SOLUTIONS – HOMEWORK # 8

Assignment: Chapter 9, pp 212–214, # 2, 5, 9, 10, 11, 16, 20

(#2) Let $f(x,y) = (x^2+y^2)^\alpha$. For $\alpha \le 1/2$ $\frac{\partial f}{\partial x}(0,0)$ doesn't exist so f is not differentiable at $(0,0)$. For $\alpha > 1/2$ f is continuously differentiable on $\mathbb{R}^2$ since $\frac{\partial f}{\partial x}(x,y) = 2x\,(x^2+y^2)^{\alpha-1}$, $\frac{\partial f}{\partial y}(x,y) = 2y\,(x^2+y^2)^{\alpha-1}$ are continuous on $\mathbb{R}^2 - \{0,0\}$, and

$$\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial x}(x,y) = \lim_{(x,y)\to(0,0)} \frac{2x}{\sqrt{x^2+y^2}}\,(x^2+y^2)^{\alpha-1/2} = 0$$

$$= \lim_{x\to 0} \frac{(x^2)^\alpha}{x} = \frac{\partial f}{\partial x}(0,0)$$

with similar results for $\frac{\partial f}{\partial y}$.

(5) Let $f : \mathcal{U}^{\text{open}} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$

(i) If there are continuous functions $A_1, \dots, A_n$ on $\mathcal{U}\times\mathcal{U}$ for which

$$(*)\quad f(x) - f(y) = \sum_{i=1}^{n} A_i(x,y)(x_i - y_i) \quad \forall x,y \in \mathcal{U},$$

it follows easily that $\frac{\partial f}{\partial x_i}(x)$ exists and is equal to the continuous function $A_i(x,x)$ for each i so f is continuously differentiable on $\mathcal{U}$

(ii) When $\underbrace{\mathcal{U}}_{f \text{ is } C^1 \text{ on } \mathcal{U} \text{ and}}$ is convex, there is a natural and useful choice of continuous functions A_i on $\mathcal{U}\times\mathcal{U}$ satisfying $(*)$, namely

$$A_i(x,y) = \int_0^1 \frac{\partial f}{\partial x_i}(y + t(x-y))\,dt$$ indeed

$$f(x) - f(y) = \int_0^1 \frac{d}{dt}\big(f(y + t(x-y))\big)\,dt$$

$$= \sum_1^n A_i(x,y)(x_i - y_i)$$ by the

chain rule and an easy argument using uniform continuity of $\frac{\partial f}{\partial x_i}$ on compact sets makes A_i continuous on $U \times U$.

In general, for $x \neq y$, $\tilde{A}_i(x,y) = \frac{f(x) - f(y)}{\|x-y\|_2^2}(x_i - y_i)$ gives ~~a~~ continuous functions on $\{(x,y) \in U \times U : x \neq y\}$ satisfying $\sum \tilde{A}_i(x,y)(x_i - y_i)$

$$= \big(f(x) - f(y)\big)\frac{\sum (x_i - y_i)^2}{\|x-y\|_2^2} = f(x) - f(y)$$

One can then concoct a weighted average of $\int_0^1 \frac{\partial f}{\partial x_i}(y + t(x-y))\,dt$ for $\|x-y\| \leq 2\varepsilon$ and $\tilde{A}_i(x,y)$ for $\|x-y\| > \varepsilon$ to come up with A_i's satisfying (*)

This is pointless and the details are unimportant since for a non-convex U, there's no useful choice of A_i's satisfying (*) and any particular choice of A_i's can always be replaced by $A_i + B_i$ for infinitely many choices of B_i's satisfying $\sum B_i(x,y)(x_i - y_i) = 0$ $\forall x, y$. Such B_i's have nothing to do with f.

The upshot is that this problem is badly phrased — it should have either specified U to be convex or asked only for A_i's continuous at points (x,x) with a construction like that on p.196.

(9) $\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial y}\right)\left(\frac{1}{\sqrt{y}}\, e^{-x^2/4y}\right)$

$$= e^{-x^2/4y}\left\{\frac{1}{\sqrt{y}}\left(-\frac{2x}{4y}\right)^2 - \frac{2}{(\sqrt{y})\,4y} - \frac{1}{2}\,\frac{1}{y^{3/2}} + \left(\frac{x^2}{4y^2}\right)\frac{1}{\sqrt{y}}\right\} = 0$$

$\forall\ (x,y) \in \mathbb{R}\times(0,\infty)$. Thus by Problem 7 and the Chain Rule

for each $f:[a,b]\to\mathbb{R}$ continuous,

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial y}\right)\left\{\int_a^b f(t)\ \frac{1}{\sqrt{y}}\, e^{-(x-t)^2/4y}\,dt\right\}$$

