(a_3^2/a_3^3)}{\partial\epsilon_0},$$

so that

$$F^*a_3^1 = a_2^1 - E^*a_3^2 - C^*a_3^3 \tag{31}$$

for some C^* independent of ϵ_0.

Lastly for $(k,i) = (1,3)$, we have

$$a_3^1\frac{\partial a_1^3}{\partial\epsilon_0} + a_3^2\frac{\partial a_1^2}{\partial\epsilon_0} + a_3^3\frac{\partial a_1^1}{\partial\epsilon_0} = a_1^1\frac{\partial a_3^3}{\partial\epsilon_0} + a_1^2\frac{\partial a_3^2}{\partial\epsilon_0} + a_1^3\frac{\partial a_3^1}{\partial\epsilon_0},$$

which is

$$F\frac{\partial(a_3^1/a_3^3)}{\partial\epsilon_0}=\frac{\partial(a_1^1/a_3^3)}{\partial\epsilon_0}-E\frac{\partial(a_3^2/a_3^3)}{\partial\epsilon_0},$$

so that

$$Fa_3^1=a_1^1-Ea_3^2-Ca_3^3 \tag{32}$$

for some C not depending on ϵ_0.

From (31) and (32), we can solve for a_1^1 and a_2^1, which, when substituted into (30), yields

$$J=C^*F-CF^*.$$

Conversely, it is easy to check that the specified a_j^i solve (9). □

4.2. **A second reduction.** With the above specialized $\left(a_j^i\right)$, we are now left with solving (8) to come up with smooth torsion-free G_3-connections. To this end we introduce a further reduction. We set $a_3^3=1, a_3^2=0, a_k^0=\epsilon_0 a_{k0}^0, 1\le k\le 3$, and $a_3^1=\epsilon_0 a_{30}^1$, where $a_{10}^0,a_{20}^0,a_{30}^0,a_{30}^1$ are independent of ϵ_0, so that

$$\begin{pmatrix}a_0^0&a_1^0&a_2^0&a_3^0\\a_0^1&a_1^1&a_2^1&a_3^1\\a_0^2&a_1^2&a_2^2&a_3^2\\a_0^3&a_1^3&a_2^3&a_3^3\end{pmatrix}=\begin{pmatrix}1&\epsilon_0a_{10}^0&\epsilon_0a_{20}^0&\epsilon_0a_{30}^0\\0&F\epsilon_0a_{30}^1+C&F^*\epsilon_0a_{30}^1+C^*&\epsilon_0a_{30}^1\\0&E&E^*&0\\0&F&F^*&1\end{pmatrix}.$$

A simple row reduction shows that

$$\det\left(a_j^i\right)=E^*C-EC^*.$$

We now analyze (8).

I: $s+t=\alpha=0$ in (8). This gives

$$a_i^0\frac{\partial a_k^0}{\partial\epsilon_0}-a_k^0\frac{\partial a_i^0}{\partial\epsilon_0}=\frac{\partial a_k^0}{\partial\epsilon_i}-\frac{\partial a_i^0}{\partial\epsilon_k}.$$

(i): $(i,k)=(2,3)$. We obtain

$$\frac{\partial a_{30}^0}{\partial\epsilon_2}=\frac{\partial a_{20}^0}{\partial\epsilon_3}. \tag{33}$$

(ii): $(i,k)=(1,3)$. We obtain

$$\frac{\partial a_{30}^0}{\partial\epsilon_1}=\frac{\partial a_{10}^0}{\partial\epsilon_3}. \tag{34}$$

(iii): $(i,k)=(1,2)$. We obtain

$$\frac{\partial a_{20}^0}{\partial\epsilon_1}=\frac{\partial a_{10}^0}{\partial\epsilon_2}. \tag{35}$$

II: $s+t=\alpha=1$. We have

$$a_i^0\frac{\partial a_k^1}{\partial\epsilon_0}+a_i^1\frac{\partial a_k^0}{\partial\epsilon_0}-a_k^0\frac{\partial a_i^1}{\partial\epsilon_0}-a_k^1\frac{\partial a_i^0}{\partial\epsilon_0}=\frac{\partial a_k^1}{\partial\epsilon_i}-\frac{\partial a_i^1}{\partial\epsilon_k}.$$

(i): $(i,k)=(1,2)$. We obtain

$$\frac{\partial C}{\partial\epsilon_2}+a_{20}^0C-\frac{\partial C^*}{\partial\epsilon_1}-a_{10}^0C^*=\epsilon_0(\frac{\partial F^*a_{30}^1}{\partial\epsilon_1}-\frac{\partial Fa_{30}^1}{\partial\epsilon_2}),$$

so that

$$\frac{\partial C}{\partial\epsilon_2}+a_{20}^0C-\frac{\partial C^*}{\partial\epsilon_1}-a_{10}^0C^*=0, \tag{36}$$

and

$$\frac{\partial F^*a_{30}^1}{\partial\epsilon_1}-\frac{\partial Fa_{30}^1}{\partial\epsilon_2}=0. \tag{37}$$

(ii): $(i,k)=(1,3)$. We obtain

$$\frac{\partial C}{\partial\epsilon_3}+a_{30}^0C=\epsilon_0(\frac{\partial a_{30}^1}{\partial\epsilon_1}-\frac{\partial Fa_{30}^1}{\partial\epsilon_3}),$$

so that

$$\frac{\partial C}{\partial\epsilon_3}+a_{30}^0C=0, \tag{38}$$

and

$$\frac{\partial a_{30}^1}{\partial\epsilon_1}-\frac{\partial Fa_{30}^1}{\partial\epsilon_3}=0. \tag{39}$$

(iii): $(i,k)=(2,3)$. We obtain

$$\frac{\partial C^*}{\partial\epsilon_3}+a_{30}^0C^*=\epsilon_0(\frac{\partial a_{30}^1}{\partial\epsilon_2}-\frac{\partial F^*a_{30}^1}{\partial\epsilon_3}),$$

so that

$$\frac{\partial C^*}{\partial\epsilon_3}+a_{30}^0C^*=0, \tag{40}$$

and

$$\frac{\partial a_{30}^1}{\partial\epsilon_2}-\frac{\partial F^*a_{30}^1}{\partial\epsilon_3}=0. \tag{41}$$

III: $s+t=\alpha=2$. It yields

$$a_i^1\frac{\partial a_k^1}{\partial\epsilon_0}+a_i^2\frac{\partial a_k^0}{\partial\epsilon_0}-a_k^1\frac{\partial a_i^1}{\partial\epsilon_0}-a_k^2\frac{\partial a_i^0}{\partial\epsilon_0}=\frac{\partial a_k^2}{\partial\epsilon_i}-\frac{\partial a_i^2}{\partial\epsilon_k}.$$

