Math 450 - Homework 8
Solutions

1. Given the generator matrix

\[
Q = \begin{pmatrix}
-2 & 1 & 1 \\
4 & -4 & 0 \\
2 & 1 & -3
\end{pmatrix}
\]

we look for \( p_{11}(t) \) of \( P(t) = e^{tQ} \). The eigenvalues of \( Q \) are the roots of

\[
\text{det}(\lambda I - Q) = (\lambda + 4)((\lambda + 2)(\lambda + 3) - 6) = 0.
\]

They are \( \lambda = 0, -4, -5 \). Therefore, it is possible to find an invertible matrix \( U \) (we do not need to have it explicitly) such that \( U^{-1}QU \) is diagonal, with diagonal entries 0, -4, -5. This gives:

\[
e^{tQ} = U \begin{pmatrix}
1 & 0 & 0 \\
0 & e^{-4t} & 0 \\
0 & 0 & e^{-5t}
\end{pmatrix} U^{-1}.
\]

Then \( p_{11}(t) \) must be a linear combination of the functions 1, \( e^{-4t} \) and \( e^{-5t} \), so there must be constants \( a, b, c \) such that

\[
p_{11}(t) = a + be^{-4t} + ce^{-5t}.
\]

We know that \( p_{11}(0) = 1 \), \( p'_{11}(0) = q_{11} = -2 \), and \( p''_{11}(0) = q^{(2)}_{11} = 10 \). (The last equation is justified by part (iv) of theorem 2.1.1) So we have three equations for \( a, b, c \):

\[
\begin{align*}
a + b + c &= 1 \\
-4b - 5c &= -2 \\
16b + 25c &= 10.
\end{align*}
\]

The solution to this system is easily obtained: \( a = 3/5, b = 0, c = 2/5 \). Therefore,

\[
p_{11}(t) = \frac{3}{5} + \frac{2}{5} e^{-5t}.
\]
2. Consider the diagram of figure 1. The infinitesimal generator $Q$ for this diagram is given by

$$Q = \begin{pmatrix}
-5 & 5 & 0 & 0 & 0 \\
2 & -7 & 5 & 0 & 0 \\
0 & 4 & -9 & 5 & 0 \\
0 & 0 & 4 & -9 & 5 \\
0 & 0 & 0 & 4 & -4
\end{pmatrix}.$$ 

The $\Pi$-matrix is

$$\Pi = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
2 & 7 & 0 & 5 & 7 \\
0 & 4 & 0 & 5 & 0 \\
0 & 0 & 4 & 0 & 5 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}.$$ 

We can draw the graphs of $p_{24}(t)$, $p_{55}(t)$, $p_{34}(t)$, and $p_{45}(t)$ for $0 \leq t \leq 2$, using the following commands:

```
Q=zeros(5,5);
Q(1,1)=-5; Q(1,2)=5;
Q(2,1)=2; Q(2,2)=-7; Q(2,3)=5;
Q(3,2)=4; Q(3,3)=-9; Q(3,4)=5;
Q(4,3)=4; Q(4,4)=-9; Q(4,5)=5;
Q(5,4)=4; Q(5,5)=-4;
t=0:0.05:2;
a=length(t);
p24=[];
p55=[];
p34=[];
p45=[];
for i=1:a
    u=expm(t(i)*Q);
    p24=[p24 u(2,4)];
p55=[p55 u(5,5)];
p34=[p34 u(3,4)];
p45=[p45 u(4,5)];
```

Figure 1: Diagram for exercise 2.
end
subplot(2,2,1)
plot(t,p24)
grid
title('p_{55}(t)')

subplot(2,2,2)
plot(t,p55)
grid
title('p_{55}(t)')

subplot(2,2,3)
plot(t,p34)
grid
title('p_{34}(t)')

subplot(2,2,4)
plot(t,p45)
grid
title('p_{45}(t)')

Figure 2: The transition functions $p_{24}(t)$, $p_{55}(t)$, $p_{34}(t)$, and $p_{45}(t)$ for the infinitesimal generator $Q$. 
3. I will order the states as $0, H_1, T_1, H_2, T_2, \ldots$. The diagram is shown in figure 3.

![Diagram for problem 4.](image)

Figure 3: Diagram for problem 4.

