Math 450 - Homework 3
Solutions

1. **Exercise 6.2, LN03.** Let X be a random variable with mean m and standard deviation σ. Chebyshev's inequality reads:

$$P(|X - m| \geq K\sigma) \leq \frac{1}{K^2}.$$

Therefore, the probability that X lies on the interval $(m - 5\sigma, m + 5\sigma)$ is

$$P(|X - m| < 5\sigma) = 1 - P(|X - m| \geq 5\sigma) \geq 1 - \frac{1}{5^2} = \frac{24}{25} = 0.96.$$

2. **Exercise 6.4, LN03.** For a random variable $X \sim \text{Geom}(p)$ with have, by definition:

$$P(X = k) = (1 - p)^{k-1}p, \quad k = 1, 2, 3, \ldots$$

The cumulative distribution function is

$$\begin{aligned} F_X(k) = P(X \leq k) &= p + (1 - p)p + (1 - p)^2 p + \cdots + (1 - p)^{k-1}p \\ &= p \sum_{i=0}^{k-1} (1 - p)^i \\ &= p \frac{1 - (1 - p)^k}{1 - (1 - p)} \\ &= 1 - (1 - p)^k. \end{aligned}$$

3. **Exercise 6.7, LN03.** We want to find the expected value and variance of an exponential random variable X with parameter λ. First the expected value. Note that $\lim_{x\to\infty} xe^{-\lambda x} = 0$.

$$\begin{aligned} E[X] &= \int_0^\infty x\lambda e^{-\lambda x} dx \\ &= \left[-xe^{-\lambda x} + \int e^{-\lambda x} \right]_0^\infty \\ &= \int_0^\infty e^{-\lambda x} dx \\ &= \frac{1}{\lambda}. \end{aligned}$$

Now, the variance. First note that

$$\int_0^\infty \left(x - \frac{1}{\lambda}\right) e^{-\lambda x} dx = 0$$

since the mean of X is $1/\lambda$. Using again an integration by parts, we obtain

$$\begin{aligned}
\mathrm{Var}(X) &= E[(x - 1/\lambda)^2] \\
&= \int_0^\infty \left(x - \frac{1}{\lambda}\right)^2 \lambda e^{-\lambda x} dx \\
&= \frac{1}{\lambda^2} + 2\int_0^\infty \left(x - \frac{1}{\lambda}\right) e^{-\lambda x} dx \\
&= \frac{1}{\lambda^2}.
\end{aligned}$$

4. **Exercise 6.10, LN03.** Let f be the function $f(x) = cx^2$ over the interval $[-1, 1]$, where c is a normalization constant. We discretize it and write in Matlab:

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
 x=[-1:0.01:1]; f=x.^2; f=f/(sum(f)*0.01);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

We now use the discrete convolution program:

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function g=convolution(f,a,b,n)
%Input  - f vector discretization of a function
%       - a and b are the left and right endpoints
%         of an interval outside of which f is zero
%       - n degree of convolution
%Output - h vector approximating the n degree
%         convolution of f with itself
%         over interval [na, nb]
N=length(f)-1;
e=(b-a)/N;
s=[a:e:b];
g=f;
for k=2:n
    x=[k*a:e:k*b];
    h=zeros(size(x));
    for j=1:k*N+1
        for i=max([j-N,1]):min([j,(k-1)*N+1])
            h(j)=h(j)+f(j-i+1)*g(i)*e;
        end
```

```
    end
    g=h;
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

To find the convolution power $f^{*n} = f * \cdots * f$ of degree n, we invoke the function `convolution` defined above. This is done with the command

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
 g=convolution(f,-1,1,n);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

When plotting `g`, keep in mind that it may be non-zero over the bigger interval $[na, nb]$. So write now `x=[-n:0.01:n]; plot(x,g)`.

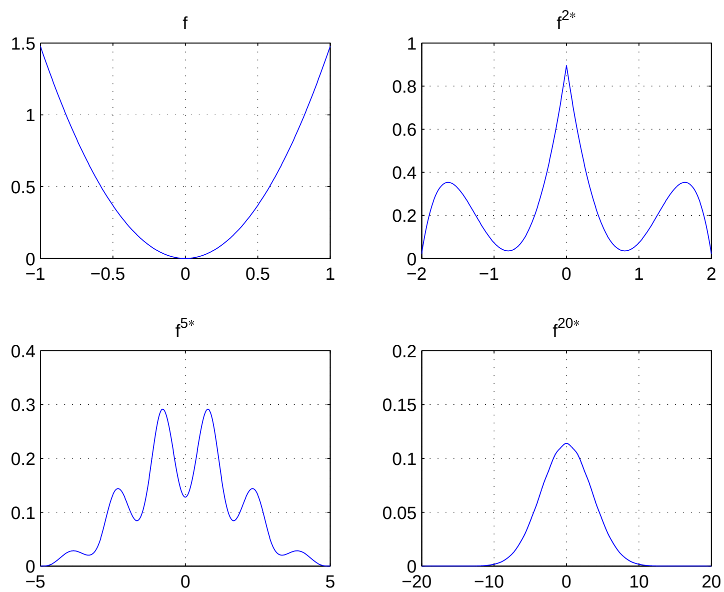

Figure 1: Graphs of f and the convolution powers of degree $n = 2, 5$ and 20.

