## Math 450 - Homework 10 Solutions

1. Consider a fleet of $N$ buses. Each bus breaks down independently at rate $\mu$ and is then sent to the depot for repair. The repair shop can only repair one bus at a time and each bus takes an exponential time of parameter $\lambda$ to repair.
(a) The Petri net diagram for this situation is given in figure 1.


Figure 1: The bus-breaks-down reaction satisfies the mass-action law, but the bus-isrepaired reaction does not.
(b) Suppose now that $N=4$. The state transitions diagram with the transition rate constants is shown in figure 2 .


Figure 2: States are represented by the number of buses in working order.
(c) The transition rates matrix can be written as follows:

$$
Q=\left(\begin{array}{cccccc}
-\lambda & \lambda & 0 & 0 & \cdots & 0 \\
\mu & -\mu-\lambda & \lambda & 0 & \cdots & 0 \\
0 & 2 \mu & -2 \mu-\lambda & \lambda & \cdots & 0 \\
0 & 0 & 3 \mu & -3 \mu-\lambda & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & \lambda \\
0 & 0 & 0 & 0 & \cdots & -(N-1) \mu-\lambda \\
0 & 0 & 0 & 0 & \cdots & N \mu
\end{array}\right)
$$

An invariant probability measure $p$ must satisfy $p Q=0$. This corresponds to the system of equations:

$$
\begin{aligned}
& \lambda p_{0}+\mu p_{1}=0 \\
& \lambda p_{0}-(\mu+\lambda) p_{1}+2 \mu p_{2}=0 \\
& \lambda p_{1}-(2 \mu+\lambda) p_{2}+3 \mu p_{3}=0 \\
& \cdots \\
& \lambda p_{N-2}-((N-1) \mu+\lambda) p_{N-1}+N \mu p_{N}=0 .
\end{aligned}
$$

The solution to this system can be obtained by a simple induction. The result is $p=\left(p_{0}, p_{1}, \ldots, p_{N}\right)$ where

$$
p_{k}=\frac{(\lambda / \mu)^{k} / k!}{\sum_{j=0}^{N}(\lambda / \mu)^{j} / j!}
$$

The expected value of the number $B$ of in service buses is given by

$$
E[B]=\frac{\sum_{j=0}^{N} j(\lambda / \mu)^{j} / j!}{\sum_{j=0}^{N}(\lambda / \mu)^{j} / j!}=\frac{\lambda}{\mu} \frac{\sum_{j=0}^{N-1}(\lambda / \mu)^{j} / j!}{\sum_{j=0}^{N}(\lambda / \mu)^{j} / j!}
$$

For large values of $N$, this expected values approaches $\lambda / \mu$.
(d) We assume that $\mu=1 / 5, \lambda=1$, and $N=50$. By the result of the previous item the expected number of in service buses can be obtained numerically and equals 5.0000.
The following program gives the time average of the number of in service buses over sample paths. The value I obtained is 5.0359 (over 10000 transition events).
Notice that it would give a different result (somewhat higher value) if the average was computed without taking into account the holding times. This is because states corresponding to a higher number of in service buses have shorter holding time.
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```
tic
rand('seed',157)
N=50; %Total number of buses
lambda=1;
mu=1/5;
M=[5]; %Initial number of in service buses is N
K=10000; %Number of transition events in simulation
q=M(1)*mu+lambda*(M(1)<N);
t=[-log(rand)/q];
for k=1:K-1
    a=length(M);
    m=M(a);
    p=(lambda/(m*mu+lambda))*(m~}=N); %Probability of transition
                                    %m->m+1
    q=m*mu+lambda*(m<N);
    M=[M m-1+2*(rand<=p)];
    t=[t - log(rand)/q];
end
E=sum(M.*t)/sum(t)
toc
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

2. In problem 3.3.1 (HW 9), the value for the expected hitting time $k_{1}^{4}$ from state 1 to state 4 was obtained by solving the appropriate linear system and the value obtained was 7 . We want to confirm this value now by numerical simulation. The $Q$ matrix of the process is

$$
Q=\left(\begin{array}{rrrr}
-1 & 1 / 2 & 1 / 2 & 0 \\
1 / 4 & -1 / 2 & 0 & 1 / 4 \\
1 / 6 & 0 & -1 / 3 & 1 / 6 \\
0 & 0 & 0 & 0
\end{array}\right)
$$

The following program gave me the approximate value $k_{1}^{4}=7.0166$.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
tic
rand('seed',371)
Q=[-1 1/2 1/2 0; 1/4 -1/2 0 1/4; 1/6 0 -1/3 1/6; 0 0 0 0];
pi=[11 0 0 0}|]
s=0;
for i=1:10000
    [t y]=ctmc(10^7,pi,Q);
    a=length(t);
    s=s+t(a);
end
s=s/10000
```

```
toc
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

The above uses the program ctmc shown below.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [t y]=ctmc(n,pi,Q)
%Obtain a sample path with n events for a
%continuous-times Markov chain with initial
%distribution pi and generator matrix Q.
%The output consists of two row vectors:
%the event times }t\mathrm{ and the vector of states y.
%Vectors t and y may be shorter than n if
%an absorbing state is found before event n.
%Uses samplefromp(pi,n).
t=[0];
y=[samplefromp(pi,1)]; %initial state
for k=1:n-1
    i=y(k);
    q=-Q(i,i);
    if q==0
            break
        else
            s=-log(rand)/(-Q(i,i)); %exponential holding time
            t=[t t(k)+s];
            p=Q(i,:);
            p(i)=0;
            p=p/sum(p);
            y=[y samplefromp(p,1)];
        end
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

3. Problem from course notes on the predator-prey model with migration.
```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
U_pre=zeros(4,10);
U_pre(1,1:4)=[\begin{array}{llll}{1}&{1}&{0}&{1}\end{array}];
U_pre(2,2:3)=[1 1];U_pre(2,6)=1;
U_pre(3,5)=1;U_pre(3,8:9)=[1 1];
U_pre(4,7:10)=[1 0 1 1];
U_post=zeros(4,10);
U_post (1,1)=2;U_post (1,5)=1;
U_post (2,2)=2;U_post (2,7)=1;
U_post (3,4)=1;U_post (3,8)=2;
```



Figure 3: Changing population size of predator at the initially unoccupied site.

```
U_post (4, 6)=1;U_post (4,9)=2;
T=150;
dt=0.05;
c=[\begin{array}{llllllllllllll}{1}&{0.005}&{0.6}&{0.01}&{0.01}&{0.1}&{0.1}&{1}&{0.005}&{0.6}\end{array}];
x_0=[100 50 0 0)}]
[t y]=gillespied(U_pre,U_post,c,x_0,T,dt);
plot(t,y(4,:))
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