(i): $(i,k)=(1,3)$. It results in

$$\frac{\partial E}{\partial\epsilon_3}+a_{30}^0E=-Ca_{30}^1. \tag{42}$$

(ii): $(i,k)=(2,3)$. It results in

$$\frac{\partial E^*}{\partial \epsilon_3}+a_{30}^0E^*=-C^*a_{30}^1. \tag{43}$$

(iii): $(i,k)=(1,2)$. It results in

$$\frac{\partial E^*}{\partial \epsilon_1}-\frac{\partial E}{\partial \epsilon_2}+a_{10}^0E^*-a_{20}^0E=(F^*C-FC^*)a_{30}^1. \tag{44}$$

IV: $s+t=\alpha=3$. We get

$$a_i^2\frac{\partial a_k^1}{\partial \epsilon_0}+a_i^3\frac{\partial a_k^0}{\partial \epsilon_0}-a_k^2\frac{\partial a_i^1}{\partial \epsilon_0}-a_k^3\frac{\partial a_i^0}{\partial \epsilon_0}=\frac{\partial a_k^3}{\partial \epsilon_i}-\frac{\partial a_i^3}{\partial \epsilon_k}.$$

(i): $(i,k)=(1,3)$. It is

$$\frac{\partial F}{\partial \epsilon_3}+a_{30}^0F-a_{10}^0=-Ea_{30}^1. \tag{45}$$

(ii): $(i,k)=(2,3)$. It is

$$\frac{\partial F^*}{\partial \epsilon_3}+a_{30}^0F^*-a_{20}^0=-E^*a_{30}^1. \tag{46}$$

(iii): $(i,k)=(1,2)$. It is

$$\frac{\partial F^*}{\partial \epsilon_1}-\frac{\partial F}{\partial \epsilon_2}+a_{10}^0F^*-a_{20}^0F=(EF^*-E^*F)a_{30}^1. \tag{47}$$

4.3. **A third reduction and a class of solutions.** Let us now set

$$\begin{aligned} C&=0,\\ C^*&=-2e^{-\phi},\\ a_{i0}^0&=\frac{\partial \phi}{\partial \epsilon_i}, \end{aligned}$$

with $i=1,2,3$ for some smooth function ϕ, depending only on $\epsilon_1,\epsilon_2,\epsilon_3$, to be chosen later. Then it is readily seen that (33) through (35), and (36), (38) and (40) are satisfied.

Next, in view of (42) and (43), we choose

$$\begin{aligned} E&=\gamma e^{-\phi},\\ E^*&=2e^{-\phi}M, \end{aligned}$$

for some smooth function γ of ϵ_2 alone, where

$$M=\int a_{30}^1 d\epsilon_3.$$

Then E and E^* solve (42) and (43).

We now impose a further condition on ϕ so that ϕ is a function of ϵ_3 alone. Then $a_{10}^0 = a_{20}^0 = 0$ and we choose

$$F = -\gamma e^{-\phi} M,$$
$$F^* = -e^{-\phi} M^2 + f e^{-\phi},$$

for some smooth function f of ϵ_2 alone to be chosen later. Then F and F^* solve (45) and (46).

In general, substituting a_{10}^0 and a_{20}^0, in terms of the remaining terms of (45) and (46), into (44) and (47), and replacing C and C^* there via (42) and (43), we end up with, respectively,

$$F^* \frac{\partial E}{\partial \epsilon_3} - F \frac{\partial E^*}{\partial \epsilon_3} + E^* \frac{\partial F}{\partial \epsilon_3} - E \frac{\partial F^*}{\partial \epsilon_3} + \frac{\partial E^*}{\partial \epsilon_1} - \frac{\partial E}{\partial \epsilon_2} = 0, \tag{48}$$

and

$$F^* \frac{\partial F}{\partial \epsilon_3} - F \frac{\partial F^*}{\partial \epsilon_3} + \frac{\partial F^*}{\partial \epsilon_1} - \frac{\partial F}{\partial \epsilon_2} = 0. \tag{49}$$

We then observe that (37) follows from (39), (41) and (49), because $\partial a_{30}^1/\partial \epsilon_1$ and $\partial a_{30}^1/\partial \epsilon_2$, in terms of the remaining terms of (39) and (41), verify (37) by (49).

We are now only left with (39), (41), (48), (49). We impose the condition that M is only a function of ϵ_2 and ϵ_3 alone. For notational convenience, we set

$$N := e^{\phi}.$$

Then (48) comes down to

$$2\gamma M \frac{\partial M}{\partial \epsilon_3} - N \frac{d\gamma}{d\epsilon_2} = 0, \tag{50}$$

whereas (49) is reduced to

$$-\gamma M^2 \frac{\partial M}{\partial \epsilon_3} - f\gamma \frac{\partial M}{\partial \epsilon_3} + \gamma N \frac{\partial M}{\partial \epsilon_2} + MN \frac{d\gamma}{d\epsilon_2} = 0. \tag{51}$$

We now set

$$\gamma = \epsilon_2.$$

(In fact, $d\gamma/d\epsilon_2 = 1$ is necessary and sufficient.) Then (50) gives

$$N = \epsilon_2 \frac{\partial M^2}{\partial \epsilon_3}, \tag{52}$$

which is then substituted into (51) to yield

$$(2\epsilon_2 M \frac{\partial M}{\partial \epsilon_2} + M^2 - f) \frac{\partial M}{\partial \epsilon_3} = 0,$$

for which we choose to solve

$$2\epsilon_2 M\frac{\partial M}{\partial \epsilon_2}+M^2-f=0.$$

We obtain

$$M^2=(\epsilon_2)^{-1}\tau+(\epsilon_2)^{-1}\int f d\epsilon_2,$$

for an arbitrary smooth function τ of ϵ_3 alone. Inserting M^2 into (52) we see

$$N=\frac{d\tau}{d\epsilon_3}.$$

Accordingly, we may now set

$$\tau=\int e^{\phi}d\epsilon_3$$

for an arbitrarily chosen ϕ of ϵ_3 alone. Now we choose

$$f=e^{\psi}$$

for an arbitrary smooth function ψ of ϵ_2 only. Since

$$a^1_{30}=\frac{\partial M}{\partial \epsilon_3},$$

we obtain, when we restrict to positive ϵ_2,

$$a^1_{30}=\frac{e^{\phi}}{2\sqrt{\epsilon_2(\int e^{\psi}d\epsilon_2+\int e^{\phi}d\epsilon_3)}}.$$

