SOLUTIONS HW # 9 MATH 4111

$\boxed{\#8, \text{p.}133}$ We want to show that if $f : [a,b] \to \mathbb{R}$ is monotonic, then f is Riemann integrable on $[a,b]$. Replacing f by $-f$ if need be, we can assume f is monotonic increasing.

Method 1: For $c \in (a,b)$, $\ell_+(c) = \lim\limits_{\substack{x \to c \\ x > c}} f(x)$ and

$\ell_-(c) = \lim\limits_{\substack{x \to c \\ x < c}} f(x)$ exist since f is monotonic.

For $c = b$, put $\ell_+(b) = f(b)$ and, for $c = a$, put $\ell_-(c) = f(c)$. In all cases $\ell_+(c) \geq f(c) \geq \ell_-(c)$ and f is continuous at $c \iff \ell_+(c) = \ell_-(c)$. When f is not continuous at c, $j(c) = \ell_+(c) - \ell_-(c)$ measures the size of the jump discontinuity at c. The set N of discontinuities of f is countable since for any $n \in \mathbb{N}$, the number of jumps $\geq 1/n$ must be $\leq n(f(b) - f(a))$. If N is finite, an easy argument shows that f is integrable. Otherwise choose an enumeration $c_1, c_2, c_3, \ldots$ of N.

Then $\sum\limits_{i=1}^{\infty} j(c_i) \leq f(b) - f(a)$ so $\sum\limits_{i=n+1}^{\infty} j(c_i) \to 0$ as $n \to \infty$. For each n, define functions $g_n(x)$ and $h_n(x)$ by

$g_n(x) = f(x) - (\text{sum of terms } j(c_i) \text{ with } i \geq n+1 \text{ and } c_i < x)$

$h_n(x) = f(x) + (\text{sum of terms } j(c_i) \text{ with } i \geq n+1 \text{ and } c_i > x)$

Then $g_n(x) \geq f(x) \geq h_n(x)$ and both g_n and h_n are monotonic with discontinuities only at $c_1, \ldots, c_n$

Clearly $0 < \int_a^b (h_n - g_n)(x) < \sum_{i=n+1}^{\infty} J(c_i)$

Also $h_n \geq h_{n+1} \geq \cdots \geq h_\infty \geq f \geq g_\infty \geq g_2 \geq \cdots$

Hence there is a number L for which the integrals of the h_n's decrease monotonically to L while the integrals of the g_n's increase monotonically to L

Since Riemann sums for f are trapped between Riemann sums for g_n and Riemann sums for h_n (using a common selection rule for each partition) it follows that $S(\mathcal{P}, f, *) \to L$ as $w(\mathcal{P}) \to 0$ so f is Riemann integrable with $\int_a^b f(x)\,dx = L$

Method 2. For any bounded $f : [a,b] \to \mathbb{R}$ ~~the upper~~ and any partition $\mathcal{P}$ of $[a,b]$ the upper Riemann sum $\overline{S}(\mathcal{P}, f)$ and lower Riemann sum $\underline{S}(\mathcal{P}, f)$ are defined to be the supremum (respectively, infimum) of the Riemann sums $S(\mathcal{P}, f, *)$ as $*$ ranges over all possible selection rules. When $\mathcal{P}$ and $\mathcal{P}'$ are two partitions of $[a,b]$ (each thought of as an ordered finite subset of $[a,b]$), it's simple to check that

$$\underline{S}(\mathcal{P}, f) \leq \underline{S}(\mathcal{P} \cup \mathcal{P}', f) \leq \overline{S}(\mathcal{P} \cup \mathcal{P}', f) \leq \overline{S}(\mathcal{P}', f)$$

Then the sup $\underline{S}(f)$ (respectively, inf $\overline{S}(f)$) of $\underline{S}(\mathcal{P}, f)$ (resp. $\overline{S}(\mathcal{P}, f)$) as $\mathcal{P}$ ranges over all partitions $\mathcal{P}$ of $[a,b]$ satisfy

$$\underline{S}(\mathcal{P}, f) \leq \underline{S}(f) \leq \overline{S}(f) \leq \overline{S}(\mathcal{P}, f)$$

for every partition $\mathcal{P}$. It follows easily that f is Riemann integrable on $[a,b] \iff \underline{S}(f) = \overline{S}(f)$ and then $\int_a^b f(x)\,dx = \underline{S}(f) = \overline{S}(f)$

When f is monotone increasing, then for

$\mathcal{P}: \quad a = x_0 < x_1 < \dots < x_N = b$ any

partition of $[a,b]$, clearly

$$\overline{S}(f,\mathcal{P}) = \sum_{i=1}^{N} f(x_i)(x_i - x_{i-1}) \quad \left(\text{right endpoint rule}\right)$$

$$\underline{S}(f,\mathcal{P}) = \sum_{i=1}^{N} f(x_{i-1})(x_i - x_{i-1}) \quad \left(\text{left endpoint rule}\right)$$