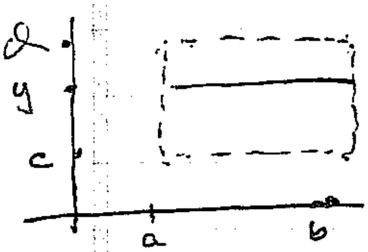
$$= \int_a^b f(t)\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial y}\right)\left(\frac{1}{\sqrt{y}}\, e^{-(x-t)^2/4y}\right)dt = 0$$

The notation in #9 is non-standard. For $x = (x_1,\dots,x_n) \in \mathbb{R}^n$ and $\|x\|^2 = \|x\|_2^2 = \sum_1^n x_i^2$ the function $G(x,t) = \frac{1}{\sqrt{t}}\, e^{-\|x\|^2/4t}$ on $\mathbb{R}^n\times\mathbb{R}^+$ is the fundamental solution of the n-dim'l heat equation $\left(\nabla^2 - \frac{\partial}{\partial t}\right)u = 0$

and, under mild hypotheses, one can prove that every solution u is generated by G via $u(x,t) = \int_{\mathbb{R}^n} f(y)\, G(x-y,t)\,dy$ for a suitable f

See books on mathematical physics for details

(10)



Suppose $f : (a,b)\times(c,d) \to \mathbb{R}$ is continuously differentiable and that $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$ exists and is 0 at each point. Then, for $c < y < d$, $x \mapsto \frac{\partial f}{\partial y}(x,y)$ is constant on (a,b). Let $g(y)$ be the constant value.

For any choice of $y_0 \in (c,d)$, the Fundamental Theorem of Calculus gives

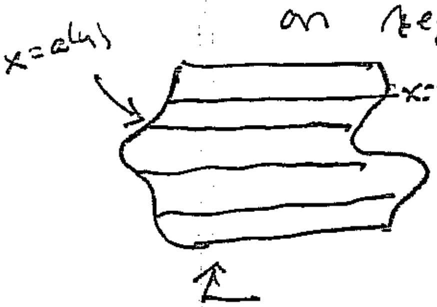
$$f(x,y) - f(x,y_0) = \int_{y_0}^{y} \frac{\partial f}{\partial y}(x,u)\,du = \int_{y_0}^{y} g(u)\,du \equiv f_2(y)$$

so

$$f(x,y) = f_1(x) + f_2(y) \quad \text{with } f_1(x) = f(x,y_0)$$

Obviously f_1 and f_2 are C^1 functions.

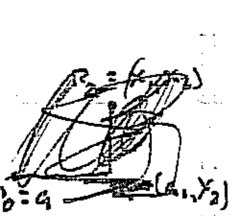
Caution: In general, when $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 0$ one can conclude that $f(x,y) = f_1(x) + f_2(y)$ only on regions of the form



$$\mathcal{U} = \bigcup_{y\in(c,d)} \{(x,y) : a(y) < x < b(y)\}$$
$=$ union of horizontal line segments

The proof is the same as above

(11) We suppose $f = (f_1,\dots,f_n)$ is a C^1 vector field on a ball $\mathcal{U} \subseteq \mathbb{R}^n$ with $\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$ and wish to find F for which $f = \nabla F$, i.e. $\frac{\partial F}{\partial x_j} = f_j$ for each j

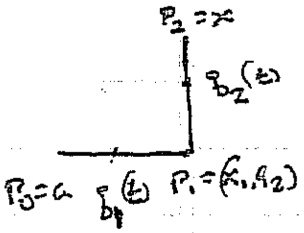


Let $a = (a_1,\dots,a_n)$ be the center of the ball

As in the theorem on continuous partials implying differentiability, for $x = (x_1,\dots,x_n) \in \mathcal{U}$, we let

$P_i = (x_1,\dots,x_{i-1},x_i,a_{i+1},\dots,a_n)$ for $0 \le i \le n$

and parametrize the line segment joining P_{i-1} and P_i



by $g_i(t) = (x_1, \dots, x_{i-1}, t, a_{i+1}, \dots, a_n)$, $a_i \le t \le x_i$

Define $F(x) = \sum_{i=1}^{n} \int_{a_i}^{x_i} f_i(g_i(t))\,dt$

Then $$\frac{\partial F}{\partial x_j} = f_j(P_j) + \sum_{i=j+1}^{n} \int_{a_i}^{x_i} \frac{\partial f_i}{\partial x_j}(g_i(t))\,dt$$

$$= f_j(P_j) + \sum_{i=j+1}^{n} \int_{a_i}^{x_i} \frac{\partial f_j}{\partial x_i}(g_i(t))\,dt$$

(Fund Thm of Calc) $$= f_j(P_j) + \sum_{i=j+1}^{n} f_j(P_i) - f_j(P_{i-1})$$

$$= f_j(P_n) = f_j(x)$$

[with more effort, a variation of this proof works for U any convex open set]

