1) A particle is moving along the x-axis and its position for $x \geq 0$ is given by the formula $x = \frac{1}{3}t^3 - 2t^2 + 3t$. On what interval(s) is the **velocity** of the particle decreasing ?

Solution: $v = dx/dt = t^2 - 4t + 3$, $dv/dt = 2t - 4 = 2(t-2)$. The velocity v is decreasing where $dv/dt < 0$. That is, on the open interval $(0, 2)$.

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2) A rock is thrown vertically upward from the edge of a stand on the moon's surface, which is 10 feet above the surface. Its height in meters after t seconds is given by

$$h(t) = 24t - 0.8t^2 + 10 \quad (\text{e.g. } h(0) = 10).$$

Find the **total distance** traveled by the rock from the time it is thrown up until the time it passes the stand on its the way down.

Solution: $v = 24 - 1.6t = 0$ when $t = 24/1.6 = 15$. That means that after 15 seconds the rock reaches its highest point. 15 seconds later it will pass the stand on the way down(you can check that $h(30) = 10$). Total distance traveled will then be $s(15) - s(0) + |s(30) - s(15)| = 180 + 180 = 360$ meters

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3) Find an equation for the **normal line** to the curve $y = x\tan(x)$ at the point $(\pi, 0)$.

Solution: $dy/dx = \tan(x) + x\sec^2(x)$. For $x = \pi$ we get $dy/dx = \pi$ ($\tan(\pi) = 0$ and $\sec(\pi) = -1$). Then slope of normal line is $-\frac{1}{\pi}$ and equation of normal line is $y = -\frac{1}{\pi}(x - \pi) = -\frac{1}{\pi}x + 1$.

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4) Eliminate the parameter to find a **Cartesian equation** for the curve

$$x = -1 + 3\sec(t) \qquad y = 2 + 3\tan(t)$$

Solution: $x + 1 = 3\sec(t)$ and $y - 2 = 3\tan(t)$. So $(x+1)^2 = 9\sec^2(t)$ and $(y-2)^2 = 9\tan^2(t)$. From the identity $1 + \tan^2(t) = \sec^2(t)$ we get that $9 + 9\tan^2(t) = 9\sec^2(t)$.

So we get the cartesian equation $9 + (y-2)^2 = (x+1)^2$. This can also be written as $(x+1)^2 - (y-2)^2 = 9$, which is a hyperbola.

5) From the parametric equations $x = t - \sin(t)$, $y = 1 - \cos(t)$, find the second derivative, $\frac{d^2y}{dx^2}$, at $t = \frac{\pi}{3}$.

Solution: $y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1+sin(t)}{1-cos(t)}$. Next we have $\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{dy'/dt}{dx/dt}$.
Now $dy'/dt = \frac{cos(t)(1-cos(t)) - (1+sin(t))(sin(t))}{(1-cos(t))^2} = \frac{cos(t) - sin(t) - 1}{(1-cos(t))^2}$ *and*
$dx/dt = (1 - cos(t))$.
Then $\frac{dy'/dt}{dx/dt} = \frac{cos(t) - sin(t) - 1}{(1-cos(t))^3} = \frac{\frac{1}{2} - \frac{\sqrt{3}}{2} - 1}{(1-\frac{1}{2})^3} = -(2\sqrt{3} + 2)$ *at* $t = \frac{\pi}{3}$.

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6) If $f(x) = x \cdot \ln(e^{\sqrt{x}})$, find $f'(1)$.

Solution: $f(x) = x \cdot \sqrt{x}\ \ln(e) = x^{\frac{3}{2}}$. Then $f'(x) = \frac{3}{2}\sqrt{x}$ and $f'(1) = \frac{3}{2}$.

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7) Find an equation for the **tangent line** to the curve $x^3 + y^3 = 9xy$ at the point $(2, 4)$.

Solution: $3x^2 + 3y^2 \frac{dy}{dx} = 9y + 9x \frac{dy}{dx}$. For $x = 2$ and $y = 4$ we get
$12 + 48 \frac{dy}{dx} = 36 + 18 \frac{dy}{dx}$. So $30 \frac{dy}{dx} = 24$ and $\frac{dy}{dx} = \frac{4}{5}$.
Equation for tangent line is $(y - 4) = \frac{4}{5}(x - 2)$.

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8) If $f(x) = (\tan^{-1}(x))^2$ then $f'(1) =$:

Solution: $f'(x) = 2(\tan^{-1}(x)) \cdot \frac{1}{1+x^2}$. Then $f'(1) = 2\,\frac{\pi}{4} \cdot \frac{1}{2} = \frac{\pi}{4}$.

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9) Find the **slope** of the tangent line to the curve $x \cdot \arctan(y) + x \cdot y = \frac{\pi+4}{4}$ at the point $(1, 1)$.