With $w(\mathcal{P}) = \max\{x_i - x_{i-1} : 1 \le i \le N\}$

$$\overline{S}(f,\mathcal{P}) - \underline{S}(f,\mathcal{P}) \le \sum_{i=1}^{N} \left(f(x_i) - f(x_{i-1})\right) w(\mathcal{P})$$

$$= \left(f(b) - f(a)\right) w(\mathcal{P})$$

As $w(\mathcal{P}) \to 0$, the right hand side goes to 0

We deduce that $\underline{S}(f) = \overline{S}(f)$. Furthermore calling $\int_a^b f(x)\,dx$ the common value, when $w(\mathcal{P}) < \delta(\varepsilon) = \dfrac{\varepsilon}{f(b) - f(a)}$

$$\left| \int_a^b f(x)\,dx - S(f,\mathcal{P},*) \right| < \varepsilon \quad \text{for every selection rule } *$$

so f is Riemann integrable on $[a,b]$

One can similarly define upper and lower Riemann integrals $\overline{S}(f)$ and $\underline{S}(f)$ for f a bounded function from a closed bounded rectangle $R \subseteq \mathbb{R}^n$ into $\mathbb{R}$ and deduce that f is Riemann integrable on R $\iff \underline{S}(f) = \overline{S}(f) = \int_R f(x)\,dx$

As above, when f is separately monotone increasing in each variable we easily get $\underline{S}(f) = \overline{S}(f)$ so f is Riem. integrable on R

#13, p. 133) With $f : [a,b] \to [0,\infty)$ continuous and $\|f\|_\infty = \max_{x\in[a,b]} f(x)$, we wish to show that

$$\|f\|_\infty = \lim_{n\to\infty} \|f\|_n = \lim_{n\to\infty} \left(\int_a^b f(x)^n dx \right)^{1/n}$$

(i) Since $f(x)^n \le \|f\|_\infty^n$

$$\|f\|_n \le \|f\|_\infty (b-a)^{1/n} \quad \text{for each } n$$

Using $\lim_{n\to\infty} c^{1/n} = 1$ for each $c > 0$

$$\limsup_{n\to\infty} \|f\|_n \le \|f\|_\infty$$

(ii) For any $\varepsilon > 0$, choosing c for which $\|f\|_\infty = f(c)$, there is an interval of length δ containing c on which $f(x) \geq \|f\|_\infty - \varepsilon$. This gives

$$\|f\|_n \geq (\|f\|_\infty - \varepsilon)\, \delta^{1/n} \quad \text{for each } n$$

and $\liminf_{n\to\infty} \|f_n\| \geq (\|f\|_\infty - \varepsilon) \lim_{n\to\infty} \delta^{1/n} = \|f\|_\infty - \varepsilon$

Since this holds $\forall \varepsilon > 0$

$$\liminf_{n\to\infty} \|f\|_n \geq \|f\|_\infty$$

(iii) By (i) and (ii), $\liminf_{n\to\infty} \|f_n\| = \limsup_{n\to\infty} \|f_n\| = \|f\|_\infty$

so $\lim_{n\to\infty} \|f\|_n$ exists and equals $\|f\|_\infty$

#16, p. 133) We define $I : (C[a,b], d_\infty) \to \mathbb{R}$ by

$$I(f) = \int_a^b f(x)\, dx.$$

Then $|I(f) - I(g)| \le \int_a^b |f(x) - g(x)|\, dx$

$$\le d_\infty(f,g)\,(b-a)$$

(recall $d_\infty(f,g) = \|f-g\|_\infty = \max_{x\in[a,b]} |f(x) - g(x)|$)

$\therefore$ I is unif cont since $\forall \varepsilon > 0$,

$$d_\infty(f,g) < \varepsilon/_{b-a} \implies |I(f) - I(g)| < \varepsilon$$

(19) Induction Hypothesis: H_n: When $f: \mathbb{R} \to \mathbb{R}$ is n times continuously differentiable on an open interval U, then for all $a, b \in U$

$$f(b) - T_{f,n-1,a}(b) = \frac{\int_a^b (b-x)^{n-1} f^{(n)}(x)\,dx}{(n-1)!}$$

where $T_{f,n-1,a}(b) = \sum_{j=0}^{n-1} \frac{f^{(j)}(a)}{j!}(b-a)^j$

(i) H_1 is true since $f(b) - f(a) = \int_a^b f'(x)\,dx$ by the Fund Thm of Calc.

(ii) $H_n \Rightarrow H_{n+1}$: $T_{f,n,a}(b) = T_{f,n-1,a}(b) + \frac{f^{(n)}(a)(b-a)^n}{n!}$

Using H_n,

$$f(b) - T_{f,n,a}(b) = \int_a^b \frac{(b-x)^{n-1} f^{(n)}(x)\,dx}{(n-1)!} - \frac{f^{(n)}(a)(b-a)^n}{n!}$$

