# Math 450 - Homework 6

Due date: Friday, 3/02/07

1. Read sections 1.7 and 1.8 (pages 33 to 46) of the textbook.

2. Exercise 1.7.1. Find all the invariant distributions of the transition matrix

$$P = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

3. Exercise 1.7.2. (Ehrenfest model) Gas molecules move about randomly in a box which is divided into two halves symmetrically by a partition. A hole is made in the partition. Suppose there are $N$ molecules in the box. (Think of the partitions as two urns containing balls labeled 1 through $N$. Molecular motion can be modeled by choosing a number between 1 and $N$ at random and moving the corresponding ball from the urn it is presently in to the other. This is a historically important physical model introduced by Ehrenfest in the early days of statistical mechanics to study thermodynamic equilibrium.)

   (a) Show that the number of molecules on one side of the partition just after a molecule has passed through the hole evolves as a Markov chain. What are the transition probabilities? (Take as the sets of states the number of molecules in one of the partitions.) Draw a transition diagram of the process.

   (b) Is the chain recurrent? (Explain your answer.)

   (c) What is the invariant distribution of this chain? (Hint: Show that $\pi_i = (p_0 p_1 \ldots p_{i-1})/(q_1 q_2 \ldots q_i)$. Obtain the binomial distribution.)

   (d) What is the number of steps it takes on average for a partition to become empty given that it was initially empty? In other words, find the expected return time to state 0. (To make this question more interesting, calculate this time in years, assuming that each transition takes about 1/400 of a second. Note that nitrogen molecules at 273 Kelvin have an average speed of approximately 400 meters per second, and we assume for this problem, more or less arbitrarily, that the molecule moves on average about one meter before changing sides inside the box.)

(e) Do a computer simulation of this process for $N = 100$. Start from state 0 (one of the partitions is empty) and follow the chain up to 1000 steps. Draw a graph of the number of molecules in the initially empty partition as a function of the number of steps. On the basis of your answer to the previous item, would you expect to observe during the course of the simulation a return to state 0?

4. Exercise 1.7.3. A particle moves on the eight vertices of a cube in the following way: at each step the particle is equally likely to move to each of the three adjacent vertices, independently of its past motion. Let $i$ be the initial vertex occupied by the particle, $o$ the vertex opposite $i$. Calculate each of the following quantities:

   (a) the expected number of steps until the particle returns to $i$;

   (b) the expected number of visits to $o$ until the first return to $i$;

   (c) the expected number of steps until the first visit to $o$.

   Confirm your answers by computer simulation of the random walk on the cube. In each case, obtain the expected value using sample means over 1000 runs of the random walk.