MATH 131 **PRACTICE FINAL EXAM (solutions)** **FALL 2007**

1) Find the **x and y coordinates** of all points on the curve $y = x^3 - 12x$ at which the **tangent line** is **horizontal**.

$y' = 3x^2 - 12 = 0 \; for \; x = \pm 2 . \; The\, points\, are\; (-2, 16) \; and \; (2, -16)$

2) For $r = (1 + \sec(\theta)) \cdot \sin(\theta)$, find $\frac{dr}{d\theta}$.

$r = sin(\theta) + tan(\theta) \; so \; \frac{dr}{d\theta} = cos(\theta) + sec^2(\theta).$

3) For $f(x) = \frac{e^x - e^{-x}}{x}$ find $f'(1)$.

$f'(x) = \frac{(e^x + e^{-x})x - (e^x - e^{-x})}{x^2} \; so \; f'(1) = \frac{2}{e}.$

4) Find an equation for the **tangent line** to the curve $y = \left(\frac{sin(x)}{1 + cos(x)}\right)^2$ at the point $\left(\frac{\pi}{2}, 1\right)$.

$\frac{dy}{dx} = 2\left(\frac{sin(x)}{1+cos(x)}\right)\left(\frac{cos(x)(1+cos(x)) + sin^2(x)}{(1+cos(x))^2}\right. . \; For\, x = \frac{\pi}{2}, \; \frac{dy}{dx} = 2.$

$So\, tangent\, line\, is \; y - 1 = 2\left(x - \frac{\pi}{2}\right).$

5) If $f(\theta) = \ln\left(\frac{e^\theta}{1 + e^\theta}\right)$ then find $f'(\theta)$.

$\ln\left(\frac{e^\theta}{1 + e^\theta}\right) = ln(e^\theta) - ln(1 + e^\theta) = \theta - ln(1 + e^\theta).$

$So \; f'(\theta) = 1 - \frac{e^\theta}{1+e^\theta} = \frac{1}{1+e^\theta}.$

6) Find an equation for the **tangent line** to the curve $x^3 + y^3 - 9xy = 0$ at the point $(2, 4)$.

$3x^2 + 3y^2 y' - 9y - 9x y' = 0 . \; If\, x = 2\, and\, y = 4\, we\, get$

$12 + 48 y' - 36 - 18 y' = 0 \; and \; y' = \frac{4}{5}.$

7) For the function $f(x) = x^{2/3}$ find the **y-coordinate** of the **absolute maximum** point of the curve on the closed interval $-2 \leq x \leq 3$.

$y' = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3x^{\frac{1}{3}}} . \; It\, never\, equals\, 0 \; and\, it\, DNE \; at \; x = 0, \; so\, x = 0$

$is\, the\, only\, critical\, point . \; f(-2) = 1.5874, \; f(0) = 0 \; and \; f(3) = 2.08.$

$So \; y = 2.08 \; is\, the\, y - coordinate\, of\, the\, absolute\, maximun\, point.$

8) Find the **x-coordinates** of all the points on the curve $y = x^3 + x^2 - 8x - 5$ which are either **local maximum** or **local minimum** points. In each case **state** clearly which one they are..

$y' = 3x^2 + 2x - 8 = (3x-4)(x+2) = 0$ *when* $x = -2$ *or* $x = \frac{4}{3}$.
Curve is inc. for $x < -2$ *and* $x > \frac{4}{3}$ *and it is decreasing* $-2 < x < \frac{4}{3}$.
So $x = -2$ *is a local maximum and* $x = \frac{4}{3}$ *is a local minimum.*

9) Find the **critical points** (both the x and the y coordinates) of $f(x) = x^{\frac{1}{3}}(x-4) = x^{\frac{4}{3}} - 4x^{\frac{1}{3}}$. Then identify the **intervals** on which f is **increasing** and **decreasing**.

$f' = \frac{4}{3}x^{\frac{1}{3}} - \frac{4}{3}x^{-\frac{2}{3}} = \frac{4x-4}{3x^{\frac{2}{3}}}$. *This equals* 0 *when* $x = 1$ *and DNE for* $x = 0$.
$(1, -3)$ *and* $(0, 0)$ *sre the two critical points and curve is decreasing for* $x < 1$ *and it is increasing for* $x > 1$. ($x^{\frac{2}{3}}$ *is always nonnegative*).

10) On the curve $y = 4x^3 - x^4$ find the intervals on which the curve is **concave up** and the intervals on which it is **concave down**.

$y' = 12x^2 - 4x^3$, $y'' = 24x - 12x^2 = 12x(2-x) = 0$ *when* $x = 0$ *or* $x = 2$. *Concave up on* $(0, 2)$ *and concave down for* $x < 0$ *and* $x > 2$.