(1b) Let $f : U^{\text{open}} \subseteq \mathbb{R}^n \to \mathbb{R}$ be a C^2 function with $a = (a_1, \dots, a_n)$ a critical point for f i.e. $\partial f / \partial x_i (a) = 0$ for $1 \le i \le n$. Then the 2^{nd} order Taylor expansion for f about a gives

$$f(x) - f(a) = \frac{1}{2}(x-a)' H (x-a) + \|x-a\|^2 E(x)$$

where $E(x) \to 0$ as $x \to a$ and H = Hessian matrix of f at a = $n \times n$ symmetric matrix with i,j entry $\frac{\partial^2 f}{\partial x_i \partial x_j}(a)$

Let m_1 and m_2 be the minimum and maximum values of the continuous function $v \mapsto v' H v$ on the compact set $S = \{v \in \mathbb{R}^n : \|v\| = 1\}$. Choose $r_0 > 0$ for which $|E(x)| < \frac{1}{2}\min(|m_1|, |m_2|)$ when $\|x - a\| < r_0$ (assuming m_1 and m_2 are non-zero)

Then, for $\|x-a\| < r_0$,

$$\left(\tfrac{1}{2} m_1 + E(x)\right)\|x-a\|^2 \leq f(x)-f(a) \leq \left(\tfrac{1}{2} m_2 + E(x)\right)\|x-a\|^2$$

Cases: (i) $m_2 \geq m_1 > 0$ (H positive definite)

$\Rightarrow$ $f(x) - f(a) > 0$ for $0 < \|x-a\| < r_0$

$\Rightarrow$ f has a local minimum at a

(ii) $m_1 \leq m_2 < 0$ (H negative definite)

$\Rightarrow$ $f(x) - f(a) < 0$ for $0 < \|x-a\| < r_0$

$\Rightarrow$ f has a local maximum at a

(iii) $m_1 < 0 < m_2$ (H neither positive semi-definite nor negative semi-definite)

With u_i a unit vector for which $u_i' H u_i = m_i$ and $0 < |t| < r_0$, $f(a+tu_1) > f(a) > f(a+tu_2)$

So f has neither a local maximum nor a local minimum at a (a is a saddle point for f)

(iv) m_1 or $m_2 = 0$. Here the test fails — H doesn't detect whether or not f has a local max/min at a.

(20) We assume that S is a closed convex set in $\mathbb{R}^n$ with f a C^1 function from an open set containing S into $\mathbb{R}^n$ and $f(S) \subseteq S$. Then, for x, y in S

$$f(y) - f(x) = \int_0^1 \frac{d}{dt} f(x + t(y-x))\, dt$$

$$= \int_0^1 (df)_{x+t(y-x)} (y-x)\, dt$$

As we'll show later, for any norm $\|\cdot\|$ on $\mathbb{R}^n$, we have

$$\| f(y) - f(x) \| \le \int_0^1 \| (df)_{x+t(y-x)} (y-x) \| \, dt$$

For each z, $\| (df)_z (v) \| \le K_z \|v\|$ $\forall v \in \mathbb{R}^n$ for K_z the maximum value of $\| (df)_z (v) \|$ on the compact set $S = \{ v : \|v\| = 1 \}$. Hence, for $c = \max_{z \in S} K_z$

$$\| f(y) - f(x) \| \le \int_0^1 c \|y-x\| \, dt = c \|y-x\|$$

When $c < 1$, f is a contraction mapping on the complete metric space $(S, \|\cdot\|)$

(i) Using $\|v\| = \|v\|_\infty = \max_{1 \le i \le n} |v_i|$, $\| (df)_z (v) \|_\infty = \max_{1 \le i \le n} | (df_i)_z (v) |$

$$= \max_{1 \le i \le n} \left| \sum_{j=1}^n \frac{\partial f_i}{\partial x_j} (z) v_j \right|$$

$$\le \left(\max_{1 \le i \le n} \sum_j \left| \frac{\partial f_i}{\partial x_j} (z) \right| \right) \|v\|_\infty$$

So we get a contraction mapping when $\sum_j \left| \frac{\partial f_i}{\partial x_j} \right| \le c < 1$ $\forall i$

(ii) Using $\|v\| = \|v\|_1 = \sum_1^n |v_i|$

$$\| (df)_z (v) \|_1 \le \sum_{i=1}^n \sum_{j=1}^n \left| \frac{\partial f_i}{\partial x_j} (z) \right| |v_j|$$

$$\le \left(n \max_{1 \le i,j \le n} \left| \frac{\partial f_i}{\partial x_j} (z) \right| \right) \|v\|_1$$

(iii) Using $\|v\| = \|v\|_2$ and Cauchy - Schwartz

$$\| (df)_z (v) \|_2^2 = \sum_{i=1}^n \left| \sum_{j=1}^n \frac{\partial f_i}{\partial x_j} (z) v_j \right|^2$$

$$\le \sum_{i=1}^n \sum_{j=1}^n \left(\frac{\partial f_i}{\partial x_j} (z) \right)^2 \|v\|^2$$

So we get a contraction mapping when $\sum_{i,j} \left(\frac{\partial f_i}{\partial x_j} (z) \right)^2 \le c^2 < 1$ $\forall z$