(parts)

$$= \underbrace{-\frac{(b-x)^n}{n!} f^{(n)}(x)\Big|_{x=a}^{x=b} - \frac{f^{(n)}(a)}{n!}(b-a)^n}_{=0}$$

$$+ \frac{1}{n!}\int_a^b (b-x)^n f^{(n+1)}(x)\,dx$$

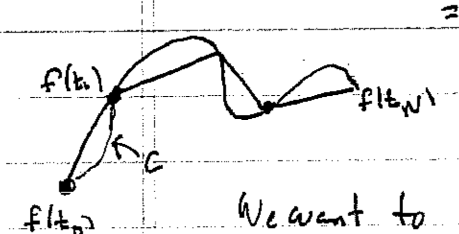
so H_{n+1} is true when H_n is true

(iii) $\therefore$ H_n is true $\forall n \in \mathbb{N}$

(20) Assume $f = (f_1, \dots, f_n)$ is cont. diff'ble from $[a,b]$ into $\mathbb{R}^n$. The arc length $L(C)$ of $C = f([a,b])$ is defined to be $\int_a^b \|f'(t)\|_2\,dt$

For each partition $\mathcal{P}$: $a = t_0 < t_1 < \dots < t_N = b$ of $[a,b]$, put $L(\mathcal{P}) = \sum_{i=1}^{N} \|f(t_i) - f(t_{i-1})\|_2$

= sum of the lengths of the chord lines successively joining the points $f(a) = f(t_0), f(t_1), \dots, f(t_N) = b$ on C



We want to prove that $L(C) = \sup\{L(\mathcal{P}) : \mathcal{P} \text{ any partition of } [a,b]\}$

(i) Given $\varepsilon > 0$, uniform continuity of $f_1', \ldots, f_n'$ allows us to choose $\delta = \delta(\varepsilon) > 0$ such that when $|t - t'| < \delta$, $|f_j'(t) - f_j'(t')| < \frac{\varepsilon}{\sqrt{n}(b-a)}$ for $1 \le j \le n$.

When $P : a = t_0 < t_1 < \cdots < t_N = b$ is a partition of $[a, b]$ with $\omega(P) = \max_{1 \le i \le N} |t_i - t_{i-1}| < \delta$, for each i, the MVT gives points $(t_i^{(1)}, \ldots, t_i^{(n)})$ in (t_{i-1}, t_i) for which

$$f(t_i) - f(t_{i-1}) = \left(f_1'(t_i^{(1)}), \ldots, f_n'(t_i^{(n)})\right)(t_i - t_{i-1})$$

and we deduce that

$$\left| \, \|f(t_i) - f(t_{i-1})\|_2 - \int_{t_{i-1}}^{t_i} \|f'(t)\|_2 \, dt \, \right|$$

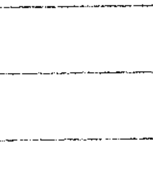
$$\le \int_{t_{i-1}}^{t_i} \left| \frac{\|f(t_i) - f(t_{i-1})\|_2}{t_i - t_{i-1}} - \|f'(t)\|_2 \right| dt$$

$$< \frac{\varepsilon}{b-a}(t_i - t_{i-1}) \quad \text{for each } i$$

Thus $|L(P) - L(C)| \le \sum_{i=1}^{N} \left| \, \|f(t_i) - f(t_{i-1})\|_2 - \int_{t_{i-1}}^{t_i} \|f'(t)\|_2 \, dt \, \right|$

$$< \frac{\varepsilon}{b-a} \sum_{i=1}^{N} (t_i - t_{i-1}) = \varepsilon$$

(ii) For P as above and $t_* \in (t_{i-1}, t_i)$, the triangle inequality for the Euclidean norm $\| \ \|_2$ gives



$$\|f(t_i) - f(t_{i-1})\|_2 \le \|f(t_i) - f(t_*)\|_2 + \|f(t_*) - f(t_{i-1})\|_2$$

and we get equality $\Leftrightarrow$ $f(t_*)$ lies on the line segment $\overline{f(t_{i-1}) f(t_i)}$. Whenever P' is a partition obtained from P by adjoining additional points, this tells us that $L(P) \le L(P')$. Also, unless $C = \overline{f(a) f(b)}$, we can choose P' for which $\|f(b) - f(a)\|_2 < L(P')$. By (i), $L(P)$ and $L(P')$ are both within ε of $L(C)$ when $\omega(P) < \delta$ and we deduce that $L(C) = \sup\{L(P) : P \text{ any partition of } [a,b]\}$ with $\|f(b) - f(a)\|_2 < L(C)$ when C is not the straight line path $\overline{f(a) f(b)}$

$\boxed{22, p.134}$ For $j \in \mathbb{N}$, let $I_j = \int_j^{j+1} \frac{dt}{t} = \ln(j+1) - \ln j$

Since $f(t) = 1/t$ is monotonic decreasing

$\frac{1}{j} > I_j > \frac{1}{j+1}$ for each j. Clearly

$I_1 + \cdots + I_{n-1} = \ln n$

Then $a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} - \ln n = \sum_{j=1}^{n-1} \frac{1}{j} - I_j > 0$ for each n

with $0 < a_1 < a_2 < \cdots < a_n < a_{n+1} < \cdots$

On the other hand, for $b_n = \frac{1}{2} + \cdots + \frac{1}{n} - \ln n = \sum_{j=1}^{n} \frac{1}{j+1} - I_j$