11) Find the x and y coordinates of all the **inflection points** on the curve $y = x^4 - 8x^3 + 18x^2$.

$y' = 4x^3 - 24x^2 + 36x$, $y'' = 12x^2 - 48x + 36 = 12(x-1)(x-3) = 0$ *for* $x = 1$ *or* $x = 3$. *Inflection points are* $(1, 11)$ *and* $(3, 27)$.

12) Find $\lim_{x \to 0} \frac{x - sin(x)}{x^3} \left(\frac{0}{0}\right) = \lim_{x \to 0} \frac{1 - cos(x)}{3x^2} \left(\frac{0}{0}\right) = \lim_{x \to 0} \frac{sin(x)}{6x} \left(\frac{0}{0}\right) = \lim_{x \to 0} \frac{cos(x)}{6} = \frac{1}{6}$.

13) For the limit $\lim_{x \to \infty} (ln(x))^{1/x}$ first **state** which type of **indeterminate form** it is and then **find the limit**.

It is the ∞^0 *type, so we look at* $ln[(ln(x))^{1/x}] = \frac{1}{x} ln[ln(x)] = \frac{ln[ln(x)]}{x}$.
Now we calculate $\lim_{x \to \infty} \frac{ln[ln(x)]}{x} \left(\frac{\infty}{\infty}\right) = \lim_{x \to \infty} \frac{1}{ln(x)} \frac{1}{x} = 0$. *Hence*
$\lim_{x \to \infty} (ln(x))^{1/x} = e^0 = 1$.

14) Estimate the area under the curve $y = \frac{1}{x}$ between $x = 1$ and $x = 7$ using M_3.
$\Delta x = 2$ *and midpoints are* $x = 2, 4, 6$. *So* $M_3 = (\frac{1}{2} + \frac{1}{4} + \frac{1}{6})\,2 = 1.8333.$

15) Evaluate $\int_{-1}^{4} |\, x - 2 \,|\, dx = \int_{-1}^{2} -(x-2)\,dx + \int_{2}^{4} (x-2)\,dx =$
$(2x - \frac{x^2}{2})\Big|_{-1}^{2} + (\frac{x^2}{2} - 2x)\Big|_{2}^{4} = \frac{9}{2} + 2 = \frac{13}{2}.$

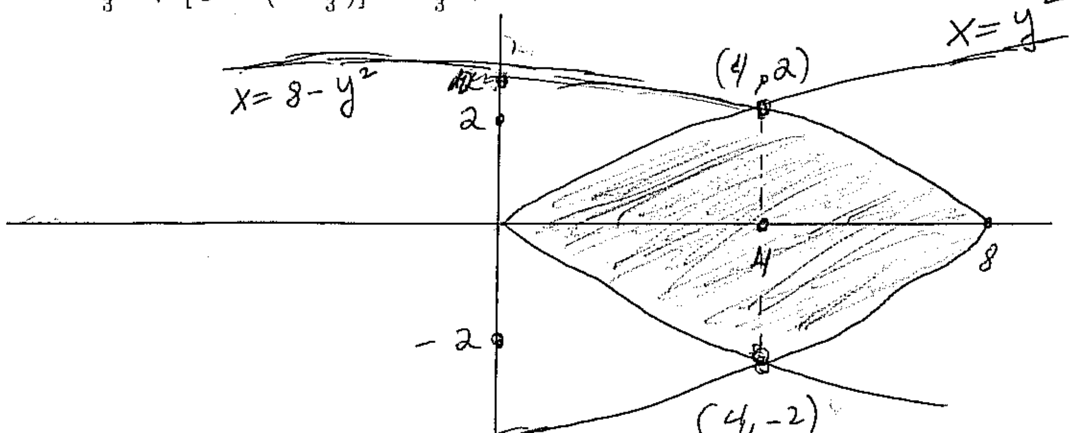
16) Find $\frac{dy}{dx}$ if $y = \int_{tan(x)}^{0} \frac{1}{1+t^2}\,dt = -\int_{0}^{tan(x)} \frac{1}{1+t^2}\,dt$. *By the Fund* Th^m *and the chain rule* $\frac{dy}{dx} = -\,(\frac{1}{1+tan^2(x)})\,(sec^2(x))$.

