Math 450 - Homework 8

Due date: Friday, 3/23/07

1. Read the Lecture Notes on continuous-time Markov chains (posted online). Also look at the sections of the text referred to in those notes that relate to this assignment.

2. Exercise 2.1.1, page 66. (Look at example 2.1.3. You will need to recall about matrix diagonalization.)

3. For the diagram of figure 1, write down the Q-matrix and the Π-matrix. (The latter is defined in section 2.6.) Draw the graphs of $p_{24}(t)$, $p_{55}(t)$, $p_{34}(t)$, and $p_{45}(t)$ for $0 \leq t \leq 2$. (Use the matlab function `expm` to exponentiate the matrix Q.)

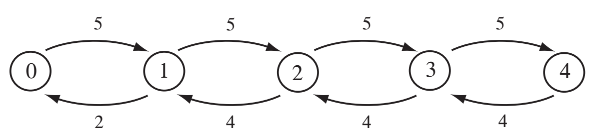

Figure 1: Transition diagram for problem 3.

4. (This is based on exercise 2.3.2, page 73.) Consider the transitions diagram of figure 2.

   (a) Explain why the following interpretation is justified: Starting at position 0, we wait an exponential (holding) time of parameter λ before advancing one unit along the set $\{0, 1, 2, \dots\}$. At the time we make the move to the next position, we toss a coin (possibly biased with probabilities $P(T) = \beta$, $P(H) = 1 - \beta$). If it comes up tail (T) we stop. If head (H) we wait again a random time, exponential with parameter λ, then move one step further. (Suggestion: compute the Π-matrix and rate constants q_i, and use the hold-and-jump interpretation of the process.)

   (b) Let N the be number of heads before obtaining a tail. This is a geometric random variable of parameter β, that is.

$$P(N = n) = \beta(1 - \beta)^{n-1}, \quad n = 1, 2, \dots.$$

Let $T_1, T_2, \ldots$ denote the holding times. Show that the total time $T = T_1 + T_2 + \cdots + T_N$ has exponential distribution of parameter $\beta\lambda$. (Assume that the holding times and coin tosses are all independent random variables.) Note: it is enough to show that

$$P(T > t) = e^{-\beta\lambda t}.$$

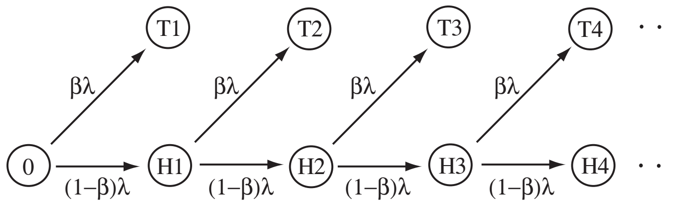

Figure 2: Diagram for problem 4.

A few hints: Let Y_t denote the process of example 2.1.4, page 66. Note that the following gives two descriptions of the same event:

$$\{T_1 + T_2 + \cdots + T_n > t\} = \{Y_t \leq n-1\}.$$

Now note that

$$P(Y_t \leq n-1) = \sum_{j=0}^{n-1} P_0(Y_t = j) = \sum_{j=0}^{n-1} p_{0j}(t),$$

where $p_{ij}(t)$ is given in that example, and

$$P\left(T > t\right) = \sum_{n=1}^{\infty} P(N = n) P\left(\sum_{i=1}^{n} T_i > t\right).$$

Then put it all together to show that

$$P\left(\sum_{i=1}^{N} T_i > t\right) = e^{-\beta\lambda t}.$$

Along the way you will need to find the sum of a double series. The key trick is to change the order of summation, recognize a geometric sum and the Taylor series of an exponential function.

(c) What is the expected time till the process reaches an absorbing state? (Recall the general properties of exponential distributions from Lecture Notes 3.)

(d) Consider now the process described by the diagram of figure 2, except that state $H16$ is also assumed to be absorbing. This means that we now have a finite state process with 33 states. Using the program `ctmc` described in the on-line course notes write a program to simulate the following experiment: first set $\lambda = 1$ and let β vary from 0.1 to 1 in steps of 0.1. For each β simulate 1000 runs (sample paths) of the continuous time chain, from state 0 at time 0 until it stops at an absorbing state. Compute the mean stopping time, τ, for each β and plot τ as a function of β. (Note: it took me about 55 seconds to do this on my laptop.) Does your graph reflect, qualitatively, the result you obtained analytically for the process with infinite states?