$b_n < 0$ for each n with $\cdots < b_{n+1} < b_n < \cdots < b_1 < 0$

But $c_n = a_n + \frac{1}{n} = 1 + b_n = \sum_1^n \frac{1}{j} - \ln n$

Since $b_n < 0$, $c_n < 1$ and since $a_n > 0$, $c_n > 0$

Also $b_{n+1} < b_n \Rightarrow c_n > c_{n+1}$. This means

that $(c_n)_{n \geq 1}$ is a monotonic decreasing sequence of

terms in $(0,1)$ with a limit γ called Euler's

constant. Since $\lim_{n \to \infty} \frac{1}{n} = 0$, $\gamma = \lim_{n \to \infty} a_n > 0$

because $(a_n)_{n \geq 1}$ is an increasing sequence

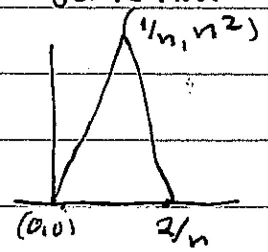
Since $0 < c_n < \cdots < c_1 = 1 + b_1 < 1$, $\gamma < 1$

Like π and e, one can show that γ is transcendental, i.e. not a root of any polynomial equation with integer coefficients. There are a variety of computer algorithms more efficient than Riemann sums to compute γ to N decimal places (as I recall, γ is between .5 and .6)

Note that $h_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ and $\ln n = I_1 + \cdots + I_{n-1}$ are partial sums of infinite series diverging to ∞

The fact that $h_n \approx \ln n + \gamma$ for large n has many combinatoric applications.

$\boxed{\#3, p.161}$ For $n \geq 1$, let $f_n : [0,1] \to (0,\infty)$ be the tent function with graph



Then $\lim_{n\to\infty} f_n(x) = 0 \quad \forall x \in (0,1]$ and

$$\int_0^1 f_n(x)\,dx = \frac{1}{2}\left(\frac{2}{n}\right)n^2 = n \to \infty \text{ as } n \to \infty$$

$\boxed{\#4 p.161}$ Let $f_n(x) = \frac{1}{\sqrt{n}} \sin n\pi x$

On $(0,1)$, $f_n \to 0$ uniformly since $|f_n(x)| \leq 1/\sqrt{n}$ for each x

However $f_n'(x) = \sqrt{n} \cos n\pi x$

with $f_n'(1/2) = \pm\sqrt{n}$ for n odd

so $(f_n')_{n\geq 1}$ doesn't converge pointwise on $[0,1]$

$\boxed{\#5, p.161}$ Let $q_1, q_2, \dots$ be an enumeration of $\mathbb{Q} \cap [0,1]$

Define $f_n(x) = \begin{cases} 1 & \text{if } x \text{ is one of } q_1, q_2, \dots, q_n \\ 0 & \text{otherwise} \end{cases}$

Each f_n is Riemann integrable with $\int_0^1 f_n(x)\,dx = 0$

But, $f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}\cap[0,1] \\ 0 & \text{otherwise} \end{cases}$

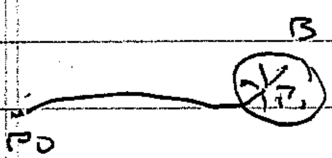
f is nowhere continuous on $[0,1]$. For any partition $\mathcal{P}$ of $[0,1]$ we can use selection rules $*$ and $*'$ for which $S(f,\mathcal{P},*) = 0$, $S(f,\mathcal{P},*') = 1$

It follows that f is not Riemann integrable on $[0,1]$

TAKE HOME EXAM SOLUTIONS

① (i) We will show that every path ~~connected~~ component S_{p_0} in a metric space $S(S, D)$ is connected. In particular, if $S = S_{p_0}$ then S is connected. Suppose R is clopen in S_{p_0} and non-empty. Since $S_{p_0} = S_{p_0^*}$ $\forall\, p_0^* \in S_{p_0}$, we can assume $p_0 \in R$. Then for any $p_1 \in S_{p_0}$ we have a connecting path $C = \{f(t) : 0 \le t \le 1\}$ joining p_0 to p_1 so $R \cap C$ is clopen in C and non-empty, giving $R \cap C = C$ and $p_1 \in R$.
$\therefore$ $R = S_{p_0}$ and S_{p_0} is connected.

(ii) Let U be open in $\mathbb{R}^n$. Fix $p_0 \in U$.
For each $p_1 \in U_{p_0}$, there is an $\varepsilon > 0$ for which $B = B_{p_1}(\varepsilon) \subset U$. Every member of B is connected to p_1 by a line segment, then connected to p_0 by a ~~path~~ composite path. Hence $B \subseteq U_{p_0}$ so U_{p_0} is open.



Then $U \setminus U_{p_0}$ is either empty or a union of path components each of which is open so $U \setminus U_{p_0}$ is open and this means U_{p_0} is clopen in U and non-empty so U connected $\Rightarrow U = U_{p_0}$ is path connected.

② Suppose $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is differentiable but not continuous at some $a \in \mathbb{R}$.
Let $\alpha = \liminf\limits_{x \to a} f'(x) \equiv \lim\limits_{\delta \to 0} \inf \left(f'(a-\delta, a+\delta) \right)$

and $\beta = \limsup_{x \to a} f'(x) = \lim_{\delta \to 0} \sup f'(a-\delta, a+\delta)$

Both of these limits are monotone limits and $\alpha \neq \beta$ since f' isn't continuous at a. From a homework problem (#7 in Assignment #7) $I_\delta = f'(a-\delta, a+\delta)$ is an interval for each δ and by the definition of α and β, $I_\delta \supseteq (\alpha, \beta)$ $\forall \delta$.