17) Evaluate $\int_{1}^{9} \frac{(\sqrt{t} - 1)^3}{\sqrt{t}}\,dt$. *Let* $u = \sqrt{t} - 1$, $du = \frac{1}{2\sqrt{t}}\,dt$.
Then $\int_{1}^{9} \frac{(\sqrt{t} - 1)^3}{\sqrt{t}}\,dt = 2\int_0^2 u^3\,du = \frac{u^4}{2}\Big|_0^2 = 8.$

18) Using the substitution $u = x^2 + 1$ solve the indefinite integral
$\int x^3 \sqrt{x^2+1}\;dx. = \int x^2\, x\, \sqrt{x^2+1}\; dx$
$du = 2x\,dx$, $dx = \frac{1}{2x}\,du$, $x^2 = u - 1.$
$\int x^2\, x\, \sqrt{x^2+1}\;dx = \frac{1}{2}\int (u-1)\sqrt{u}\;du = \frac{1}{2}\int u^{\frac{3}{2}} - u^{\frac{1}{2}}\,du =$
$\frac{1}{2}\,(\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}) + C = \frac{1}{5}(x^2+1)^{\frac{5}{2}} - \frac{1}{3}(x^2+1)^{\frac{3}{2}} + C$

19) Find the **area** of the **region** enclosed by the parabolas $x = 8 - y^2$ and $x = y^2$. *Intersection points are* $(4\,,2)$ *and* $(4,\ -2)$.
Area $= \int_{y=-2}^{y=2} (8 - y^2) - y^2\,dy = \int_{y=-2}^{y=2} 8 - 2y^2\,dy = 2\int_0^2 8 - 2y^2\,dy =$
$2(8y - \frac{2}{3}y^3)\Big|_0^2 = 2\,(16 - \frac{16}{3}) = \frac{64}{3}.$

Another possibility is integral in $x -$ *coordinate. We will need two integrals.*
Area $= \int_{x=0}^{x=4} \sqrt{x} - (-\sqrt{x})\,dx + \int_{x=4}^{x=8} \sqrt{8-x} - (-\sqrt{8-x})\,dx =$
$\int_{x=0}^{x=4} 2\sqrt{x}\,dx + \int_{x=4}^{x=8} 2\sqrt{8-x}\,dx = \frac{4}{3}x^{\frac{3}{2}}\Big|_0^4 + (-\frac{4}{3}(8-x)^{\frac{3}{2}}\Big|_4^8 =$
$\frac{32}{3} + [0 - (-\frac{32}{3})] = \frac{64}{3}$.



20) Find the **area** of the **region** enclosed by the curve $y = x^3$ and the line $y = x$.

(Hint : In this problem the top and bottom curves change position in the middle) .

Intersection points are $(-1, -1)$, $(0, 0)$, $(1, 1)$.

For $-1 \leq x \leq 0$ $y = x^3$ *is top and* $y = x$ *is the bottom and for* $0 \leq x \leq 1$ *it reverses ,* $y = x$ *is top and* $y = x^3$ *is bottom. Hence*

Area $= \int_{-1}^{0} x^3 - x\, dx + \int_0^1 x - x^3\, dx = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

Problem done in class :

Which definite integral is defined by the following Riemann sum :

$$\lim_{n \to \infty} \sum_{k=1}^{n} \sqrt{2 + k\left(\frac{2}{n}\right)}\ \left(\frac{2}{n}\right) .$$

If you are given a number of choices, $\int_a^b f(x)\, dx$, here is what to look for.

Since in the above Riemann sum $\Delta x = \frac{2}{n}$ *we know that* $b - a$ *must be equal to* 2 . *Also need to check the intergrand,* $f(x)$.

The first term in the summand is $\sqrt{2 + \frac{2}{n}}$ $(k = 1)$ *so we need that* $f(a + \frac{2}{n}) = \sqrt{2 + \frac{2}{n}}$, *similarly* $f(a + \frac{4}{n}) = \sqrt{2 + \frac{4}{n}}$ $(k = 2)$, *etc.*

For example $\int_0^2 \sqrt{2 + x}\, dx$ *works since* $2 - 0 = 2$. Also *with* a = 0 and $f(x) = \sqrt{2 + x}$, $f(a + \frac{2}{n}) = f(0 + \frac{2}{n}) = \mathrm{f}(\frac{2}{n}) = \sqrt{2 + \frac{2}{n}}$, *OK* .

Simalarly $\int_1^3 \sqrt{1 + x}\, dx$ *works. First* $3 - 1 = 2$. *Then with* $a = 1$, $f(x) = \sqrt{1 + x}$, $f(a + \frac{2}{n}) = f(1 + \frac{2}{n}) = \sqrt{1 + (1 + \frac{2}{n})} = \sqrt{2 + \frac{2}{n}}$, *OK* .

If the integral were $\int_1^4 f(x)\, dx$, *for any* $f(x)$, *since* $4 - 1 = 3$, <u>*not*</u> *OK*.

Also $\int_1^3 \sqrt{2 + x}\, dx$ *is no good since* $f(x) = \sqrt{2 + x}$, *then* $f(a + \frac{2}{n}) = f(1 + \frac{2}{n}) = \sqrt{2 + (1 + \frac{2}{n})} = \sqrt{3 + \frac{2}{n}}$ *which is not* $\sqrt{2 + \frac{2}{n}}$ (<u>*not*</u> *OK*).