Math 450 - Homework 4
Solutions

1. The cat and mouse problem.

   (a) Directed graph representing a Markov chain for the mouse's random walk.

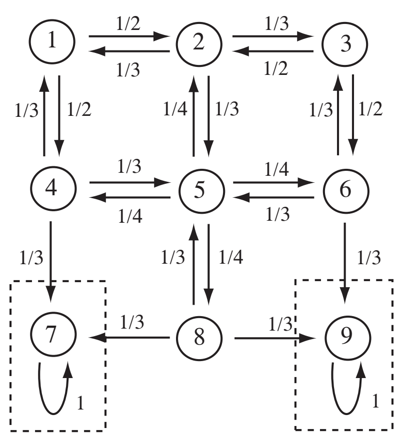

Figure 1: Directed diagram for the mouse random walk. States 7 and 9 are absorbing.

   (b) Transition probability matrix P.

$$P = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(c) Let $u_7(n)$ (respectively, $u_9(n)$) represent the probability that the mouse will be visiting room 7 (respectively, 9) at the nth room change. On a same coordinate system, plot the graphs of $u_7(n)$ and $u_9(n)$ for n from 1 to 50. (Note: these numbers can be obtained from the powers P^n.) Also find approximate values for

$$\lim_{n\to\infty} u_7(n) \quad \text{and} \quad \lim_{n\to\infty} u_7(n).$$

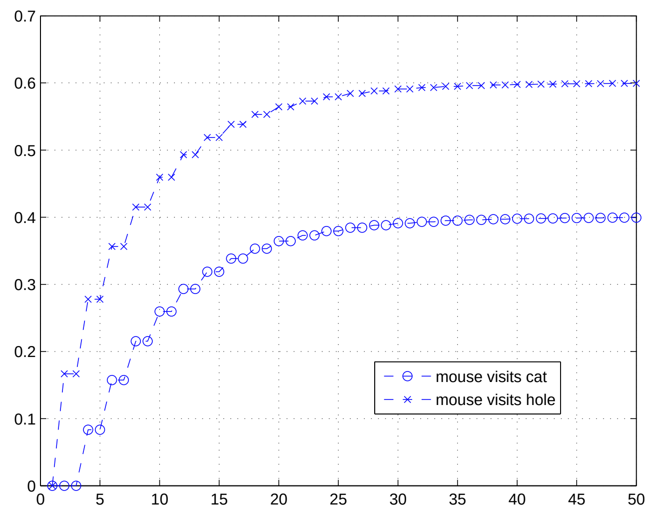

Figure 2: Probabilities $u_7(n)$ and $u_9(n)$ for n from 1 to 50. From the graph it seems that $\lim_{n\to\infty} u_7(n) = 0.4$ and $\lim_{n\to\infty} u_9(n) = 0.6$.

The above graph was obtained with the following script:

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
P=zeros(9,9);
P(1,[2 4])=1/2;
P(2,[1 3 5])=1/3;
P(3,[2 6])=1/2;
P(4,[1 5 7])=1/3;
P(5,[2 4 6 8])=1/4;
P(6,[3 5 9])=1/3;
P(7,7)=1;
P(8,[5 7 9])=1/3;
P(9,9)=1;
```

```
pi=zeros(1,9);
pi(3)=1;

u7=[];
u9=[];
for i=1:50
    p=pi*P^i;
    u7=[u7 p(7)];
    u9=[u9 p(9)];
end
n=1:50;
plot(n,u7,'--o')
hold on
plot(n,u9,'--x')
grid
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

(d) With probability 1, the mouse eventually visits one of the absorbing states, 7 and 9. All the other rooms correspond to transient states. This is clear from the graph since the sum of the two probabilities (for 7 and 9) is 1. It is also apparent of visiting room 9 before 7 is 0.6.

(e) To simulate the Markov chain with stop at hitting a set A, we use the following program.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function X=stop_at_set(p,P,A,n)
%Inputs - p probability distribution of initial state
%       - P transition probability matrix
%       - A set of states where chain is stopped
%       - n maximal time. (If A is empty, stop at time n.)
%Output - X sample chain till hitting time min{T_A,n}.
%
%Note: need function samplefromp.m
q=p;
i=samplefromp(q,1);
X=[i];
m=0;
while length(find(A==i))==0 & m<n-1
q=P(i,:);
i=samplefromp(q,1);
X=[X i];
m=m+1;
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

For the problem, $A = \{7, 9\}$. We run the above program 1000 times and count the number of times we arrive at 9. (Take `A=[7 9]` and a large arbitrary n such as 10000, to insure that we will have hit A before the deadline n.) This is done with the following program.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
rand('seed',121)
a=0;
for i=1:1000
    X=stop_at_set(pi,P,[7 9],10000);
    s=length(X);
    a=a+(X(s)==9);
end
a=a/1000;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

This gave proportion 0.5780 of reaching 9 before 7. Running the same experiment (with same seed) 10000 times gives the value 0.5998.

2. Exercise 1.2.1 on page 11 of Norris' book.

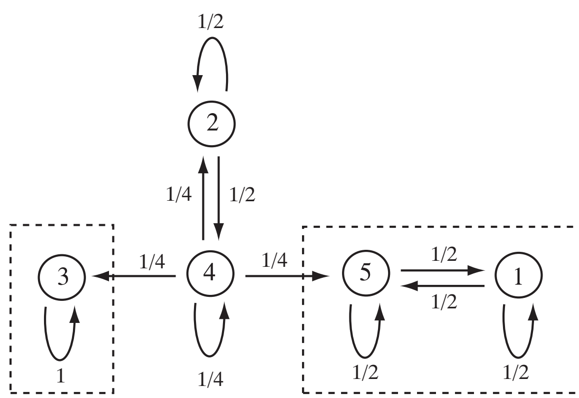

Figure 3: Diagram for Exercise 1.2.1. The communication classes are $\{1,5\}$, $\{2,4\}$, and $\{3\}$. The first and last (boxed in the figure) are closed classes.