Lastly, a straightforward calculation shows that

$$\begin{aligned}F&=-(2a^1_{30})^{-1},\\ F^*&=(a^1_{30})^{-1}\int\frac{\partial a^1_{30}}{\partial \epsilon_2}d\epsilon_3=(a^1_{30})^{-1}\frac{\partial M}{\partial \epsilon_2},\end{aligned}$$

so that F and F^* satisfy the differential equations

$$\begin{aligned}\frac{\partial F}{\partial \epsilon_3}+\frac{\partial\log a^1_{30}}{\partial \epsilon_3}F&=\frac{\partial\log a^1_{30}}{\partial \epsilon_1},\\ \frac{\partial F^*}{\partial \epsilon_3}+\frac{\partial\log a^1_{30}}{\partial \epsilon_3}F^*&=\frac{\partial\log a^1_{30}}{\partial \epsilon_2},\end{aligned}$$

respectively, which are exactly (39) and (41).

In conclusion, we have found a class of solutions. Namely, $a^i_0=\delta^i_0, 0\le i\le 3, a^0_j=\delta^3_j\epsilon_0\phi', 1\le j\le 3$. Moreover, $a^3_3=1, a^2_3=0$,

$a^1_3 = \epsilon_0 e^{\phi}/(2\sqrt{\epsilon_2}\sqrt{R})$, and

$$\begin{pmatrix} a^1_1 & a^1_2 \\ a^2_1 & a^2_2 \\ a^3_1 & a^3_2 \end{pmatrix} =$$

$$\begin{pmatrix} -\epsilon_0/2 & -2e^{-\phi} + \epsilon_0 e^{\psi}/(2\sqrt{\epsilon_2}\sqrt{R}) - \epsilon_0\sqrt{R}/(2(\sqrt{\epsilon_2})^3) \\ \epsilon_2 e^{-\phi} & 2e^{-\phi}\sqrt{R}/\sqrt{\epsilon_2} \\ -\sqrt{\epsilon_2}e^{-\phi}\sqrt{R} & e^{\psi-\phi} - e^{-\phi}R/\epsilon_2 \end{pmatrix},$$

where ψ and ϕ are arbitrary smooth functions of ϵ_2 and ϵ_3, respectively, ϕ' denotes the derivative of ϕ, and

$$R = \int e^{\psi} d\epsilon_2 + \int e^{\phi} d\epsilon_3. \tag{53}$$

This class gives rise to torsion-free G_3-connections by Theorem 8.

Remark 10. We can now exhibit in full the solution to (4). We set $\zeta_i = x^i, 0 \le i \le 3$. Then we have

$$r = e^{\phi - \epsilon_1 x/2 + \sqrt{R}x/\sqrt{\epsilon_2}},$$

and

$$\begin{aligned} y &= r(\epsilon_0 - 2a^2_1 x - 2a^3_1 x^2) \\ &= e^{\phi - \epsilon_1 x/2 + \sqrt{R}x/\sqrt{\epsilon_2}}(\epsilon_0 - 2\epsilon_2 e^{-\phi}x + 2\sqrt{\epsilon_2}e^{-\phi}\sqrt{R}x^2), \end{aligned}$$

which verify, in clear retrospect, the correctness of our solution to the PDEs encountered in this section.

5. Existence of torsion-free G_3-connections that are not analytic

5.1. **The holonomy group of the constructed class of connections is G_3.**

We have seen in Proposition 7 that (25) yields

$$\Omega^3_{00} = \Omega^3_{01} = \Omega^3_{02} = \Omega^2_{00} = \Omega^2_{01} = \Omega^1_{00} = 0.$$

Now with the matrix $\left(a^i_j\right)$ specified at the end of the preceding section, we have

$$\Omega^1_{01} = \Omega^2_{02} = \Omega^3_{03} = 0;$$

this follows from the identity

$$\Omega^3_{03}a^3_i = \frac{\partial a^3_i}{\partial \epsilon_0},$$

which is a variation of the third identity in (25), and the fact that a^3_i in our specified data are independent of ϵ_0. Likewise,

$$\Omega^1_{02} = \Omega^2_{03} = 0$$

because of the identities

$$\Omega_{0s}^3 a_i^1 + \Omega_{0s}^2 a_i^3 + \Omega_{0s}^1 a_i^3 = \frac{\partial a_i^3}{\partial \epsilon_0}\delta_s^1 + \frac{\partial a_i^2}{\partial \epsilon_0}\delta_s^2 + \frac{\partial a_i^1}{\partial \epsilon_0}\delta_s^3,$$

and

$$\Omega_{0s}^3 a_i^2 + \Omega_{0s}^2 a_i^3 = \frac{\partial a_i^3}{\partial \epsilon_0}\delta_s^2 + \frac{\partial a_i^2}{\partial \epsilon_0}\delta_s^3,$$

which are variations of the first and the second identities of (25).

As a consequence, the connection matrix form ω in Theorem 8 is reduced to

$$\omega = \begin{pmatrix} -\Omega_0^0 & \frac{1}{9}\Omega_{03}^1\theta^1 & 0 & 0 \\ -\Omega_{03}^1\theta^3 & -\Omega_0^0 + \frac{2}{9}\Omega_{03}^1\theta^2 & \frac{2}{9}\Omega_{03}^1\theta^1 & 0 \\ 0 & -\frac{2}{3}\Omega_{03}^1\theta^3 & -\Omega_0^0 + \frac{4}{9}\Omega_{03}^1\theta^2 & \frac{1}{3}\Omega_{03}^1\theta^1 \\ 0 & 0 & -\frac{1}{3}\Omega_{03}^1\theta^3 & -\Omega_0^0 + \frac{2}{3}\Omega_{03}^1\theta^2 \end{pmatrix}.$$

Let

$$\Theta := d\omega + \omega \wedge \omega$$

be the curvature form of the torsion-free G_3-connection associated with our specified (a_j^i). Then we calculate to see

$$tr(\Theta) = \frac{4}{3}d(a_{30}^1\theta^2) = -\frac{4}{3}(a_{30}^1)^2\theta^1 \wedge \theta^3 + \cdots \neq 0,$$

where we only display the $\theta^1 \wedge \theta^3$ term. Hence the holonomy of this class of connections cannot be reduced to H_3, and so must be G_3.

5.2. **Infinitesimal symmetries of the class.** Since the a_j^i specified do not depend on ϵ_1, the translations along ϵ_1 clearly form a 1-parameter family of symmetries. It turns out something more remarkable is true. In the following we impose the maximal domain, which is the half Euclidean space $\epsilon_2 > 0$.

