Math 450 - Homework 7
Solutions

1. Find the invariant distribution of the transition matrix:

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} \\ p & 1-p & 0 \end{pmatrix}.$$

The equation $\pi P = \pi$, and the normalization condition $\pi_1 + \pi_2 + \pi_3 = 1$ give the system:

$$\begin{aligned} \pi_1 &= p\pi_3 \\ \pi_2 &= \pi_1 + \frac{2}{3}\pi_2 + (1-p)\pi_3 \\ \pi_3 &= \frac{1}{3}\pi_2 \\ \pi_1 + \pi_2 + \pi_3 &= 1. \end{aligned}$$

We can disregard one of the first 3 equations, since they are dependent; let us discard the second equation. It is now a simple matter to solve the non-homogeneous linear system. The result is

$$\pi = \left(\frac{p}{4+p}, \frac{3}{4+p}, \frac{1}{4+p} \right).$$

2. Exercise 1.8.4, page 46 of textbook. There are 6 states for this problem, corresponding to the 6 permutations of the books on the shelf. I will represent a state by triples such as 312, or 213, and will order them in the following (arbitrary) order:

$$s_1 = 132,\ s_2 = 312,\ s_3 = 213,\ s_4 = 123,\ s_5 = 321,\ s_6 = 231.$$

For this ordering of states, the transition probabilities matrix is

$$P = \begin{pmatrix} \alpha_1 & \alpha_3 & \alpha_2 & 0 & 0 & 0 \\ \alpha_1 & \alpha_3 & 0 & 0 & 0 & \alpha_2 \\ 0 & 0 & \alpha_2 & \alpha_1 & \alpha_3 & 0 \\ 0 & \alpha_3 & \alpha_2 & \alpha_1 & 0 & 0 \\ \alpha_1 & 0 & 0 & 0 & \alpha_3 & \alpha_2 \\ 0 & 0 & 0 & \alpha_1 & \alpha_3 & \alpha_2 \end{pmatrix}.$$

The transition diagram is given if figure 1. The probability that on day n the student finds the books in the order 123 is now $p_n = P(X_n = 4)$. As n goes to ∞, this number must converge to π_4, where π is the stationary probability distribution of P.

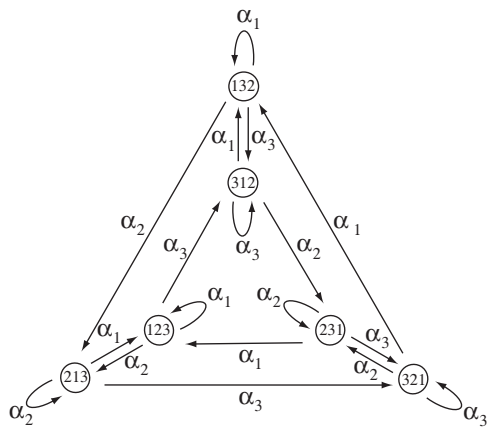

Figure 1: Diagram for the book shuffling exercise. Note that each state leads to itself in one step, so the chain is necessarily aperiodic. It is also clear from the diagram that the chain is irreducible. From this, and a theorem shown in class, we obtain $P(X_n = s) \to \pi_s$, where π is the unique stationary distribution for the chain.

The condition $\pi P = \pi$ consists of the the six equations below, where I write $x_i = \pi_i$:

$$\begin{aligned}
x_1 &= \alpha_1(x_1 + x_2 + x_5) \\
x_2 &= \alpha_3(x_1 + x_2 + x_4) \\
x_3 &= \alpha_2(x_1 + x_3 + x_4) \\
x_4 &= \alpha_1(x_3 + x_4 + x_6) \\
x_5 &= \alpha_3(x_3 + x_5 + x_6) \\
x_6 &= \alpha_2(x_2 + x_5 + x_6).
\end{aligned}$$

We can discard the last equation since these are linearly dependent equations, and we add the normalization condition

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1.$$

Solving this system by hand is a rather tedious but elementary problem. After quite a bit of work, I got the following answer. For convenience, define $\eta_i = \alpha_i/(1 - \alpha_i)$. Then

$$x_1 = \alpha_3\eta_1,\ x_2 = \alpha_1\eta_3,\ x_3 = \alpha_1\eta_2,\ x_4 = \alpha_2\eta_1,\ x_5 = \alpha_2\eta_3,\ x_6 = \alpha_3\eta_2.$$

Therefore, the probability that the order of books is 123 on day n converges to

$$\pi_4 = \frac{\alpha_1 \alpha_2}{1 - \alpha_1}.$$

Note that if $\alpha_1 = \alpha_2 = \alpha_3 = 1/3$, the probabilities simplify to $\pi_s = 1/6$ for all s.

3. Exercise 1.9.1, (a), (b) and (c), page 51 of textbook. In each case, determine whether the stochastic matrix is reversible.

