Math 450 - Homework 9
Solutions

1. Exercise 3.3.1, page 114. Consider the Markov chain on $\{1, 2, 3, 4\}$ with generator matrix

$$\begin{pmatrix} -1 & 1/2 & 1/2 & 0 \\ 1/4 & -1/2 & 0 & 1/4 \\ 1/6 & 0 & -1/3 & 1/6 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Calculate (a) the probability of hitting 3 starting from 1, and (b) the expected time to hit 4 starting from 1.

For (a), let $h = (h_1, h_2, h_3, h_4)$ be the vector where h_i denotes the probability of hitting 3 starting from i. Then by Theorem 3.3.1 h is the minimal non-negative solution to the system of equations

$$\begin{aligned} h_3 &= 1 \\ q_{11}h_1 + q_{12}h_2 + q_{13}h_3 + q_{14}h_4 &= 0 \\ q_{21}h_1 + q_{22}h_2 + q_{23}h_3 + q_{24}h_4 &= 0 \\ q_{41}h_1 + q_{42}h_2 + q_{43}h_3 + q_{44}h_4 &= 0. \end{aligned}$$

Substituting the values of q_{ij} and $h_3 = 1$ gives:

$$\begin{aligned} -h_1 + \frac{1}{2}h_2 + \frac{1}{2} &= 0 \\ \frac{1}{4}h_1 - \frac{1}{2}h_2 + \frac{1}{4}h_4 &= 0. \end{aligned}$$

The general solution to this system is $h = (h_1, 2h_1 - 1, 1, 3h_1 - 2)$. The minimal non-negative solution is the one for which $h_1 = 2/3$. This yields

$$h = (2/3, 1/3, 1, 0).$$

For part (b), let $k = (k_1, k_2, k_3, k_4)$ be the vector where k_i denotes the expected time to hit 4 starting from i. Note that $q_i > 0$ for all $i \neq 4$, so Theorem 3.3.3 applies. It says that k is the minimal non-negative solution to the system of equations:

$$\begin{aligned} k_4 &= 0 \\ q_{11}k_1 + q_{12}k_2 + q_{13}k_3 + q_{14}k_4 &= -1 \\ q_{21}k_1 + q_{22}k_2 + q_{23}k_3 + q_{24}k_4 &= -1 \\ q_{31}k_1 + q_{32}k_2 + q_{33}k_3 + q_{34}k_4 &= -1. \end{aligned}$$

Substituting the values of q_{ij} and $k_4 = 0$ gives:

$$\begin{aligned} -k_1 + \frac{1}{2}k_2 + \frac{1}{2}k_3 &= -1 \\ \frac{1}{4}k_1 - \frac{1}{2}k_2 &= -1 \\ \frac{1}{6}k_1 - \frac{1}{3}k_3 &= -1. \end{aligned}$$

This system has a unique solution, which is:

$$k = (7, 11/2, 13/2, 0).$$

2. Exercise 3.6.2, page 123.

   (a) The chain having generator matrix

$$Q = \begin{pmatrix} -2 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

   is easily seen to be irreducible. (Use, e.g., Theorem 3.2.1) By Theorem 3.6.2 we have that $p_{ij}(t) \to \lambda_j$ as $t \to \infty$, where λ is an invariant distribution, $\lambda Q = 0$. This last condition leads to the system

$$\begin{aligned} -2\lambda_1 + \lambda_4 &= 0 \\ \lambda_1 - \lambda_2 &= 0 \\ \lambda_1 + \lambda_2 - \lambda_3 &= 0 \\ \lambda_3 - \lambda_4 &= 0. \end{aligned}$$

   The solution is $\lambda = (\lambda_1, \lambda_1, 2\lambda_1, 2\lambda_1)$. After normalization we obtain

$$\lambda = (1/6, 1/6, 1/3, 1/3).$$

   Therefore,

$$p_{12}(t) \to 1/6.$$

(b) The matrix

$$Q = \begin{pmatrix} -2 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is not irreducible, so Theorem 3.6.2 does not apply. We will solve for $p_{12}(t)$ using Kolmogorov's equations. But first not that, intuitively, the limit should be 0 since the chain should eventually (with probability 1) reache the absorbing state 4 and never again return to 2. The below calculation will bear this out.