Proposition 11. *The space of infinitesimal symmetries of a connection in the class is* 4*-dimensional. All these vector fields are tangent to the hypersurface*

$$\epsilon_0 = \frac{\epsilon_2^{3/2}e^{-\phi}}{2\sqrt{\int e^{\psi}d\epsilon_2 + \int e^{\phi}d\epsilon_3}}.$$

As a consequence, the connection is not locally homogeneous as a whole, though it is so away from the hypersurface.

Proof. The proof is fairly long and some symbolic calculations are employed along the way to facilitate the computations.

Let $X := \sum_{i=0}^{3} \chi^i X_i$ be a vector field. Define χ^i_j by

$$\sum_{j=0}^{3} \chi^i_j \theta^j := d\chi^i + \sum_{j=0}^{3} \omega^i_j \chi^j. \tag{54}$$

The matrix (χ^i_j) belongs to $\mathcal{G}_3$. This follows from the fact that the vertical component of X^*, the natural lift of X to the principal G_3-bundle of the connection, at the point (p, g) relative to the trivialization $(X_0, \cdots, X_3)$, where p is the base point and $g \in G_3$, is exactly

$$g^{-1}(\chi^i_j)g.$$

X is an infinitesimal symmetry if and only if [15]

$$d\chi^i_j + \sum_s \chi^s_j \omega^i_s - \sum_s \chi^i_s \omega^s_j = -\iota_X \Theta^i_j. \tag{55}$$

Set $R^i_{jkl} := \Theta^i_j(X_k, X_l)$ and $\omega^i_{jk} := \omega^i_j(X_k)$. Taking exterior derivative of the preceding equation, one ends up with

$$\begin{aligned} 0 = &\sum_s (\chi^i_s R^s_{jkl} - \chi^s_j R^i_{skl} + \chi^s_l R^i_{jsk} - \chi^s_k R^i_{jsl}) \\ &+ \sum_{s\alpha} (\chi^\alpha R^s_{j\alpha k}\omega^i_{sl} - \chi^\alpha R^s_{j\alpha l}\omega^i_{sk} + \chi^\alpha R^i_{s\alpha l}\omega^s_{jk} - \chi^\alpha R^i_{s\alpha k}\omega^s_{jl}) \\ &+ \sum_{s\alpha} (\chi^s R^i_{j\alpha l}\omega^\alpha_{sk} - \chi^s R^i_{j\alpha k}\omega^\alpha_{sl} + \chi^s R^i_{js\alpha}\omega^\alpha_{lk} - \chi^s R^i_{js\alpha}\omega^\alpha_{kl}) \\ &+ \sum_s \chi^s (dR^i_{jsk}(X_l) - dR^i_{jsl}(X_k)). \end{aligned} \tag{56}$$

For $(i, j) = (0, 1)$ and $(k, l) = (0, 2)$, the preceding identity yields

$$\chi^1_0 = a^1_{30}\chi^3, \tag{57}$$

and therefore,

$$\begin{aligned} \chi^2_1 &= \frac{2}{3} a^1_{30}\chi^3, \\ \chi^3_2 &= \frac{1}{3} a^1_{30}\chi^3, \end{aligned}$$

since (χ^i_j) belongs to $\mathcal{G}_3$. With (57) in place, we derive, for $(i, j) = (3, 2)$ and $(k, l) = (2, 3)$,

$$\chi^0_0 + 2\chi^3_3 = \frac{e^\phi(5\chi^2 R + 9\chi^3 \epsilon_2^{1/2} R^{1/2})}{6\epsilon_2^{1/2} R^{3/2}}. \tag{58}$$

Likewise,

$$\chi_0^0 + \frac{6R^{1/2}}{\epsilon_2^{1/2}}\chi_1^0 + 5\chi_3^3 = \frac{14\chi^1\epsilon_2^{1/2}R^{3/2} + 34\chi^2\epsilon_2 R - 9\chi^3\epsilon_0 e^{\phi}R + 54\chi^3\epsilon_2^{3/2}R^{1/2}}{12e^{-\phi}\epsilon_2^{3/2}R^{3/2}} \tag{59}$$

when $(i,j) = (3,3)$ and $(k,l) = (2,3)$. Here R is given in (53). (In fact, (57) through (59) are the only nontrivial equations out of (56).) We can now solve (58) and (59) for χ_0^0 and χ_1^0 in terms of χ_3^3 and χ^1, χ^2, χ^3. As a result of the fact that (χ_j^i) belongs to $\mathcal{G}_3$, all the χ_j^i thus have been determined by χ_3^3 and χ^1, χ^2, χ^3. Differentiating (59) with respect to X_3 and keeping (54) and (55) in mind, we obtain

$$\chi_3^3 = \frac{P\chi^0 + Q\chi^1 + U\chi^2 + V\chi^3}{12e^{-\phi}R(\epsilon_2^2 - 2e^{\phi}\epsilon_0\epsilon_2^{1/2}R^{1/2})}, \tag{60}$$

where $P := 12\epsilon_2^{1/2}R^{3/2}, Q := 6\epsilon_2 R, U := -10\epsilon_0 Re^{\phi} + 8\epsilon_2^{3/2}R^{1/2}$ and $V := 12\epsilon_2^2 - 21e^{\phi}\epsilon_0\epsilon_2^{1/2}R^{1/2}$. Hence, all χ_j^i are now in terms of $\chi^0, \cdots, \chi^3$, which, when incorporated with (54), enables one to derive differential equations of the form

$$\frac{\partial \chi^i}{\partial \epsilon_j} = F_j^i, \tag{61}$$

where F_j^i are certain complicated functions of $\epsilon_0, \cdots, \epsilon_3$ and $\chi^0, \cdots, \chi^3$. (We shall not exhibit explicitly all of F_j^i.) Among these equations we have

$$\frac{\partial \chi^0}{\partial \epsilon_0} = \frac{\chi^0}{\epsilon_0 - s} + \frac{\sqrt{\epsilon_2}\chi^1}{2\sqrt{R}(\epsilon_0 - s)} + \frac{\epsilon_2\chi^2}{4R(\epsilon_0 - s)} + \frac{\epsilon_2^{3/2}\chi^3}{4R^{3/2}(\epsilon_0 - s)} - \frac{\epsilon_0 e^{\phi}\chi^3}{4R(\epsilon_0 - s)}, \tag{62}$$

where

$$s := \frac{\epsilon_2^{3/2}e^{-\phi}}{2\sqrt{R}},$$

and

$$(63)\qquad \begin{aligned} \frac{\partial \chi^1}{\partial \epsilon_0} &= \frac{e^{\phi}\chi^3}{2\sqrt{\epsilon_2}\sqrt{R}} \\ \frac{\partial \chi^2}{\partial \epsilon_0} &= 0 \\ \frac{\partial \chi^3}{\partial \epsilon_0} &= 0. \end{aligned}$$