5. **Exercise 6.11, LN03.** We wish to simulate a random variable X with values in $[0, 1]$ and PDF $f(x) = 3x^2$ using the transformation method. The cumulative distribution function is

$$F(x) = \int_0^x 3s^2 ds = x^3.$$

Its inverse $F^{-1} : [0, 1] \to [0, 1]$ is given by $F^{-1}(u) = u^{1/3}$. Let now U be a random variable with the uniform distribution over $[0, 1]$. By the transformation method, $X = U^{1/3}$ will have the desired PDF, $f(x) = 3x^2$.

The following program simulates n independent realizations of X.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function y=hw3_1(n)
%Simulates n independent realizations of
%a random variable with PDF f(x)=3x^2.
%The output is a row vector of length n.
u=rand(1,n);
y=u.^(1/3);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

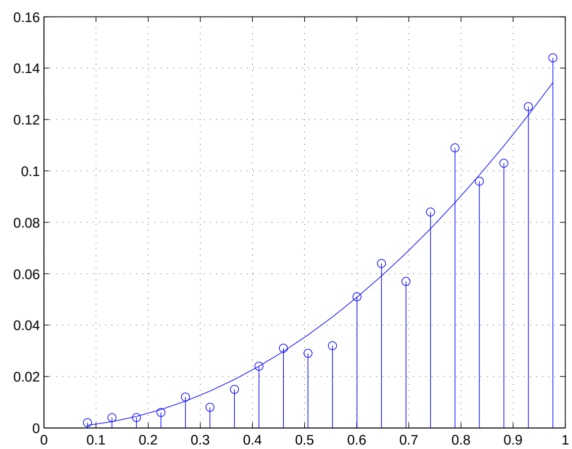

Figure 2: Stem plot with 20 bins for 1000 realizations of a random variable with PDF $f(x) = 3x^2$. Superposed to it is the graph of $f(x)$, suitably normalized so as to give the right occupation proportion in each bin.

The above graph was obtained using the following commands.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
y=hw3_1(1000);
[n,xout]=hist(y,20);
stem(xout,n/1000)
hold on
dx=xout(2)-xout(1);
f=3*xout.^2;
```

```
plot(xout,f*dx)
grid
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

6. **Exercise 6.12, LN03.** We wish now to simulate a random variable X over $[-1, 1]$ with PDF
$$f(x) = \frac{3}{4}(1 - x^2)$$
using the uniform rejection method.

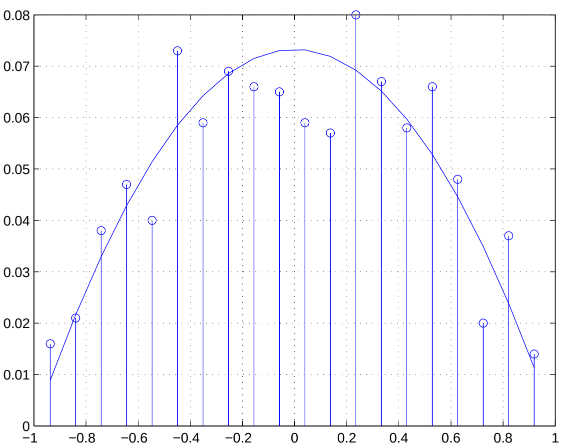

Figure 3: Stem plot with 20 bins for 1000 realizations of a random variable with PDF $f(x) = (3/4)(1 - x^2)$. Superposed to it is the graph of $f(x)$, suitably normalized so as to give the right occupation proportion in each bin.

The following program simulates n independent realizations of X.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function x=hw3_2(n)
%Simulates n independent realizations of
%a random variable with PDF (3/4)*(1-x^2)
%over the interval [-1,1] using the uniform
%rejection method.
x=[];
for i=1:n
    U=0;
    Y=1;
    while Y>=(3/4)*(1-U^2)
        U=2*rand-1;
```

```
        Y=(3/4)*rand;
    end
    x=[x U];
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

The above graph was obtained using the following commands.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
y=hw3_2(1000);
[n,xout]=hist(y,20);
stem(xout,n/1000)
grid
hold on
dx=xout(2)-xout(1);
f=(3/4)*(1-xout.^2);
plot(xout,f*dx)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```