(3) (i) When $f : [a,b] \to [a,b]$ is continuous and $g(x) = f(x) - x$, $g(a) \geq 0 \geq g(b)$ so the intermediate value theorem gives $x^* \in [a,b]$ for which $g(x^*) = 0$ i.e. $f(x^*) = x^*$. If f is differentiable on (a,b) with $f'(x) \neq 1$ $\forall x$,

$f(x^{**}) = x^{**} \Rightarrow x^* - x^{**} = f(x^*) - f(x^{**})$

$= f'(x)(x^* - x^{**})$ for some x

between x^* and x^{**} so $f'(x) \neq 1$ dictates that $x^* = x^{**}$.

(ii) For $f : \mathbb{R} \to \mathbb{R}$ differentiable with $|f'(x)| < 1$ $\forall x$, the same argument as in (i) implies that f has at most one fixed point. If $c = \sup\{|f'(x)| : x \in \mathbb{R}\} < 1$, the MVT implies that f is a contraction mapping on the complete metric space $(\mathbb{R}, |\cdot|)$ so f has a fixed point by the Contraction Mapping Lemma. But when $c = 1$, f need not have a fixed point. For example,

$f(x) = \ln(1+e^x) > x = \ln(e^x)$ $\forall x$ and

$f'(x) = \frac{e^x}{1+e^x} \in (0,1)$ $\forall x$

(4) We wish to show that if $(A_i)_{i \in I}$ is an open cover of a closed bounded interval $[a,b] \subset \mathbb{R}$ then $[a,b]$ is covered by finitely many of the A_i's. Putting $J = \{x \in [a,b] : [a,x]$ is covered by finitely many of the A_i's $\}$, it's enough to show that $J = [a,b]$ for then $b \in J$. This follows by showing that J is clopen in $[a,b]$ and $\neq \emptyset$. Obviously $a \in J$ so $J \neq \emptyset$. For any $x \in [a,b]$ $x \in A_{i_0}$ for some index i_0 and then $\exists \varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subset A_{i_0}$. If $x' \in J$ for some $x' \in (x-\varepsilon, x]$, we deduce

that each $x'' \in [x, x+\varepsilon)$ is in J from

$$[a,x''] = [a,x'] \cup (x',x''] \subseteq [a,x'] \cup A_{i_0}$$

If $x \in J$, $(x-\varepsilon, x+\varepsilon) \cap [a,b]$ is an open neighbourhood of x in $[a,b]$ so J is open in $[a,b]$. If $x \notin J$, no member of $(x-\varepsilon, x+\varepsilon)$ can be in J so $[a,b] \setminus J$ is also open in $[a,b]$ and J is then clopen in $[a,b]$.

(5) Let $S = [a,b] \times [c,d]$ and $f : S \to \mathbb{R}$ a function for which $g(x,y) = \frac{\partial f}{\partial y}(x,y)$ exists for all $(x,y) \in S^0$ and $\frac{\partial g}{\partial x}(x,y)$ exists and is 0 $\forall (x,y) \in S^0$. Put $h_x(y) = f(x,y) - f(a,y)$. By two applications of the MVT, $(f(x,y) - f(a,y)) - (f(x,b) - f(a,b))$

$$= h_x(y) - h_x(b) = \left(\frac{\partial f}{\partial y}(x,y^*) - \frac{\partial f}{\partial y}(a,y^*)\right)(y-b)$$

$= \left(g(x,y^*) - g(a,y^*) \right) (y-b)$

$= \frac{\partial g}{\partial x}(x^*,y^*)(x-a)(y-b) \quad = 0$

(with y^* between y and b, x^* between x and a)

Hence $\forall x,y \in S$

$f(x,y) = \varphi(y) + \psi(x)$

with $\varphi(y) = f(a,y)$ differentiable

and $\psi(x) = f(x,b) - f(a,b)$ possibly not even continuous

(6) We assume that $(f_n)_{n\geq 1}$ is a sequence of monotonic increasing functions from $[a,b]$ to $\mathbb{R}$ and that $f(x) = \lim_{n\to\infty} f_n(x)$ exists and is finite $\forall x \in [a,b]$. Then f is monotonic increasing since $y \geq x$

$\Rightarrow f(y) = \lim_{n\to\infty} f_n(y) \geq \lim_{n\to\infty} f_n(x) = f(x)$

Now assume f is continuous. Then f is uniformly continuous and for each $\varepsilon > 0$ we can choose a partition $a = x_0 < x_1 < \dots < x_N = b$ of $[a,b]$ for which $x \in [x_{i-1}, x_i] \Rightarrow |f(x) - f(x_{i-1})| < \varepsilon/3$