First note that Q is an upper-triangular matrix. The exponential of Q is also upper-triangular (think about the definition of e^{tQ} in terms of its Taylor series), so we have

$$P(t) = \begin{pmatrix} e^{-2t} & * & * & * \\ 0 & e^{-t} & * & * \\ 0 & 0 & e^{-t} & * \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

By Kolmogorov's backward equation we have (note: $p_{32} = p_{42} = 0$)

$$\begin{aligned} p_{12}' &= q_{11}p_{12} + q_{12}p_{22} + q_{13}p_{32} + q_{14}p_{42} \\ &= -2p_{12} + p_{22} \\ &= -2p_{12} + e^{-t}. \end{aligned}$$

Thus p_{12} is the solution of the initial value problem

$$p_{12}' + 2p_{12} = e^{-t}, \quad p_{12}(0) = 0.$$

This is a standard first order linear equation. It can be solved by the integrating factor method. The solution is

$$p_{12}(t) = e^{-t} - e^{-2t} \to 0 \text{ as } t \to \infty.$$

(c) The generator matrix

$$Q = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 2 & -2 \end{pmatrix}$$

has block form. From the Taylor series definition of e^{tQ} it is easy to see that

$$P(t) = e^{tQ} = \begin{pmatrix} A(t) & 0 \\ 0 & B(t) \end{pmatrix}$$

where $A(t) = (a_{ij}(t))$ is the exponential of the upper-left 2-by-2 block and $B(t)$ the exponential of the lower-right 2-by-2 block. So we have $p_{12}(t) = a_{12}(t)$. Now, the smaller Q-matrix

$$Q' = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

is irreducible, so we can apply Theorem 3.6.2 to it. The solution to $\lambda Q' = 0$ is $\lambda = (\lambda_1, \lambda_2)$ such that $\lambda_1 = \lambda_2 = 1/2$. Therefore

$$p_{12}(t) \to 1/2 \quad \text{as } t \to \infty.$$

(d) The matrix

$$Q = \begin{pmatrix} -2 & 1 & 0 & 1 \\ 0 & -2 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is not irreducible. Again, we try to find the function $p_{12}(t)$ explicitly. Using Kolmogorov's forward equations we obtain for p_{12} and p_{13} the following equations (note that $p_{11}(t) = e^{-2t}$):

$$\begin{aligned} p_{12}' &= e^{-2t} - 2p_{12} + p_{13} \\ p_{13}' &= 2p_{12} - p_{13}. \end{aligned}$$

To solve this system, notice that $(p_{12} + p_{13})' = e^{-2t}$, so

$$p_{12} + p_{13} = \left(1 - e^{-2t}\right)/2.$$

Substituting the value for $p13$ from this equation into the first equation in the above system gives:

$$p_{12}' + 3p_{12} = \left(1 + e^{-2t}\right)/2.$$

This is again an ordinary first order linear differential equation, which we can solve by the method of integrating factor. The initial condition is $p_{12}(0) = 0$. The result is:

$$p_{12}(t) = \frac{1}{6}\left(1 + 3e^{-2t} + 4e^{-3t}\right) \to 1/6.$$

3. Perform the following simulated experiment. A number of balls, some red and some black, are distributed between two urns. The balls are numbered from 1 to N. At random times one person picks a number between 1 and N and transfers the corresponding ball from its urn to the other. Independent of the first person and also at random times, a second person picks a number between 1 and N and replaces the corresponding ball with a red one if the chosen ball is from urn I, or black if the chosen ball is from urn II, keeping the new ball in the same urn as the old one.

(Note that if, for example, the chosen ball is red and it was in urn I, then the action does not change anything.) We assume that the sequence of actions of the first person have independent holding times which are exponential of parameter q_1, and the actions of the second person have independent holding times which are exponential of parameter q_2. We wish to find the long term fraction, b, of the N balls that are black and lie in urn II. Do this for the values $q_1 = 1$ and $q_2 = 0.1, 2$, and 20. For each value of q_2, draw a graph of the fraction of balls that are black and in urn II as a function of time. I suggest taking the following parameters: total number of balls: 100, total number of events (actions of the two persons): 20000. (It took about 45 seconds for each of the three runs of 20000 events.) What are the approximate values b for each q_2? Give a qualitative explanation for the values you obtain.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
tic
rand('seed',123)
N=100;          %number of balls
u=zeros(2,N);   %u(1,j)=0 if j in urn I, u(1,j)=1 if in urn II
                %u(2,j)=0 if j ball is red, u(2,j)=1 if black
q1=1;           %exponential rate for type 1 event
q2=2;           %exponential rate for type 2 event
t=[0];          %time sequence of events
b=[0];          %number of black balls in urn II at each time
K=20000;        %number of events in one run of the experiment
for k=1:K-1
    s=-log(rand)/(q1+q2);
    i=ceil(N*rand);      %choose ball at random
    x=(rand<q1/(q1+q2)); %decide if event 1 (x=1) or 2 (x=0)
    if x==1
        u(1,i)=rem(u(1,i)+1,2); %transfer ball i to other urn
    else
        u(2,i)=u(1,i); %change color of ball i according to urn
    end
    t=[t t(k)+s];
    b=[b sum(u(1,:).*u(2,:))/N];
end
stairs(t,b)
toc
%We get for q1=1 and
%q2=0.1 : b approaches approx. 0.25
%q2=2   : b approaches approx. 0.35
%q2=20  : b approaches approx. 0.5
%each run takes approx. 45 sec.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

