Math 450 - Homework 2 - Solutions (Exercises from Lecture Notes 2)

1. **Exercise 4.1** To count the number of ways to get a 9, I'll count how many triples (n_1, n_2, n_3) of numbers from 1 to 6 such that $n_1 \leq n_2 \leq n_3$ and $n_1 + n_2 + n_3 = 9$, and then the number of permutations leading to different triples. Here they are:

$$\begin{array}{ll} (1 \quad 2 \quad 6) & \#\text{ of permutations: } 6 \\ (1 \quad 3 \quad 5) & \#\text{ of permutations: } 6 \\ (1 \quad 4 \quad 4) & \#\text{ of permutations: } 3 \\ (2 \quad 2 \quad 5) & \#\text{ of permutations: } 3 \\ (2 \quad 3 \quad 4) & \#\text{ of permutations: } 6 \\ (3 \quad 3 \quad 3) & \#\text{ of permutations: } 1 \end{array}$$

This gives a total of 25 triples. So the probability of obtaining a 9 is $25/216 = 0.1157$. Doing the same thing for a 10 gives:

$$\begin{array}{ll} (1 \quad 3 \quad 6) & \#\text{ of permutations: } 6 \\ (1 \quad 4 \quad 5) & \#\text{ of permutations: } 6 \\ (2 \quad 2 \quad 6) & \#\text{ of permutations: } 3 \\ (2 \quad 3 \quad 5) & \#\text{ of permutations: } 6 \\ (2 \quad 4 \quad 4) & \#\text{ of permutations: } 3 \\ (3 \quad 3 \quad 4) & \#\text{ of permutations: } 3 \end{array}$$

This gives a total of 27 triples. So the probability of obtaining a 10 is $27/216 = 0.1250$. This shows that it is more likely to obtain a 10 than a 9.

I use below the `samplefromp(p,n)` program of lecture notes 1.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
rand('seed',121)
p=[1/6 1/6 1/6 1/6 1/6 1/6];
x1=samplefromp(p,50000);
x2=samplefromp(p,50000);
x3=samplefromp(p,50000);
s=x1+x2+x3;
fraction_of_09=sum(s== 9)/50000
```

```
fraction_of_10=sum(s==10)/50000
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

This gives the proportions: 0.1149 of 9s and 0.1246 of 10s.

2. **Exercise 4.2** The experiment consists of tossing 3 coins. The sample space is $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$. We designate the various events by the sets below.

   Exactly two heads: $A = \{HHT, HTH, THH\}$

   The first outcome is a head: $E_1 = \{HHH, HHT, HTH, HTT\}$

   The first outcome is a tail: $E_2 = \{THH, THT, TTH, TTT\}$

   The first two outcomes are heads: $E_3 = \{HHH, HHT\}$

   The first two outcomes are tails: $E_4 = \{TTH, TTT\}$

   The first and third outcomes are heads: $E_5 = \{HHH, HTH\}$

   Now recall the definition:

   $$P(A|E_i) = \frac{P(A \cap E_i)}{P(E_i)}.$$

   It is now straightforward to obtain:

   $P(A|E_1) = \frac{P(HHT,HTH)}{P(HHH,HHT,HTH,HTT)} = 1/2.$

   $P(A|E_2) = \frac{P(THH)}{P(THH,THT,TTH,TTT)} = 1/4.$

   $P(A|E_3) = \frac{P(HHT)}{P(HHH,HHT)} = 1/2.$

   $P(A|E_4) = \frac{P(\emptyset)}{P(TTH,TTT)} = 0.$

   $P(A|E_5) = \frac{P(HTH)}{P(HHH,HTH)} = 1/2.$

3. **Exercise 4.5** We consider quadratic equations $ax^2 + bx + c = 0$ with random coefficients a, b, c so that a and c are uniform random variables over $[0, 1]$ and b is a uniform random variables over $[0, 2]$. So the sample space is $S = [0, 1] \times [0, 2] \times [0, 1]$. The set of points corresponding to equations with real roots are:

   $$E = \{(a, b, c) : b \geq 2\sqrt{ac}\}.$$

   Thus we need to find $P(E)$. I claim that the probability is 5/9 of having only real roots. We can see this as follows.

$$
\begin{aligned}
P(E) &= \frac{1}{2}\iiint_E da\,db\,dc \\
&= \frac{1}{2}\int_0^1\int_0^1 (2-2\sqrt{ac})da\,dc \\
&= 1-\int_0^1 a^{\frac{1}{2}}\,da\int_0^1 c^{\frac{1}{2}}\,dc \\
&= 1-(2/3)^2 \\
&= 5/9.
\end{aligned}
$$

We can simulate choosing at random 50000 quadratic equations and testing whether $b \geq 4\sqrt{ac}$ as follows.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
rand('seed',121);
x=rand(50000,3);      %a, b, c correspond to columns 1, 2, 3 of x
x(:,2)=2*x(:,2);      %multiply second column by 2
y=sum(x(:,2)>=2*sqrt(x(:,1).*x(:,3)))/50000
%y gives proportion of the 50000 with non-negative discriminant
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

This gave me the value 0.5559, which should be compared with the exact value $5/9 = 0.5556$.

4. **Exercise 4.11** A certain disease is present in about one out of 10000 persons. We denote by D the event that a person has the disease, no other information being given. So $P(D) = 1/10000$. The clinical test gives a positive reading 98% of the time given that a person has the disease. If R represents the event that the test has a positive reading, then $P(R|D) = 0.98$. Similarly, it is given that $P(R|\overline{D}) = 0.01$, where $\overline{D}$ represents the event that a person is healthy. We wish to find $P(D|R)$, the probability that a person has the disease given a positive test reading. By Bayes theorem,

$$
\begin{aligned}
P(D|R) &= \frac{P(R|D)P(D)}{P(R|D)P(D)+P(R|\overline{D})P(\overline{D})} \\
&= \frac{0.98\times 0.0001}{0.98\times 0.0001+0.01\times 0.9999} \\
&= 0.0097
\end{aligned}
$$

This means that a positive reading implies a less than 1% chance that the person actually has the disease. Note that, although the test is fairly reliable with a 98% accuracy in detecting the disease and a 1% chance of a false positive reading, the probability of actually having the disease having tested positive for it is still small due to the fact that the disease is fairly rare.

5. **Exercise 4.14** The PDF of an exponentially distributed random variable with parameter λ is

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

The convolution integral is

$$(f * f)(x) = \int_{-\infty}^{\infty} f(x-s) f(s) ds.$$

Note that $f(s)$ is 0 for $s \leq 0$ and $f(x-s)$ is 0 for $s \geq x$. Thus we only need to integrate from 0 to x. So we have:

$$\begin{aligned} (f * f)(x) &= \int_0^x \left(\lambda e^{-\lambda(x-s)}\right) \left(\lambda e^{-\lambda s}\right) ds \\ &= \lambda^2 e^{-\lambda x} \int_0^x ds \\ &= \lambda^2 x e^{-\lambda x}. \end{aligned}$$

Therefore,

$$(f * f)(x) = \lambda^2 x e^{-\lambda x}.$$