Solution: $\arctan(y) + \frac{x}{1+y^2}\frac{dy}{dx} + y + x\frac{dy}{dx} = 0$. For $x = 1$ and $y = 1$ we get
$\frac{\pi}{4} + \frac{1}{2}\frac{dy}{dx} + 1 + \frac{dy}{dx} = 0$. $\frac{3}{2}\frac{dy}{dx} = -\left(\frac{\pi+4}{4}\right)$. Then $\frac{dy}{dx} = -\left(\frac{\pi+4}{6}\right)$.

10) If $f(x) = x \cdot \log_3(2^{\sqrt{x}})$, find $f'(1)$.

Solution: $f'(x) = \log_3(2^{\sqrt{x}}) + x \cdot \frac{1}{\ln(3)} \cdot \frac{1}{2^{\sqrt{x}}} \cdot \ln(2)\ 2^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} =$
$\log_3(2^{\sqrt{x}}) + \frac{1}{2}\frac{\ln(2)}{\ln(3)}\sqrt{x}$. $f'(1) = \log_3(2) + \frac{1}{2}\frac{\ln(2)}{\ln(3)} = \frac{3}{2}\frac{\ln(2)}{\ln(3)}$.
Another way of doing this : $f(x) = x \cdot \frac{1}{\ln(3)} \ln(2^{\sqrt{x}}) = x \cdot \frac{1}{\ln(3)} \cdot \sqrt{x} \cdot \ln(2) =$
$\frac{\ln(2)}{\ln(3)} \cdot x^{3/2}$. So $f'(x) = \frac{3}{2}\frac{\ln(2)}{\ln(3)} \cdot x^{\frac{1}{2}}$ and $f'(1) = \frac{3}{2}\frac{\ln(2)}{\ln(3)}$ (as before).

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11) If $f(x) = x^{e^x}$ then find $f'(1)$.

Solution: By logarithmic differentiation we have $\ln(y) = \ln(x^{e^x}) = e^x \ln(x)$.
Then $\frac{1}{y}\frac{dy}{dx} = e^x \ln(x) + e^x \frac{1}{x}$ and $f'(x) = x^{e^x}(e^x \ln(x) + e^x \frac{1}{x})$.
Finally we get that $f'(1) = e$ ($\ln(1) = 0$ and $1^e = 1$).

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12) If $f(x) = \sin^{-1}(\tan(x))$ then find $f'(x)$.

Solution: $f'(x) = \frac{1}{\sqrt{1-\tan^2(x)}} \cdot \sec^2(x) = \frac{\sec^2(x)}{\sqrt{1-\tan^2(x)}}$.

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13) If $f(x) = x \cdot 4^{-x^2}$ then find $f'(x)$.

Solution: $f'(x) = 4^{-x^2} + x \cdot \ln(4)\, 4^{-x^2} \cdot -2x = 4^{-x^2}(1 - 2\ln(4)\, x^2)$.

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14) Use logarithmic differentiation to find $\frac{dy}{dx}$ if $y = \sqrt[4]{\frac{x^3+1}{\tan(x) \cdot \sec(x)}}$.

Solution: $\ln(y) = \frac{1}{4} \cdot \ln(\frac{x^3+1}{\tan(x) \cdot \sec(x)}) = \frac{1}{4}(\ln(x^3+1) - \ln(\tan(x)) - \ln(\sec(x)))$.
$\left(\text{We're using} \quad \ln\left((\frac{x^3+1}{\tan(x) \cdot \sec(x)})^{\frac{1}{4}}\right) = \frac{1}{4}\, ln\left((\frac{x^3+1}{\tan(x) \cdot \sec(x)})\right.\right)$
$\frac{1}{y}\frac{dy}{dx} = \frac{1}{4}\left(\frac{3x^2}{x^3+1} - \frac{sec^2(x)}{tan(x)} - \frac{sec(x)\,tan(x)}{sec(x)}\right)$
Then $\frac{dy}{dx} = \frac{1}{4} \cdot \sqrt[4]{\frac{x^3+1}{\tan(x) \cdot \sec(x)}} \cdot \left(\frac{3x^2}{x^3+1} - \frac{\sec^2(x)}{\tan(x)} - \tan(x)\right)$.

15) For $f(x) = 12\log_8(\ln(x))$, find $f'(e)$.

Solution: $f'(x) = \frac{12}{\ln(8)}\frac{1}{\ln(x)}\frac{1}{x}$. Then $f'(e) = \frac{12}{\ln(8)\cdot e}$.

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16) There are two points where the curve $x^2 + xy + y^2 = 9$ crosses the x-axis .
At those two points the **tangent lines** are parallel. Find the common **slope.**
(Hint: Point on the x-axis has coordinates $(a, 0)$).