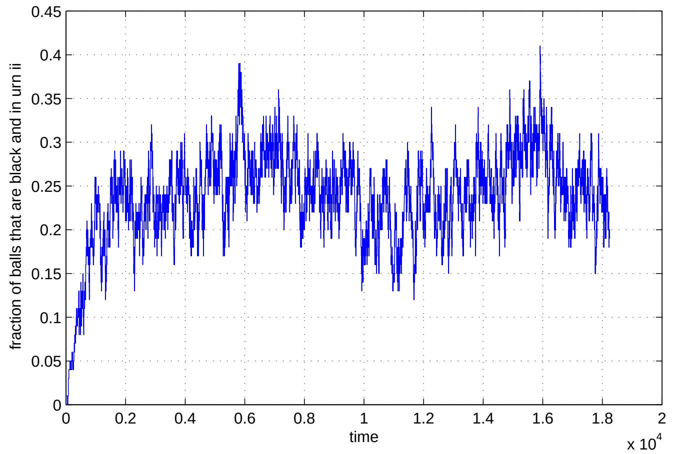

Figure 1: $q_1 = 1, q_2 = 0.1$. Black balls in urn II comprise approximately one-fourth of all balls.

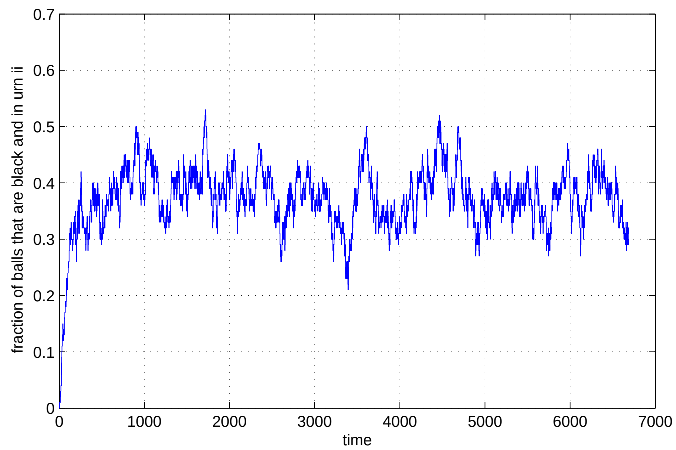

Figure 2: $q_1 = 1, q_2 = 2$. Black balls in urn II comprise approximately 0.35 of all balls.

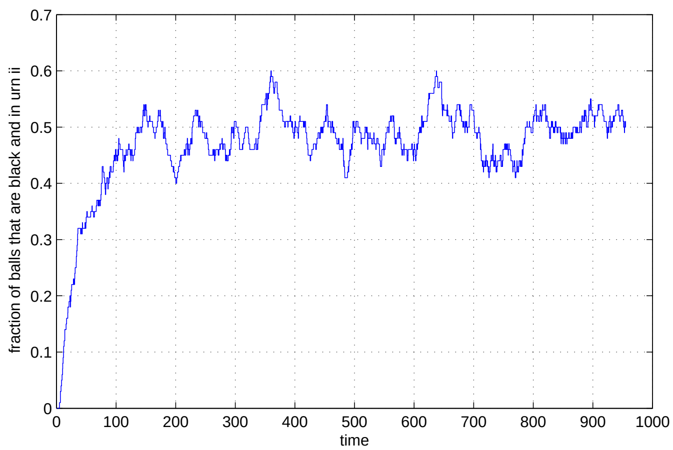

Figure 3: $q_1 = 1, q_2 = 2$. Black balls in urn II comprise approximately half of all balls.