χ^2 and χ^3 are independent of ϵ_0 in view of the second and third equations of (63), so that one can solve the first equation of (63) to see

$$(64)\qquad \chi^1 = \frac{\epsilon_0 e^{\phi}\chi^3}{2\sqrt{\epsilon_2}\sqrt{R}} + K$$

for some K depending only on $\epsilon^1, \epsilon^2, \epsilon^3$. Inserting (64) into (62), one obtains

$$(65)\qquad \frac{\partial \chi^0}{\partial \epsilon_0} = \frac{\chi^0}{\epsilon_0 - s} + \frac{\sqrt{\epsilon_2}K}{2\sqrt{R}(\epsilon_0 - s)} + \frac{\epsilon_2\chi^2}{4R(\epsilon_0 - s)} + \frac{\epsilon_2^{3/2}\chi^3}{4R^{3/2}(\epsilon_0 - s)},$$

where s, K, R, χ^2, χ^3 are all independent of ϵ_0. Hence we can easily solve (65), with (64) in hand, to derive

$$(66)\qquad \chi^0 + \frac{\epsilon_2^{1/2}}{2R^{1/2}}\chi^1 + \frac{\epsilon_2}{4R}\chi^2 + \left(\frac{\epsilon_2^{3/2}}{4R^{3/2}} - \frac{\epsilon_0 e^{\phi}}{4R}\right)\chi^3 = (\epsilon_0 - s)A,$$

for some A independent of ϵ_0. In particular, over the hypersurface

$$(67)\qquad \epsilon_0 = \frac{\epsilon_2^{3/2}e^{-\phi}}{2\sqrt{R}},$$

we have

$$\chi^0 + \frac{\epsilon_2^{1/2}}{2R^{1/2}}\chi^1 + \frac{\epsilon_2}{4R}\chi^2 + \left(\frac{\epsilon_2^{3/2}}{4R^{3/2}} - \frac{\epsilon_0 e^{\phi}}{4R}\right)\chi^3 = 0,$$

which says exactly that the infinitesimal symmetry $X = \sum_s \chi^s X_s$ is tangent to the hypersurface. Meanwhile, (64) and (66) assert that

$$(68)\qquad \chi^0 = A\epsilon_0 + B,$$

where

$$\frac{\epsilon_2^{1/2}}{2R^{1/2}}\chi^1 + \frac{\epsilon_2}{4R}\chi^2 + \left(\frac{\epsilon_2^{3/2}}{4R^{3/2}} - \frac{\epsilon_0 e^{\phi}}{4R}\right)\chi^3 = -As - B,$$

so that

$$(69)\qquad \chi^1 = -\frac{\sqrt{\epsilon_2}}{2\sqrt{R}}\chi^2 - \left(\frac{\epsilon_2}{2R} - \frac{\epsilon_0 e^{\phi}}{2\sqrt{R}\sqrt{\epsilon_2}}\right)\chi^3 - \frac{2\sqrt{R}}{\sqrt{\epsilon_2}}(As + B).$$

Substituting (68) and (69) into $\partial\chi^0/\partial\epsilon_1$ in (61), one derives

$$\frac{\partial B}{\partial\epsilon_1} = 0,$$
$$\frac{\partial A}{\partial\epsilon_1} = \frac{Be^{\phi}}{4\epsilon_2},$$

so that

$$\chi^0 = (\frac{Be^{\phi}\epsilon_1}{4\epsilon_2} + C)\epsilon_0 + B,$$

where B and C are functions of ϵ_2 and ϵ_3 alone. Likewise, substituting the new χ^0 and χ^1 into $\partial\chi^0/\partial\epsilon_2$ and $\partial\chi^0/\partial\epsilon_3$ in (61), we obtain, respectively,

$$\frac{\partial B}{\partial\epsilon_2} = \frac{B}{\epsilon_2},$$
$$\frac{\partial B}{\partial\epsilon_3} = -\frac{d\phi}{d\epsilon_3}B,$$

so that

$$B = \gamma e^{-\phi}\epsilon_2 \tag{70}$$

for some constant γ, and so

$$\chi^0 = (\frac{\gamma\epsilon_1}{4} + C)\epsilon_0 + \gamma e^{-\phi}\epsilon_2.$$

Repeating the same process, one inserts again the new χ^0 and χ^1 into $\partial\chi^0/\partial\epsilon_2$ and $\partial\chi^0/\partial\epsilon_3$ in (61) to obtain, respectively,

$$\frac{\partial C}{\partial\epsilon_2} = \frac{\gamma\sqrt{R}}{4\epsilon_2^{3/2}} - \frac{\gamma e^{\psi}}{4\sqrt{R}\sqrt{\epsilon_2}},$$
$$\frac{\partial C}{\partial\epsilon_3} = -\frac{\gamma e^{\phi}}{4\sqrt{\epsilon_2}\sqrt{R}},$$

from which one solves to see

$$C = \beta - \frac{\gamma\sqrt{R}}{2\sqrt{\epsilon_2}} \tag{71}$$

for some constant β, so that now

$$\chi^0 = (\frac{\gamma\epsilon_1}{4} + \beta - \frac{\gamma\sqrt{R}}{2\sqrt{\epsilon_2}})\epsilon_0 + \gamma e^{-\phi}\epsilon_2. \tag{72}$$

Consider now the function

$$D := \sqrt{R}\chi^2 + \sqrt{\epsilon_2}\chi^3 \tag{73}$$

that depends only on $\epsilon_1, \epsilon_2, \epsilon_3$. A calculation with the new χ^0 and χ^1 gives

$$\frac{\partial D}{\partial \epsilon_1} = \frac{\gamma R\sqrt{\epsilon_2}e^{-\phi}}{2},$$
$$\frac{\partial D}{\partial \epsilon_2} = (\frac{1}{2\epsilon_2} + \frac{e^{\psi}}{R})D - \frac{\gamma e^{-\phi}(\epsilon_2 Re^{\psi} - R^2)}{2\epsilon_2\sqrt{R}},$$
$$\frac{\partial D}{\partial \epsilon_3} = (\frac{e^{\phi}}{R} - \frac{d\phi}{d\epsilon_3})D - \frac{\gamma\sqrt{R}}{2}.$$