Next choose $n_0 = n_0(\varepsilon)$ s.t. $n \geq n_0$

$\Rightarrow |f_n(x_i) - f(x_i)| < \varepsilon/3$ for $0 \leq i \leq N$

Claim: $n \geq n_0(\varepsilon) \Rightarrow |f(x) - f_n(x)| < \varepsilon \quad \forall x \in [a,b]$

so $f_n \to f$ [unif] on $[a,b]$

To see this, for $x \in [a,b]$ there is some i for which $x \in [x_{i-1}, x_i]$. We then have

$f(x_{i-1}) \leq f(x) \leq f(x_i)$

$f(x_{i-1}) - \varepsilon/3 \leq f_n(x_{i-1}) \leq f_n(x) \leq f_n(x_i) \leq f(x_i) + \varepsilon/3$

so $|f_n(x) - f(x)| \leq 2\varepsilon/3 + |f(x_i) - f(x_{i-1})| < \varepsilon$

(7) We suppose that $(f_n)_{n\geq 1}$ is a sequence of functions from (E_1, d_1) to (E_2, d_2) which converges uniformly on E_1 to a function $f: E_1 \to E_2$. Thus $d_\infty(f_n, f)$

$$= \sup_{x\in E_1} d_2(f_n(x), f(x)) \longrightarrow 0 \text{ as } n\to\infty$$

We also suppose $g: (E_2, d_2) \to (E_3, d_3)$ is continuous

(i) When E_2 is compact, g is uniformly continuous so $\forall \varepsilon > 0 \;\exists\, \delta > 0$ s.t. $d_2(y, y') < \delta \Rightarrow d_3(g(y), g(y')) < \varepsilon$. We can then pick n_0 s.t. $d_\infty(f_n, f) < \delta \;\forall n \geq n_0$ and obtain $d_\infty(g\circ f_n, g\circ f) < \varepsilon \;\forall n \geq n_0$

so $g\circ f_n \longrightarrow g\circ f$ uniformly on E_1

(ii) For $E_1 = E_2 = E_3 = \mathbb{R}$ and $f_n(x) = x + 1/n$, $f(x) = x$, $g(x) = x^2$, $\|f_n - f\|_\infty = 1/n \to 0$ as $n\to\infty$ but $(g\circ f_n)(x) - (g\circ f)(x)$

$$= (x + 1/n)^2 - x^2 = \frac{2}{n}x + 1/n^2 \geq 2n \text{ when } x \geq n$$

so $g\circ f_n \not\to g\circ f$ [unif] on $\mathbb{R}$

(8) (i) For $f_n(x) = e^{x^2/n}$, and $f(x) = 1$

$\lim_{n\to\infty} f_n(x) = f(x) \quad \forall x \in \mathbb{R}$ since $x^2/n \to 0$ as $n\to\infty$

On any closed bounded interval $[a,b]$ each f_n is monotonic if a and b have the same sign and otherwise is monotonic decreasing on $[a,0]$ and monotonic increasing on $[0,b]$

$\therefore\; f_n(x) - 1 \leq \max(f_n(a) - 1, f_n(b) - 1) \quad \forall x \in [a,b]$

and $f_n \to f$ [unif] on $[a,b]$. On any unbounded interval I, each f_n is unbounded so $f_n \not\to f$ [unif] on I.

(ii) With $f_n(x) = x(1-x)^n$, $f_n(0) = 0$ and

$\lim_{n\to\infty} f_n(x)$ doesn't exist for $x < 0$ or $x \geq 2$

For $0 < x < 2$ and $c = |1-x|$, $c < 1$ so $1/c > 1$ and

$\lim_{n\to\infty} |f_n(x)| = x \lim_{n\to\infty} \frac{n}{(1/c)^n} = 0$

by L'Hôpital's Rule. For each n,

$f_n'(x) = n(1-x)^{n-1}((1-x) - nx)$ is 0 for $x = \frac{1}{n+1}$

and < 0 for $x > 1/n+1$. Since

$\lim_{n\to\infty} f_n\left(\frac{1}{n+1}\right) = \lim_{n\to\infty}\left(1 - 1/n+1\right)^{n+1} = 1/e$

$f_n \not\to f = 0$ unif on $[0,a]$ for any $a > 0$

For $0 < a < b < 2$, $f_n \to 0$ [unif] on a, b

since $\max_{x\in[a,b]} |f_n(x)| = \max_{n\to\infty} (f_n(a), f_n(b)) \to 0$ as $n\to\infty$

(iii) $f_n(x) = \sum_{k=1}^{n} x^k(1-x) = x(1-x)(1+x+\dots+x^{n-1})$

$= x(1-x^n) = x - x^{n+1}$

With $f(x) = x$, $f_n \to f$ [unif] on $[a,b]$

for $-1 < a < b < 1$ since $\max_{x\in[a,b]} |f_n(x) - f(x)|$

$= \max(|a|^{n+1}, |b|^{n+1}) \to 0$ as $n\to\infty$

$\lim_{n\to\infty} f_n(x)$ doesn't exist for $x \leq -1$ or $x > 1$

Although $f_n(1) = 0$ $\forall n$, $f_n \not\to f$ (unif) on $[b,1]$ since putting $f(1) = 0$ gives a function f not continuous at 0

with $\sup_{b\leq x\leq 1} |f_n(x) - f(x)| = 1$ $\forall n$

(9) We suppose that $P(t) = (x_1(t), \dots, x_n(t))$ is differentiable from an open interval I into $\mathbb{R}^n$ with $x_1'(t) \neq 0$ $\forall t \in I$

