Math 450 - Homework 8
Solutions

1. Given the generator matrix

$$Q = \begin{pmatrix} -2 & 1 & 1 \\ 4 & -4 & 0 \\ 2 & 1 & -3 \end{pmatrix}$$

we look for $p_{11}(t)$ of $P(t) = e^{tQ}$. The eigenvalues of Q are the roots of

$$\det(\lambda I - Q) = (\lambda + 4)((\lambda + 2)(\lambda + 3) - 6) = 0.$$

They are $\lambda = 0, -4, -5$. Therefore, it is possible to find an invertible matrix U (we do not need to have it explicitly) such that $U^{-1}QU$ is diagonal, with diagonal entries $0, -4, -5$. This gives:

$$e^{tQ} = U \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-4t} & 0 \\ 0 & 0 & e^{-5t} \end{pmatrix} U^{-1}.$$

Then $p_{11}(t)$ must be a linear combination of the functions 1, e^{-4t} and e^{-5t}, so there must be constants a, b, c such that

$$p_{11}(t) = a + be^{-4t} + ce^{-5t}.$$

We know that $p_{11}(0) = 1$, $p'_{11}(0) = q_{11} = -2$, and $p''_{11}(0) = q^{(2)}_{11} = 10$. (The last equation is justified by part (iv) of theorem 2.1.1) So we have three equations for a, b, c:

$$\begin{aligned} a + b + c &= 1 \\ -4b - 5c &= -2 \\ 16b + 25c &= 10. \end{aligned}$$

The solution to this system is easily obtained: $a = 3/5, b = 0, c = 2/5$. Therefore,

$$p_{11}(t) = \frac{3}{5} + \frac{2}{5}e^{-5t}.$$

2. Consider the diagram of figure 1. The infinitesimal generator Q for this diagram is given by

$$Q = \begin{pmatrix} -5 & 5 & 0 & 0 & 0 \\ 2 & -7 & 5 & 0 & 0 \\ 0 & 4 & -9 & 5 & 0 \\ 0 & 0 & 4 & -9 & 5 \\ 0 & 0 & 0 & 4 & -4 \end{pmatrix}.$$

The Π-matrix is

$$\Pi = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{2}{7} & 0 & \frac{5}{7} & 0 & 0 \\ 0 & \frac{4}{9} & 0 & \frac{5}{9} & 0 \\ 0 & 0 & \frac{4}{9} & 0 & \frac{5}{9} \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

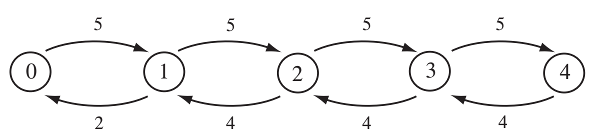

Figure 1: Diagram for exercise 2.

We can draw the graphs of $p_{24}(t)$, $p_{55}(t)$, $p_{34}(t)$, and $p_{45}(t)$ for $0 \leq t \leq 2$, using the following commands:

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
Q=zeros(5,5);
Q(1,1)=-5; Q(1,2)=5;
Q(2,1)=2; Q(2,2)=-7; Q(2,3)=5;
Q(3,2)=4; Q(3,3)=-9; Q(3,4)=5;
Q(4,3)=4; Q(4,4)=-9; Q(4,5)=5;
Q(5,4)=4; Q(5,5)=-4;
t=0:0.05:2;
a=length(t);
p24=[];
p55=[];
p34=[];
p45=[];
for i=1:a
    u=expm(t(i)*Q);
    p24=[p24 u(2,4)];
    p55=[p55 u(5,5)];
    p34=[p34 u(3,4)];
    p45=[p45 u(4,5)];
```

```
end
subplot(2,2,1)
plot(t,p24)
grid
title('p_{55}(t)')

subplot(2,2,2)
plot(t,p55)
grid
title('p_{55}(t)')

subplot(2,2,3)
plot(t,p34)
grid
title('p_{34}(t)')

subplot(2,2,4)
plot(t,p45)
grid
title('p_{45}(t)')
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

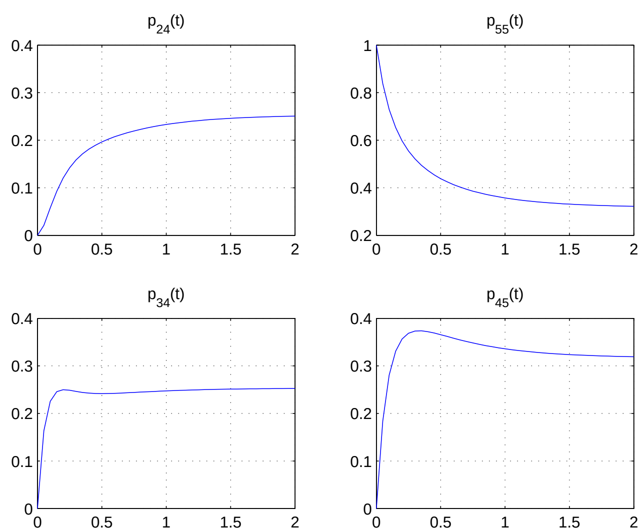

Figure 2: The transition functions $p_{24}(t)$, $p_{55}(t)$, $p_{34}(t)$, and $p_{45}(t)$ for the infinitesimal generator Q.

3. I will order the states as $0, H1, T1, H2, T2, \ldots$. The diagram is shown in figure 3.

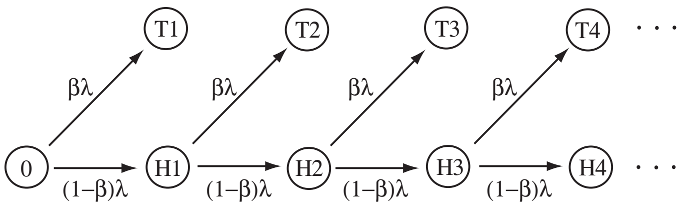

Figure 3: Diagram for problem 4.