We then solve the last two equations to come up with

$$D = E\sqrt{\epsilon_2}Re^{-\phi} - \gamma e^{-\phi}R^{3/2} \tag{74}$$

for some E depending on ϵ_1 alone. The first equation finally gives

$$E = \frac{\gamma\epsilon_1}{2} + \alpha$$

for some constant α. Now χ^3 can be expressed in terms of χ^2 in view of (73) and (74). A calculation shows

$$\frac{\partial \chi^2}{\partial \epsilon_1} = \frac{\sqrt{\epsilon_2}e^{-\phi}(\alpha\sqrt{\epsilon_2}\sqrt{R} - 2\beta\sqrt{\epsilon_2}\sqrt{R} + 2\gamma R)}{4\sqrt{R}}.$$

As a consequence

$$\chi^2 = \frac{\sqrt{\epsilon_2}e^{-\phi}(\alpha\sqrt{\epsilon_2}\sqrt{R} - 2\beta\sqrt{\epsilon_2}\sqrt{R} + 2\gamma R)\epsilon_1}{4\sqrt{R}} + F$$

for some F dependent only on ϵ_2 and ϵ_3. Substituting χ^2 into $\partial\chi^2/\partial\epsilon_2$ and $\partial\chi^2/\partial\epsilon_3$ we obtain

$$\frac{\partial F}{\partial \epsilon_2} = \frac{1}{\epsilon_2}F + \frac{(2\beta+\alpha)\sqrt{\epsilon_2}e^{\psi-\phi}}{4\sqrt{R}} - \frac{(2\beta+\alpha)\sqrt{R}e^{-\phi}}{4\sqrt{\epsilon_2}} + \frac{\gamma Re^{-\phi}}{\epsilon_2} - \gamma e^{\psi-\phi},$$
$$\frac{\partial F}{\partial \epsilon_3} = -\frac{d\phi}{d\epsilon_3}F + \frac{(2\beta+\alpha)\sqrt{\epsilon_2}}{4\sqrt{R}} - \gamma.$$

We solve the second equation to get

$$F = \frac{2\beta+\alpha}{2}\sqrt{\epsilon_2}\sqrt{R}e^{-\phi} + e^{-\phi}G - \gamma e^{-\phi}\int e^{\phi}d\epsilon_3$$

for some function depending only on ϵ_2, which, when substituted into the first equation, yields

$$\frac{dG}{d\epsilon_2} = \frac{1}{\epsilon_2}G - \gamma e^{\psi} + \frac{\gamma}{\epsilon_2}\int e^{\psi}d\epsilon_2,$$

so that

$$G = \delta\epsilon_2 - \gamma\int e^{\psi}d\epsilon_2$$

for some constant δ.

In summary, we have found all the infinitesimal symmetries, which depend on four independent constants. Explicitly,

(75)

$$\begin{aligned}
\chi^0 &= (\frac{\gamma\epsilon_1}{4} + \beta - \frac{\gamma\sqrt{R}}{2\sqrt{\epsilon_2}})\epsilon_0 + \gamma e^{-\phi}\epsilon_2,\\
\chi^1 &= -8^{-1}e^{-\phi}R^{-1/2}\epsilon_2^{-1/2}((4\epsilon_2^{3/2}\sqrt{R} + \epsilon_0\epsilon_1\sqrt{\epsilon_2}\sqrt{R}e^{\phi} - 2\epsilon_0 Re^{\phi})\alpha\\
&\quad + (8\epsilon_2^{3/2}\sqrt{R} + 4\epsilon_0 Re^{\phi} - 2\epsilon_0\epsilon_1\sqrt{\epsilon_2}\sqrt{R}e^{\phi})\beta + (8\epsilon_2 R + 4\epsilon_1\epsilon_2^{3/2}\sqrt{R})\gamma\\
&\quad + 4\epsilon_0\sqrt{\epsilon_2}\sqrt{R}e^{\phi}\delta),\\
\chi^2 &= -4^{-1}e^{-\phi}R^{-1/2}((-2\sqrt{\epsilon_2}R - \epsilon_1\epsilon_2\sqrt{R})\alpha + (2\epsilon_1\epsilon_2\sqrt{R} - 4\sqrt{\epsilon_2}R)\beta\\
&\quad + (4R^{3/2} - 2\epsilon_1\sqrt{\epsilon_2}R)\gamma - 4\epsilon_2\sqrt{R}\delta),\\
\chi^3 &= 4^{-1}e^{-\phi}\epsilon_2^{-1/2}((2\sqrt{\epsilon_2}R - \epsilon_1\epsilon_2\sqrt{R})\alpha + (2\epsilon_1\epsilon_2\sqrt{R} - 4\sqrt{\epsilon_2}R)\beta\\
&\quad - 4\epsilon_2\sqrt{R}\delta).
\end{aligned}$$

Setting one of the constants equal to 1 and the others equal to 0, in the order of $\alpha, \beta, \gamma, \delta$, we get four infinitesimal symmetries, which we shall show to be pointwise linearly dependent precisely on the hypersurface given in (67). To this end, we introduce a new coordinate system $(\eta, \epsilon_1, \epsilon_2, \epsilon_3)$, where

$$\eta = \epsilon_0 - \frac{\epsilon_2^{3/2}e^{-\phi}}{2\sqrt{R}}. \tag{76}$$

Then with respect to the new coordinate vectors $\partial/\partial\eta, \partial/\partial\epsilon_1, \partial/\partial\epsilon_2, \partial/\partial\epsilon_3$, the four infinitesimal symmetries U_0, U_1, U_2, U_3 read