(a) From the diagram we obtain the $Q$-matrix of the process:

$$Q = \begin{pmatrix}
-\lambda & (1-\beta)\lambda & \beta \lambda & 0 & 0 & 0 & \ldots \\
0 & -\lambda & (1-\beta)\lambda & \beta \lambda & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & -\lambda & (1-\beta)\lambda & \beta \lambda & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

and from the $Q$-matrix we obtain the $\Pi$-matrix:

$$\Pi = \begin{pmatrix}
0 & 1-\beta & \beta & 0 & 0 & 0 & \ldots \\
0 & 0 & 1-\beta & \beta & 0 & 0 & \ldots \\
0 & 0 & 1 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 1-\beta & \beta & 0 & \ldots \\
0 & 0 & 0 & 0 & 1 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.$$ 

The $\Pi$-matrix shows that the jump process amounts to moving one step forward with probability $1-\beta$ or stopping, with probability $\beta$. The holding times are given by $q_k = \lambda$ for all the odd-numbered states ($H_k$), and $\infty$ for the even (absorbing) states ($T_k$). In terms of the hold-and-jump interpretation, this means that at each stage of the process, with probability $\beta$ we remain forever at the current position (indexed by the set $\{0,1,2,\ldots\}$) or, after an exponential (with parameter $\lambda$) holding time we jump one step ahead.

(b) Let $N$ be the number of heads before obtaining a tail. This is a geometric random variable of parameter $\beta$, that is.

$$P(N = n) = \beta(1-\beta)^{n-1}, \quad n = 1, 2, \ldots.$$ 

Let $T_1, T_2, \ldots$ denote the holding times. We want to show that the total time $T = T_1 + T_2 + \cdots + T_N$ has exponential distribution of
parameter $\beta \lambda$. Using the notation $Y_t$ of example 2.1.4 on page 66 we have:

\[
P_0(T > t) = P_0 \left( \sum_{i=1}^{N} T_i > t \right)
\]

\[
= \sum_{n=1}^{\infty} P_0 \left( \sum_{i=1}^{n} T_i > t \right) P(N = n)
\]

\[
= \sum_{n=1}^{\infty} P_0 \left( Y_t \leq n - 1 \right) \beta (1 - \beta)^{n-1}
\]

\[
= \sum_{n=1}^{\infty} \left( \sum_{j=0}^{n-1} P_0 (Y_t = j) \right) \beta (1 - \beta)^{n-1}
\]

\[
= \sum_{n=1}^{\infty} \left( \sum_{j=0}^{n-1} p_{nj}(t) \right) \beta (1 - \beta)^{n-1}
\]

\[
= \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} \frac{(\lambda t)^j e^{-\lambda t}}{j!} \beta (1 - \beta)^{n-1}
\]

\[
= \beta e^{-\lambda t} \sum_{j=0}^{\infty} \sum_{n=j+1}^{\infty} \frac{(\lambda t)^j}{j!} (1 - \beta)^{n-1}
\]

\[
= \beta e^{-\lambda t} \sum_{j=0}^{\infty} \frac{((1 - \beta)\lambda t)^j}{j!} \sum_{n=j+1}^{\infty} (1 - \beta)^{n-1-j}
\]

\[
= \beta e^{-\lambda t} e^{(1 - \beta)\lambda t} \frac{1}{\beta}
\]

\[
= e^{-\beta \lambda t}.
\]

Therefore, $T$ is exponential with parameter $\lambda \beta$.

(c) We know from our discussion in the lecture notes set 3 that the expected value of an exponential random variable with parameter $\lambda$ is $1/\lambda$. From the result of the previous item we conclude that the expected value of $T$ is $1/\lambda \beta$.

(d) The following program took almost 1 minute to run. The steps go from 0 to 16, so there are a total of 33 states.

\[
\text{tic}
\text{rand('seed',213)}
\text{pi=zeros(1,33)};
\text{b=0.1:1/10:1;}
\text{pi(1)=1;}
\text{mean_time=[];}
\]
for k=1:10
    c=b(k);
    Q=zeros(33,33);
    Q(1,1:3)=[-1 1-c c];
    for k=1:15
        Q(2*k,2*k:2*k+3)=[-1 0 1-c c];
    end
    s=0;
    for j=1:1000
        [t,y]=ctmc(10^6,pi,Q);
        last=length(t);
        s=s+t(last);
    end
    mean_time=[mean_time s/1000];
end
toc

Figure 4: Note that for the infinite state case, the mean stopping time as a function of $\beta$ would be $1/\beta$ (assuming $\lambda = 1$). In the finite chain we see a similar qualitative behavior, except that the time does not go to infinity for small $\beta$. If $\beta = 0$, the mean time to hitting absorbing state $H16$ is exactly 16.