(i) Then $t \mapsto x_1(t)$ is 1-1 by the MVT & x_1' has constant sign so x_1 maps I strictly monotonically open

~~onto~~ onto an interval $x_1(I)$. The inverse function

$x_1^{-1}: x_1(I) \to I$ is differentiable

since for $y_0 = x_1(t_0)$

$$(x_1^{-1})'(y_0) = \lim_{y\to y_0} \frac{x_1^{-1}(y) - x_1^{-1}(y_0)}{y - y_0} \underset{y=x_1(t)}{=} \lim_{t\to t_0} \frac{t-t_0}{x_1(t)-x_1(t_0)} = \frac{1}{x_1'(t_0)}$$

(ii) $p(t_1) = p(t_2) \Rightarrow x_1(t_1) = x_1(t_2) \Rightarrow t_1 = t_2$

so p is 1-1 from I onto $C = p(I) \subseteq \mathbb{R}^n$

There are many ways to show that p^{-1} is continuous

Method 1: $\pi(x_1, \dots, x_n)$ is continuous

with $p^{-1}(p(t)) = (x_1^{-1} \circ \pi)(p(t))$

so $p^{-1} = x_1^{-1} \circ \pi$ is the composition of continuous functions

Method 2: For each $t_0 \in I$ and $\delta > 0$,

$\mathcal{U} = \{(x_1, \dots, x_n) : x_1 \in (x_1(t_0-\delta), x_1(t_0+\delta))\}$ is open

in $\mathbb{R}^n$. For $t \notin (t_0-\delta, t_0+\delta)$, $x_1(t) \notin (x_1(t_0-\delta), x(t_0+\delta))$

so $p(t) \in \mathcal{U} \iff$ ~~$t_0-\delta$~~ $t_0 - \delta < t < t_0+\delta$

$\mathcal{U} \cap C$ is then open in C and is the image of

$(t_0-\delta, t_0+\delta)$ under $t \mapsto p(t)$. This means p maps

open sets in $\mathbb{R}$ to open sets in C

Given that p^{-1} is continuous, ~~for~~ $q : J \to \mathbb{R}^n$

with $q(J) = p(I)$, $\varphi = p^{-1} \circ q : J \to I$

with $q(s) = p(\varphi(s))$ and q is cont.

$\iff \varphi$ is cont

(iii) For q as in (ii) with $q(s) = (y_1(s), \dots, y_n(s))$

$= p(\varphi(s)) = (x_1(\varphi(s)), \dots, x_n\,\varphi(s))$, $\varphi = x_1^{-1} \circ y_1$

so q differentiable $\Rightarrow s \mapsto y_1(s)$ differentiable

$\Rightarrow \varphi$ is differentiable. Vice versa, φ differentiable

$\Rightarrow q$ differentiable with $q'(s) = p'(\varphi(s))\,\varphi'(s)$

so $q'(s)$ is a positive multiple of $p'(\varphi(s)) \iff \varphi'(s) > 0 \ \forall s$

(10) Notations:

(*) $(A, V, \langle -,- \rangle^{1/2})$ is an n-dimensional Euclidean space with $p_0 \in A$ a fixed reference point

(*) For $\vec{e} \in V$ with $\|e\| = 1$,
$\mathrm{Ray}(\vec{e}) = \{p_0 + t\vec{e} : t > 0\}$ is the ray emanating from p_0 in the direction $\vec{e}$

(*) For $p \in A \setminus \{p_0\}$, $r(p) = \|p - p_0\|$ is the Euclidean distance from p_0 to p and
$\vec{e}(p) = \frac{p - p_0}{\|p - p_0\|}$ is the unit vector in the direction of p

(*) For $r_0 > 0$, $S(p_0, r) = \{p : r(p) = r_0\}$ is the $n-1$ dimensional sphere of radius r about p_0

Properties of $F(p) = p_0 + f(r(p))\,(p - p_0)$
$$= p_0 + r(p)\, f(r(p))\, \vec{E}(p)$$
for any differentiable function $f : (0,\infty) \to (0,\infty)$

(i) $F(\mathrm{Ray}(\vec{e})) \subseteq \mathrm{Ray}(\vec{e})$ for all unit vectors $\vec{e}$
(this is the reason why F is said to be radial)

This follows from $F(p_0 + t\vec{e}) = p_0 + tf(t)\,\vec{e}$
and we note in passing that F is 1-1 from $A \setminus \{p_0\}$ onto $A \setminus \{p_0\}$ $\Leftrightarrow$ $t\,f(t)$ is 1-1 onto.