(77)

$$\begin{aligned}
U_0 &= \eta(\epsilon_2 e^{\psi} - 3R)e^{-\phi}\phi'\frac{\partial}{\partial\eta} + \epsilon_1\frac{\partial}{\partial\epsilon_1} + \epsilon_2\frac{\partial}{\partial\epsilon_2} - (\epsilon_2 e^{\psi} - 3R)e^{-\phi}\frac{\partial}{\partial\epsilon_3},\\
U_1 &= \eta(R\phi' + \epsilon_2 e^{\psi}\phi' + 2e^{\phi})e^{-\phi}\frac{\partial}{\partial\eta} - \epsilon_1\frac{\partial}{\partial\epsilon_1} + \epsilon_2\frac{\partial}{\partial\epsilon_2} - (\epsilon_2 e^{\psi} + R)e^{-\phi}\frac{\partial}{\partial\epsilon_3},\\
U_2 &= \eta\epsilon_2^{-1/2}R^{-1/2}e^{-\phi}(6R^2\phi' + \epsilon_1\sqrt{\epsilon_2}\sqrt{R}e^{\phi} - \epsilon_1\sqrt{\epsilon_2}R^{3/2}\phi' + 2\epsilon_2 Re^{\psi}\phi'\\
&\quad - 2Re^{\phi} + \epsilon_1\epsilon_2^{3/2}\sqrt{R}e^{\psi}\phi')\frac{\partial}{\partial\eta} - 8\epsilon_2^{-1}R\frac{\partial}{\partial\epsilon_1} + \sqrt{\epsilon_2}(\epsilon_1\sqrt{\epsilon_2} + 2\sqrt{R})\frac{\partial}{\partial\epsilon_2}\\
&\quad - \epsilon_2^{-1/2}R^{-1/2}e^{-\phi}(2\epsilon_2 Re^{\psi} + \epsilon_1\epsilon_2^{3/2}\sqrt{R}e^{\psi} + 6R^2 - \epsilon_1\sqrt{\epsilon_2}R^{3/2})\frac{\partial}{\partial\epsilon_3},\\
U_3 &= \frac{\partial}{\partial\epsilon_1},
\end{aligned}$$

where ϕ' is the derivative of ϕ. The components of these vectors form a 4-by-4 matrix whose determinant is

$$\eta\sqrt{\epsilon_2}R^{3/2}e^{-\phi},$$

which vanishes precisely on the hypersurface $\eta = 0$, or equivalently, on the one defined in (67).

In conclusion, away from the hypersurface the four pointwise linearly independent infinitesimal symmetries integrate to generate a 4-parameter family of local automorphisms, free of fixed points, that gives rise to a local group manifold structure of the base space with respect to which the torsion-free G_3-connection is left-invariant, although globally the connection is not locally homogeneous, because the infinitesimal symmetries become pointwise linearly dependent over the hypersurface. □

Remark 12. A further calculation shows that

$$[U_0, U_1] = 0, [U_0, U_2] = \frac{1}{4}U_2, [U_0, U_3] = -\frac{1}{4}U_3,$$
$$[U_1, U_2] = -\frac{1}{2}U_2, [U_1, U_3] = \frac{1}{2}U_3, [U_2, U_3] = -\frac{1}{2}U_0 - \frac{1}{4}U_1.$$

If we set

$$E_1 := U_0 + \frac{1}{2}U_1, E_2 := U_0 - \frac{1}{2}U_1, E_3 := U_2, E_4 := U_3,$$

then we see

$$[E_1, E_2] = [E_1, E_3] = [E_1, E_4] = 0,$$
$$[E_2, E_3] = \frac{1}{2}E_3, [E_2, E_4] = -\frac{1}{2}E_4, [E_3, E_4] = -\frac{1}{2}E_1.$$

Hence $E_1, \cdots, E_4$ generate a solvable Lie algebra whose typical element is of the form

$$\begin{pmatrix} 0, a, b \\ 0, c, d \\ 0, 0, 0 \end{pmatrix}.$$

See [16] for the classification of left-invariant torsion-free G_3-connections on the corresponding Lie group.

5.3. **Existence of torsion-free G_3-connections that are not smoothly equivalent to any analytic ones.** Suppose the associated connection is analytic with respect to an analytic structure $\mathcal{A}$. The infinitesimal symmetries will be analytic with respect to $\mathcal{A}$ as well. Since the hypersurface $\eta = 0$ is precisely the zero locus of the determinant of the components of four independent infinitesimal symmetries by the preceding proposition, it follows that the hypersurface $\eta = 0$ is analytic with respect to $\mathcal{A}^*$, the induced analytic structure from $\mathcal{A}$. Therefore, the restriction of the infinitesimal symmetries to $\eta = 0$, in particular of $U_0, \cdots, U_3$ in (77) with $\eta = 0$, are all analytic with respect to $\mathcal{A}^*$, due to the fact they are tangent to the hypersurface in view of the same proposition.

The fourth identity in (77) says that we can introduce ϵ_1 as one of the coordinates of an analytic chart $(\epsilon_1, \mu_2, \mu_3)$ of $\mathcal{A}^*$. The third identity in (77) then implies that R/ϵ_2, which is the $\partial/\partial\epsilon_1$-component in the analytic chart, is analytic with respect to $\mathcal{A}^*$.

Now on the hypersurface $\eta = 0$, we utilize the first, second and the fourth identity to obtain

$$\frac{\partial}{\partial\epsilon_2} = (\frac{1}{\epsilon_2} + \frac{\epsilon_2 e^{\psi} - 3R}{4R\epsilon_2})U_0 - \frac{\epsilon_2 e^{\psi} - 3R}{4R\epsilon_2}U_1 - (\frac{\epsilon_1}{\epsilon_2} + \frac{2\epsilon_1\epsilon_2 e^{\psi} - 6R\epsilon_1}{4R\epsilon_2})U_3,$$
$$\frac{\partial}{\partial\epsilon_3} = \frac{e^{\phi}}{4R}U_0 - \frac{e^{\phi}}{4R}U_1 - \frac{2\epsilon_1 e^{\phi}}{4R}U_3.$$

Inserting this into the third identity in (77), we can express U_2 as a linear combination of U_0, U_1 and U_3, each of whose components are analytic functions in terms of ϵ_1 and R/ϵ_2, as can be easily verified. We are thus left with the first two equations. Consider the two vector fields

$$\begin{aligned} U_0 - U_1 - 2\epsilon_1 U_3 &= 4Re^{-\phi}\frac{\partial}{\partial\epsilon_3}, \\ U_0 + 3U_1 + 2\epsilon_1 U_3 &= 4\epsilon_2\frac{\partial}{\partial\epsilon_2} - 4\epsilon_2 e^{\psi-\phi}\frac{\partial}{\partial\epsilon_3}, \end{aligned} \tag{78}$$

which are analytic with respect to $\mathcal{A}^*$. The change of coordinates from $\epsilon_1, \epsilon_2, \epsilon_3$ to the analytic chart ϵ_1, μ_2, μ_3 in $\mathcal{A}^*$ converts the two vector fields in (78) into linear combinations of $\partial/\partial\mu_2$ and $\partial/\partial\mu_3$, with the $\partial/\partial\mu_2$ and $\partial/\partial\mu_3$ components being analytic functions with respect to $\mathcal{A}^*$; we therefore have