Solution: For $y = 0$ we get $x^2 = 9$. The points are then $(-3, 0)$, $(3, 0)$.
By implicit differentiation we have $2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$. For $y = 0$,
$2x + x\frac{dy}{dx} = 0$. For $x = \pm 3$ we have $\frac{dy}{dx} = -2$. Same slope.

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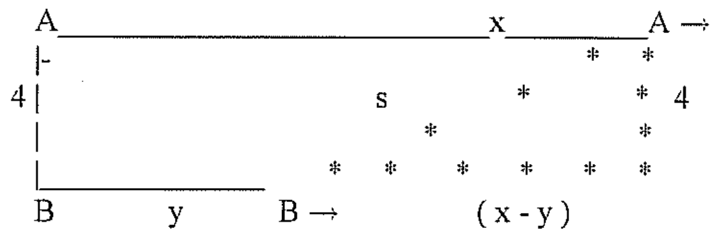
17) Find $\lim_{\theta \to 0} \cos\left(\frac{\pi\theta}{\sin(\theta)}\right)$. $\left(\text{Recall that } \lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1\right)$

Solution: By limit laws for composites we have that
$\lim_{\theta \to 0} \cos\left(\frac{\pi\theta}{\sin(\theta)}\right) = \cos\left(\lim_{\theta \to 0}\left(\frac{\pi\theta}{\sin(\theta)}\right)\right) = \cos\left(\pi \cdot \lim_{\theta \to 0}\left(\frac{\theta}{\sin(\theta)}\right)\right)$.
Finally $\lim_{\theta \to 0}\left(\frac{\theta}{\sin(\theta)}\right) = \frac{1}{\lim_{\theta \to 0}\frac{\sin(\theta)}{\theta}} = \frac{1}{1} = 1$ and we get that
$\lim_{\theta \to 0} \cos\left(\frac{\pi\theta}{\sin(\theta)}\right) = \cos(\pi) = -1$.

18) At 2:00 PM sailboat **B** is 4 km south of sailboat **A** . After that **A** starts moving east at 4 km/hr and **B** starts moving east at 1 km/hr . Find the **rate of change** of the distance between the two boats at 3:00 PM .

Solution: In this case t =0 is 2:00 PM and we want the result at t = 1 , 3:00 PM.



In the upper diagram the letters on the left represent the positions of the two boats at t = 0, boat A above and boat B below. The two letters on the right represent the positions at a later time . The arrows are the directions that boats travel.
x is distance A traveled and y is the distance B traveled. (x and y vary in time).
We are given that always, $\frac{dx}{dt} = 4$ km/hr *and* $\frac{dy}{dt} = 1$ km/hr .
The problem is to find $\frac{ds}{dt}$ when t = 1. The distance between them, s , is the hypotenuse of a right triangle with the other sides being 4 and (x − y) . So $s^2 = 4^2 + (x-y)^2$. When t =1 we have x = 4 , y = 1 giving us s = 5 .
By implicit differentiation $2\,s\,\frac{ds}{dt} = 2\,(x-y)(\frac{dx}{dt} - \frac{dy}{dt})$. For $t = 1$ we have that $10\,\frac{ds}{dt} = 6\,(4-1)$. *So* $\frac{ds}{dt} = \frac{9}{5}$ km/hr at 3:00 PM.

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19) When a circular plate of metal is heated in an oven, its radius increases at the rate of 0.01 cm/min. At what rate is the plate's **area** increasing when the radius is 50 cm ?
Solution: We have a circle of radius r and are given $\frac{dr}{dt} = 0.01$. The area $A = \pi r^2$, so we have $\frac{dA}{dt} = 2\,\pi\, r\, \frac{dr}{dt}$. When r = 50 cm $\frac{dA}{dt} = 100\,\pi\,(0.01) = \pi$ cm^2/min .

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20) The length of a rectangle is decreasing at the rate of 5 cm/sec while the width is increasing at the rate of 3 cm/ses. Find the rate of change of the **diagonal** when the length is 10 cm and the width is 15 cm. Is it increasing or decreasing ?

Solution: If x = length and y = width then we are given $\frac{dx}{dt} = -5$, $\frac{dy}{dt} = 3$, both represent cm/sec. If s is the diagonal it is the hypotenuse of a right triangle with the other sides x and y . So $x^2 + y^2 = s^2$ and by implicit differentiation $2\,x\,\frac{dx}{dt} + 2\,y\,\frac{dy}{dt} = 2\,s\,\frac{ds}{dt}$. If x = 10 and y = 15 then $s = \sqrt{325} = 5\sqrt{13}$.
For those values we get $-100 + 90 = 10\sqrt{13}\,\frac{ds}{dt}$ *and*
$\frac{ds}{dt} = -\frac{1}{\sqrt{13}}$ cm/sec (e.g. it is decreasing).