   (a) I claim that the stochastic matrix

   $$P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$$

   is reversible. First, as already noted in class in different notation, P has invariant distribution $\pi = \left(\frac{q}{p+q}, \frac{p}{p+q}\right)$. Now observe that

   $$\pi_1 p_{12} = \frac{pq}{p+q} = \pi_2 p_{21}.$$

   The other cases of $\pi_i p_{ij} = \pi_j p_{ji}$ are trivial since they involve $i = j$. Therefore, P is reversible.

   (b) Consider now the stochastic matrix

   $$P = \begin{pmatrix} 0 & p & 1-p \\ 1-p & 0 & p \\ p & 1-p & 0 \end{pmatrix}.$$

   I claim that it is reversible if and only if $p = 1/2$. Note that P is doubly-stochastic, i.e., the entries in each column also add up to 1, so $\pi = (1/3, 1/3, 1/3)$ (all entries equal) is an invariant vector for P. It is unique by theorem 1.7.6, since the chain is irreducible (and recurrent since it has a finite number of states). The condition $\pi_i p_{ij} = \pi_j p_{ji}$, for all i, j, imply $p_{ij} = p_{ji}$ for all i, j, since all the π_i are equal. Therefore, P is reversible if and only if P is symmetric, i.e., it is equal to its own transpose. But this is only the case if $p = 1 - p$, or $p = 1/2$.

   (c) Let the set of states be $\{0, 1, \ldots, N\}$ and $p_{ij} = 0$ if $|j - i| \geq 2$. I claim that P is reversible in this case. Again, in this case, there is a unique invariant probability distribution, which is given by the following set of equations:

   $$\begin{aligned} \pi_0 &= \pi_0 p_{00} + \pi_1 p_{10} \\ \pi_1 &= \pi_0 p_{01} + \pi_1 p_{11} + \pi_2 p_{21} \\ \pi_2 &= \pi_1 p_{12} + \pi_2 p_{22} + \pi_3 p_{32} \\ &\ldots \\ \pi_N &= \pi_{N-1} p_{N-1,N} + \pi_N p_{NN}. \end{aligned}$$

It suffices to show that

$$\pi_i p_{i,i+1} = \pi_{i+1} p_{i+1,i}$$

for $i = 0, 1, \ldots, N-1$. In all other cases the reversibility condition holds trivially, either because p_{ij} are 0 or because $i = j$. We prove reversibility by induction. From the first equation, we have

$$\pi_1 p_{10} = \pi_0 - \pi_0 p_{00} = \pi_0(1 - p_{00}) = \pi_0 p_{01}.$$

Suppose that the equation

$$\pi_{i-1} p_{i-1,i} = \pi_i p_{i,i-1}$$

holds, and look at the equation

$$\pi_i = \pi_{i-1} p_{i+1,i} + \pi_i p_{ii} + \pi_{i+1} p_{i+1,i}.$$

We combine these two equations with $p_{i,i-1} + p_{ii} + p_{i,i+1} = 1$ and obtain:

$$\begin{aligned}
\pi_i p_{i,i-1} + \pi_{i+1} p_{i+1,i} &= \pi_{i-1} p_{i-1,i} + \pi_{i+1} p_{i+1,i} \\
&= \pi_i (1 - p_{ii}) \\
&= \pi_i (p_{i,i-1} + p_{i,i+1}) \\
&= \pi_i p_{i,i-1} + \pi_i p_{i,i+1}.
\end{aligned}$$

Canceling the term $\pi_i p_{i,i-1}$ gives

$$\pi_{i+1} p_{i+1,i} = \pi_i p_{i,i+1}.$$

This is the induction step. The last step is shown similarly. Therefore, the transition matrix P in this case is reversible.

4. (Book shuffling) Do a simulation of the situation described in exercise 1.8.4. More precisely, assume that each morning the student takes one of $n = 3$ books from his shelf, each with equal probabilities, independently of the previous day's choice. In the evening he replaces the book at the left-hand end of the shelf. We want to find how often the shelf returns to the initial state. In your simulation, assume that the books are initially ordered as $1, 2, \ldots, n$ from left to right. Each move consists of a permutation of the form

$$(1, 2, \ldots, i-1, i, i+1, \ldots, n) \mapsto (i, 1, 2, \ldots, i-1, i+1, \ldots, n).$$

Repeat the operation 100000 times. (This took about 9 seconds on my laptop.)

The following program gave for the relative number of occurrences of the initial state: $f = 0.1660$, or $1/f = 6.0245$. The correct value is $1/f = 6$.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
tic
rand('seed',317)
n=3;     %number of books on the shelf
N=100000;%total number of shufflings
s=1:n;   %initial configuration of books in shelf
f=0;     %fraction of times state (1 2 ... n) is observed
s_new=s;
for k=1:N
    i=ceil(n*rand);
    a=find(s_new==i);
    s_new=[s_new(a) s_new(1:a-1) s_new(a+1:n)];
    f=f+(sum(s_new==s)==n);
end
f=f/N;
toc
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```