$$\begin{aligned} Re^{-\phi}\frac{\partial\mu_2}{\partial\epsilon_3} &= A_1, \\ Re^{-\phi}\frac{\partial\mu_3}{\partial\epsilon_3} &= A_2, \\ \epsilon_2\frac{\partial\mu_2}{\partial\epsilon_2} - \epsilon_2 e^{\psi-\phi}\frac{\partial\mu_2}{\partial\epsilon_3} &= A_3, \\ \epsilon_2\frac{\partial\mu_3}{\partial\epsilon_2} - \epsilon_2 e^{\psi-\phi}\frac{\partial\mu_3}{\partial\epsilon_3} &= A_4, \end{aligned}$$

for some analytic functions $A_1, \cdots, A_4$ with respect to $\mathcal{A}^*$. We solve to see

$$\begin{aligned}\frac{\partial \mu_2}{\partial \epsilon_3} &= \frac{A_1 e^{\phi}}{R},\\ \frac{\partial \mu_3}{\partial \epsilon_3} &= \frac{A_2 e^{\phi}}{R},\\ \frac{\partial \mu_2}{\partial \epsilon_2} &= \frac{A_3 + A_5 e^{\psi}}{\epsilon_2},\\ \frac{\partial \mu_3}{\partial \epsilon_2} &= \frac{A_4 + A_6 e^{\psi}}{\epsilon_2},\end{aligned}$$

where

$$\begin{aligned}A_5 &= A_1 \epsilon_2 / R,\\ A_6 &= A_2 \epsilon_2 / R,\end{aligned}$$

are analytic with respect to $\mathcal{A}^*$. Therefore,

$$\begin{aligned}\frac{\partial \epsilon_2}{\partial \mu_2} &= \frac{A_2 \epsilon_2}{A_2 A_3 - A_1 A_4},\\ \frac{\partial \epsilon_2}{\partial \mu_3} &= -\frac{A_1 \epsilon_2}{A_2 A_3 - A_1 A_4},\\ \frac{\partial \epsilon_3}{\partial \mu_2} &= -\frac{A_4 + A_6 e^{\psi}}{\epsilon_2 \Delta},\\ \frac{\partial \epsilon_3}{\partial \mu_3} &= \frac{A_3 + A_5 e^{\psi}}{\epsilon_2 \Delta},\end{aligned} \tag{79}$$

where

$$\Delta = \frac{(A_2 A_3 - A_1 A_4) e^{\phi}}{R \epsilon_2}.$$

We line-integrate the first two equations of (79) to obtain the "potential" function $\log(\epsilon_2)$ and see that ϵ_2 is of the form

$$\epsilon_2 = A_7 / B,$$

where B is a smooth function of ϵ_1 alone and A_7 is analytic with respect to $\mathcal{A}^*$. So now we can extend ϵ_1 and μ, where

$$\mu := B \epsilon_2,$$

to an analytic chart (ϵ_1, μ, ν) in $\mathcal{A}^*$. However, since U_3 ($= \partial / \partial \epsilon_1$ in the $(\epsilon_1, \epsilon_2, \epsilon_2)$ chart) is analytic with respect to $\mathcal{A}^*$, the $\partial / \partial \mu$-component of U_3, which is

$$\frac{\partial \mu}{\partial \epsilon_1} = \epsilon_2 \frac{dB}{d\epsilon_1} = \frac{\mu}{B} \frac{dB}{d\epsilon_1},$$

must be analytic with respect to $\mathcal{A}^*$. Consequently, $\log(B)$, B and $\epsilon_2 = A_7/B$ are all analytic with respect to $\mathcal{A}^*$. So now we can extend ϵ_1, ϵ_2 to an analytic chart $(\epsilon_1, \epsilon_2, \zeta)$ in $\mathcal{A}^*$. In view of (78), we see

$$\begin{aligned}4Re^{-\phi}\frac{\partial}{\partial\epsilon_3} &= U_0 - U_1 - 2\epsilon_1 U_3,\\ -4\epsilon_2 e^{\psi-\phi}\frac{\partial}{\partial\epsilon_3} &= U_0 + 3U_1 + 2\epsilon_1 U_3 - 4\epsilon_2\frac{\partial}{\partial\epsilon_2},\end{aligned}$$

are analytic with respect to $\mathcal{A}^*$. Then the same reasoning following (78) establishes

$$\begin{aligned}Re^{-\phi}\frac{\partial\zeta}{\partial\epsilon_3} &= A_8,\\ \epsilon_2 e^{\psi-\phi}\frac{\partial\zeta}{\partial\epsilon_3} &= A_9,\end{aligned}$$

for some analytic A_8 and A_9 with respect to $\mathcal{A}^*$. Inserting the first equation into the second one and employing the analyticity of R/ϵ_2 with respect to $\mathcal{A}^*$, we see that e^{ψ}, and so ψ, must be analytic with respect to $\mathcal{A}^*$. Let us introduce the new variable

$$\epsilon_3^* = \int e^{\phi(\epsilon_3)}d\epsilon_3.$$

Then the fact that ϵ_2, ψ and R are analytic with respect to $\mathcal{A}^*$ results in the analyticity of ϵ_3^* with respect to $\mathcal{A}^*$. Thus $(\epsilon_1, \epsilon_2, \epsilon_3^*)$ form an analytic chart in $\mathcal{A}^*$. As a consequence, we conclude that ψ is an analytic function in the variable ϵ_2.

Conversely, suppose ψ is analytic in ϵ_2. We introduce the coordinate system (x_0, x_1, x_2, x_3), where

$$x_0 := e^{\phi}\epsilon_0, x_1 := \epsilon_1, x_2 := \epsilon_2, x_3 := \epsilon_3^*.$$

Then ψ is analytic in $x_0, \cdots, x_3$ clearly, and so is R because

$$R = x_3 + \int e^{\psi(x_2)}dx_2.$$

In view of Theorem 8, it can be checked directly that the Christoffel symbols Γ^i_{jk} of the associated torsion-free G_3-connection in the $\epsilon_0, \cdots, \epsilon_3$ coordinates are rational functions in terms of $\epsilon_2, \epsilon_3, \psi, R$, and powers of $\epsilon_0 e^{\phi}$ up to the second degree. In other words, the Christoffel symbols are analytic in the coordinates $x_0, \cdots, x_3$. We have thus arrived at the following.

Theorem 13. *The torsion-free G_3-connection associated with ψ and ϕ is smoothly equivalent to an analytic one if and only if $\psi(\epsilon_2)$ is an analytic function in ϵ_2. Moreover, by the aforementioned coordinate change, we may assume $\phi = 0$.*

The theorem thus produces smooth torsion-free G_3-connections that are not equivalent to any analytic ones by suitable choices of ψ.

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