As a special case, for $f(r) = 1/r^2$ and $R \equiv F$
$R(p_0 + t\vec{e}) = p_0 + 1/t\, \vec{e}$. R is an inversion in the sense that $R \circ R$ = identity map and R maps the part of $\mathrm{Ray}(\vec{e})$ inside $S(p_0, 1)$ to the part of $\mathrm{Ray}(\vec{e})$ outside $S(p_0, 1)$ and vice versa

(ii) Using differentiability of $p \mapsto r(p)$ and f, F can be described as a composition of differentiable functions so F is differentiable at each $p \in A \setminus \{p_0\}$. Since V is the direct sum of the line $\mathbb{R}\,\vec{e}(p)$ and the $(n-1)$ dimensional subspace $(e(\vec{p}))^{\perp}$ consisting vectors ~~perpect~~ perpendicular to $\overrightarrow{e(p)}$, $(dF)_p$ is uniquely determined by $(dF)_p(\overrightarrow{e(p)})$ and the restriction of $(dF)_p$ to $(\overrightarrow{e(p)})^{\perp}$.

For $v \in (\overrightarrow{e(p)})^{\perp}$, $r(p+tv) = (r(p)^2 + t^2\|v\|^2)^{1/2}$ has derivative equal to 0 at $t=0$ so the Chain Rule gives

$$(dF)_p(v) = \left(\frac{d}{dt}\right)_{t=0} F(p+tv) = \left(\frac{d}{dt}\right)_{t=0}\{f(r(p+tv))(p-p_0+tv)\}$$
$$= f(r(p))\,v$$

On the other hand, $r(p + t\overrightarrow{e(p)}) = r(p) + t$ for t near 0 and

$$(dF)_p(\overrightarrow{e(p)}) = \left(\frac{d}{dt}\right)_{t=0}\{(r(p)+t)f(r(p)+t)\,\overrightarrow{e(p)}\}$$
$$= \{f(r(p)) + r(p)f'(r(p))\}\,\overrightarrow{e(p)}$$
$$= g'(r(p))\,\overrightarrow{e(p)}$$

for $g(r) = r f(r)$. Hence $(dF)_p$ is invertible $\iff$ $g'(r(p)) \neq 0$ [Indeed $\det (dF)_p = f(r(p))^{n-1} g'(r(p))$]

For $p \in S(p_0,1)$ and $f(1)=1$, $(dF)_p(v) = v$ for all $v \in (\overrightarrow{e(p)})^{\perp}$ and $(dF)_p(\overrightarrow{e(p)}) = g'(1)\,\overrightarrow{e(p)}$.

In the special case when $f(r) = 1/r^2$, $g'(r) = -1/r^2$ and for $p \in S(p_0,1)$, $(dR)_p(v + t\overrightarrow{e(p)}) = v - t\overrightarrow{e(p)}$ so $(dR)_p^2$ = identity map. This also follows without calculation of $(dR)_p$ from $R \circ R$ = identity on A so, by the Chain Rule, identity map on $V = d(R \circ R)_p$

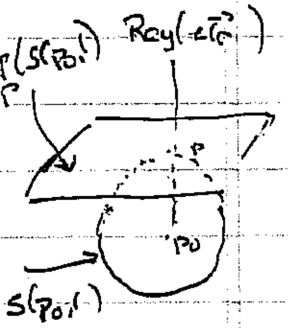
$$= (dR)_{R(p)}(dR)_p = (dR)_p^2 \quad \text{since } R(p) = p$$

(iii) <u>Geometrical Interpretation</u>: For any differentiable function φ from an interval I into $S(p_0, r_0) \subseteq A$

$$0 = (d/dt)\, \underbrace{\|\varphi(t) - p_0\|^2}_{r_0^2} = \langle \varphi'(t), \varphi(t) - p_0 \rangle .$$

With $\varphi(t_0) = p$, each $q = p + t\varphi'(t_0)$ on the tangent line at p to $\varphi(I)$ satisfies $\langle q - p, p - p_0 \rangle$ $= t\langle \varphi'(t_0), p - p_0 \rangle = 0$ so $q - p \in (e(p))^{\perp}$

Thus $T_p(S(p_0, r_0)) =$ union of these tangent lines $\equiv$ tangent space at p to $S(p_0, r_0) = \{ q = p + v : v \in (e(p))^{\perp} \}$



For F as above, $(\mathrm{Aff}\, F)_p(q) = F(p) + (dF)_p(q-p)$ is the affine approximation to $F(q)$. For $p \in S(p_0,1)$ with $f(1) = 1$, $F(p) = p$ and $(dF)_p(q - p) = q - p$ so $(\mathrm{Aff}\, F)_p$ fixes each $q \in T_p(S(p_0,1))$

In the special case $F = R$, every $\tilde{p} \in A \setminus \{p_0\}$ has the unique description $\tilde{p} = q + t\vec{e}_p$ for some $q \in T_p(S(p_0,1))$ and some $t \in \mathbb{R}$

$$\text{Then } (\mathrm{Aff}\, R)_p(\tilde{p}) = p + (dR)_p(q - p + t\, e(p)) = q - t\vec{e}_p$$

is the orthogonal reflection of $\tilde{p}$ through $T_p(S(p_0,1))$

<u>Remark</u>: In mathematical physics, the most useful functions from A to $\mathbb{R}$, A to A, or A to V are those which are radial relative to some p_0, i.e., in the notation above, functions $p \mapsto f(r(p))$, $p \mapsto F(p) = p_0 + f(r(p))(p - p_0)$, or $p \mapsto F(p) - p_0$. Properties of radial functions like those described above are very important