(a) From the diagram we obtain the Q-matrix of the process:

$$Q = \begin{pmatrix} -\lambda & (1-\beta)\lambda & \beta\lambda & 0 & 0 & 0 & \ldots \\ 0 & -\lambda & (1-\beta)\lambda & \beta\lambda & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & -\lambda & (1-\beta)\lambda & \beta\lambda & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and from the Q-matrix we obtain the Π-matrix:

$$\Pi = \begin{pmatrix} 0 & 1-\beta & \beta & 0 & 0 & 0 & \ldots \\ 0 & 0 & 1-\beta & \beta & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 1-\beta & \beta & 0 & \ldots \\ 0 & 0 & 0 & 0 & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The Π-matrix shows that the jump process amounts to moving one step forward with probability $1-\beta$ or stopping, with probability β. The holding times are given by $q_i = \lambda$ for all the odd-numbered states (Hk), and ∞ for the even (absorbing) states (Tk). In terms of the hold-and-jump interpretation, this means that at each stage of the process, with probability β we remain forever at the current position (indexed by the set $\{0, 1, 2, \ldots\}$) or, after an exponential (with parameter λ) holding time we jump one step ahead.

(b) Let N the be number of heads before obtaining a tail. This is a geometric random variable of parameter β, that is.

$$P(N = n) = \beta(1-\beta)^{n-1}, \quad n = 1, 2, \ldots.$$

Let $T_1, T_2, \ldots$ denote the holding times. We want to show that the total time $T = T_1 + T_2 + \cdots + T_N$ has exponential distribution of parameter $\beta\lambda$. Using the notation Y_t of example 2.1.4 on page 66 we have:

$$\begin{aligned}
P_0(T > t) &= P_0\left(\sum_{i=1}^{N} T_i > t\right) \\
&= \sum_{n=1}^{\infty} P_0\left(\sum_{i=1}^{n} T_i > t\right) P(N = n) \\
&= \sum_{n=1}^{\infty} P_0\left(Y_t \le n-1\right)\beta(1-\beta)^{n-1} \\
&= \sum_{n=1}^{\infty}\left(\sum_{j=0}^{n-1} P_0\left(Y_t = j\right)\right)\beta(1-\beta)^{n-1} \\
&= \sum_{n=1}^{\infty}\left(\sum_{j=0}^{n-1} p_{0j}(t)\right)\beta(1-\beta)^{n-1} \\
&= \sum_{n=1}^{\infty}\sum_{j=0}^{n-1}\frac{(\lambda t)^j e^{-\lambda t}}{j!}\beta(1-\beta)^{n-1} \\
&= \beta e^{-\lambda t}\sum_{j=0}^{\infty}\sum_{n=j+1}^{\infty}\frac{(\lambda t)^j}{j!}(1-\beta)^{n-1} \\
&= \beta e^{-\lambda t}\sum_{j=0}^{\infty}\frac{((1-\beta)\lambda t)^j}{j!}\sum_{n=j+1}^{\infty}(1-\beta)^{n-j-1} \\
&= \beta e^{-\lambda t}e^{(1-\beta)\lambda t}\frac{1}{\beta} \\
&= e^{-\beta\lambda t}.
\end{aligned}$$

Therefore, T is exponential with parameter $\lambda\beta$.

(c) We know from our discussion in the lecture notes set 3 that the expected value of an exponential random variable with parameter λ is $1/\lambda$. From the result of the previous item we conclude that the expected value of T is $1/\lambda\beta$.

(d) The following program took almost 1 minute to run. The steps go from 0 to 16, so there are a total of 33 states.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
tic
rand('seed',213)
pi=zeros(1,33);
b=0.1:1/10:1;
pi(1)=1;
mean_time=[];
```

```
for k=1:10
    c=b(k);
    Q=zeros(33,33);
    Q(1,1:3)=[-1 1-c c];
    for k=1:15
        Q(2*k,2*k:2*k+3)=[-1 0 1-c c];
    end
    s=0;
    for j=1:1000
        [t,y]=ctmc(10^6,pi,Q);
        last=length(t);
        s=s+t(last);
    end
    mean_time=[mean_time s/1000];
end
toc
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

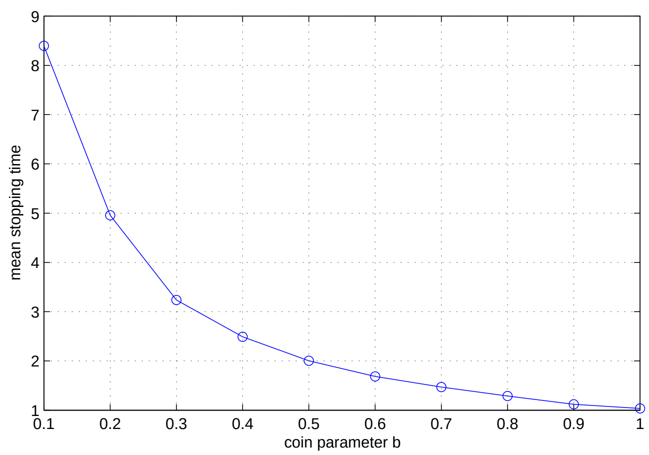

Figure 4: Note that for the infinite state case, the mean stopping time as a function of β would be $1/\beta$ (assuming $\lambda = 1$). In the finite chain we see a similar qualitative behavior, except that the time does not go to infinity for small β. If $\beta = 0$, the mean time to hitting absorbing state $H16$ is exactly 16.